NOTES ON BERNOULLI NUMBERS AND EULER'S SUMMATION FORMULA

MARK WILDON

1. Bernoulli numbers

1.1. **Definition.** We define the Bernoulli numbers B_m for $m \geq 0$ by

$$\sum_{r=0}^{m} \binom{m+1}{r} B_r = [m = 0] \tag{1}$$

Bernoulli numbers are named after Johann Bernoulli (the most prolific Bernoulli, and the discoverer of the Bernoulli effect).

1.2. **Exponential generating function.** If $f(z) = \sum a_n z^n/n!$ and $g(z) = \sum b_n z^n/n!$ then $f(z)g(z) = \sum c_n z^n/n!$ where the coefficients c_n are given by $c_n = \sum \binom{n}{r} a_r b_{n-r}$. Thus if we put $\beta(z) = \sum B_n z^n/n!$ then

$$\begin{aligned} [z^n]\beta(z)\exp z &= \frac{1}{n!}\sum B_r \binom{n}{n-r} \\ &= \frac{1}{n!}\sum_{r=0}^{n-1}\binom{n}{r}B_r + \frac{B_n}{n!} \\ &= [n = 1] + \frac{B_n}{n!} \end{aligned}$$

from which it follows that $\beta(z)\exp z = z + \beta(z)$. Therefore

$$\beta(z) = \sum B_n \frac{z^n}{n!} = \frac{z}{\exp z - 1}. \tag{2}$$

This shows that the radius of convergence of $\beta(z)$ is π. So although it is tempting to make a substitution in

$$\log P(\mathrm{e}^{-t}) = -\sum_n \log(1 - \mathrm{e}^{-tn}) = \sum_n \sum_m \frac{\mathrm{e}^{-tmn}}{m} = \sum_m \frac{1}{tm^2}\frac{tm}{\mathrm{e}^{tm} - 1}$$

this will not work, as tm is eventually more than π.

1.3. **Some values.** If we add $z/2$ to both sides of (2) we obtain

$$\beta(z) + \frac{z}{2} = \frac{z}{2}\frac{\exp z + 1}{\exp z - 1} = z/2 \coth z/2, \tag{3}$$

which is an even function of z. So $B_0 = 1$, $B_1 = -1/2$ and $B_n = 0$ if $n \geq 3$ is odd. Some further values are given below.

n	0	1	2	3	4	5	6	7	8
B_n	1	$-1/2$	$1/6$	0	$-1/30$	0	$1/42$	0	$-1/30$

Date: November 6, 2014.

Note that the numerators need not be ± 1. For instance $B_{10} = 5/66$.

1.4. **Connection with FLT.** Kummer proved that there are no positive integral solutions of $x^p + y^p = z^p$ whenever p is an odd prime not dividing the numerators of any of the Bernoulli numbers $B_2, B_4, \ldots, B_{p-3}$. Such primes are said to be *regular*. For instance as $B_{12} = 691/2730$, the prime 691 is *not* regular.

1.5. **Estimate for B_n.** In [1] it is noted that $B_{22} = \frac{854513}{138} > 6192$; indeed the Bernoulli numbers are unbounded. The authors explain that using Euler's formula for $\cot z$ (which has a nice elementary proof *via* the so-called Herglotz trick),

$$z \cot z = 1 - 2 \sum_{k \geq 1} \frac{z^2}{k^2 \pi^2 - z^2}$$

one gets

$$\begin{aligned} \beta(2z) + z &= z \coth z \\ &= 1 + 2 \sum_{k=1}^{\infty} \frac{z^2}{k^2 \pi^2 + z^2} \\ &= 1 - 2 \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \left(\frac{-z^2}{k^2 \pi^2} \right)^m \end{aligned}$$

for $|z| < \pi$. Hence, comparing coefficients of z^{2n},

$$B_{2n} = (-1)^{n-1} \frac{2(2n)!}{(2\pi)^{2n}} \zeta(2n).$$

Thus

$$\frac{2}{(2\pi)^{2n}} \leq \frac{|B_{2n}|}{(2n)!} \leq \frac{4}{(2\pi)^{2n}}.$$

Note this shows that $|B_{2n}|/(2n)!$ is a decreasing sequence. As these are coefficients in the convergent generating function $\beta(z)$, $|B_{2n}|/(2n)!$ tends to 0 as n tends to ∞.

1.6. **Connection with uniform distribution.** Let X be distributed uniformly on $[0, 1]$. Then the moment generating function for X is

$$E\mathrm{e}^{zX} = \int_0^1 \mathrm{e}^{zt} \mathrm{d}t = \frac{1}{x}(\mathrm{e}^x - 1) = \frac{1}{\beta(z)}.$$

The coefficients of $1/\beta(z)$ give us a recurrence satisfied by the Bernoulli numbers – but as

$$1/\beta(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!}$$

this is just the defining recurrence (1).

1.7. **Sums of powers.** We derive from first principles a special case of the Euler summation formula. Let

$$S_m(n) = 0^m + 1^m + \ldots + (n-1)^m = \sum_{k=0}^{n-1} k^m.$$

Consider the following generating function where m varies and n is fixed: $\hat{S}(z) = \sum_{m=0}^{\infty} S_m(n) \frac{z^m}{m!}$. We have

$$\begin{aligned} \hat{S}(z) &= \sum_{m=0}^{\infty} \left(\sum_{k=0}^{n-1} k^m \right) \frac{z^m}{m!} = \sum_{k=0}^{n-1} \sum_{m=0}^{\infty} \frac{(kz)^m}{m!} = \frac{\exp nz - 1}{\exp z - 1} \\ &= \beta(z) \frac{\exp nz - 1}{z} = \left(\sum_{n=0}^{\infty} \frac{B_n}{n!} z^n \right) \left(\sum_{m=0}^{\infty} \frac{n^{m+1}}{(m+1)!} z^m \right) \end{aligned}.$$

Comparing coefficients of $z^m/m!$ gives

$$S_m(n) = m! \sum_{r=0}^{m} \frac{B_r}{r!} \frac{n^{m-r+1}}{(m-r+1)!} = \sum_{r=0}^{m} \frac{B_r}{m+1} \binom{m+1}{r} n^{m+1-r}. \tag{4}$$

1.8. **Bernoulli polynomials.** Define

$$B_m(z) = \sum_k \binom{m}{k} B_k z^{m-k}.$$

The first few Bernoulli polynomials are $B_0(z) = 1$, $B_1(z) = z - \frac{1}{2}$, $B_2(z) = z^2 - z + \frac{1}{6}$, $B_3(z) = z^3 - \frac{3}{2}z + \frac{1}{2}$, on so on. Note that $B_m(0) = B_m$ and that

$$B_{m+1}(n) = \sum_k \binom{m+1}{k} B_k n^{m+1-k} = (m+1) S_m(n) + B_{m+1}$$

so

$$S_m(n) = \frac{B_{m+1}(n) - B_{m+1}(0)}{m+1}$$

which is the obvious definition for $S_m(z)$.

Some useful properties:

$$B_m(1) = \sum_k \binom{m}{k} B_k = \sum_{k=1}^{m-1} \binom{m}{k} B_k + B_m = [m=1] + B_m$$

so $B_m(1) = B_m(0) = B_m$ unless $m = 1$ in which case $B_1(1) = -\frac{1}{2} = -B_1$. For $m \geq 1$ we have

$$\begin{aligned} B_m(z)' = \sum_k \binom{m}{k} B_k (m-k) z^{m-k-1} &= m \sum_k \binom{m-1}{k} B_k z^{m-k-1} \\ &= m B_{m-1}(z). \end{aligned}$$

2. The Euler summation formula

From now on let $f : \mathbb{R} \to \mathbb{R}$ be a function with as many derivatives as needed. Euler's summation formula is:

Theorem 2.1 (Euler). *Let $m \geq 1$ and let $a < b$ be integers. Then*

$$\sum_{a \leq k < b} f(k) = \int_a^b f(x)\mathrm{d}x + \sum_{k=1}^{m} \frac{B_k}{k!} f^{(k-1)}(x)\Big|_a^b + R_m \tag{5}$$

where the remainder R_m is given by

$$R_m = (-1)^{m+1} \int_a^b \frac{B_m(\{x\})}{m!} f^{(m)}(x)\mathrm{d}x.$$

2.1. **Application to sums of powers.** Set $f(x) = x^r$ and take $m > r$. Then $f^{(m)} = 0$ so the remainder term vanishes. Putting $a = 0$ and $b = n$ we obtain

$$\sum_{k=1}^{n-1} k^r = \frac{n^{r+1}}{r+1} + \sum_{k=1}^{r+1} \frac{B_k}{k!} r^{\underline{k-1}} n^{r+1-k} = \sum_{k=0}^{r+1} \frac{B_k}{r+1}\binom{r+1}{k} n^{r+1-k}$$

which agrees with (4).

2.2. **First proof.** As all the terms telescope nicely it is sufficient to prove the formula when $a = 0$ and $b = 1$. This has the advantage that we can use $B_m(x)$ rather than $B_m(\{x\})$. We proceed by induction on m. The case $m = 1$ states that

$$f(0) = \int_0^1 f(x)\mathrm{d}x - \tfrac{1}{2}(f(1) - f(0)) + \int_0^1 (x - \tfrac{1}{2}) f'(x)\mathrm{d}x.$$

This follows from a simple integration by parts:

$$\int_0^1 (x - \tfrac{1}{2}) f'(x)\mathrm{d}x = f(1) - \int_0^1 f(x)\mathrm{d}x - \tfrac{1}{2}(f(1) - f(0)).$$

For $m > 1$ we can write the right-hand-side as

$$\int_0^1 f(x)\mathrm{d}x + \sum_{k=1}^{m-1} \frac{B_k}{k!}(f^{(k-1)}(1) - f^{(k-1)}(0)) \\ + \frac{B_m}{m!}(f^{(m-1)}(1) - f^{m-1}(0)) + (-1)^{m+1} \int_0^1 \frac{B_m(x)}{m!} f^{(m)}(x)\mathrm{d}x$$

Applying the inductive hypothesis gives:

$$f(0) + (-1)^{m+1} \int_0^1 \frac{B_{m-1}(x)}{(m-1)!} f^{(m-1)}(x) \\ + \frac{B_m}{m!}(f^{(m-1)}(1) - f^{(m-1)}(0)) + (-1)^{m+1} \int_0^1 \frac{B_m(x)}{m!} f^{(m)}(x)\mathrm{d}x$$

Integrating the final term by parts using the results in §1.8 gives

$$f(0) + (-1)^{m+1} \int_0^1 \frac{B_{m-1}(x)}{(m-1)!} f^{(m-1)}(x) + \frac{B_m}{m!}(f^{(m-1)}(1) - f^{(m-1)}(0)) \\ + (-1)^{m+1} \frac{B_m(x)}{m!} f^{(m-1)}(x)\Big|_0^1 + (-1)^m \int_0^1 \frac{B_{m-1}(x)}{(m-1)!} f^{(m-1)}(x)\mathrm{d}x.$$

The outermost two terms obviously cancel. And so do the inner two, as if m is odd, then as $m > 1$, $B_m = 0$. $\square$

2.3. **An alternative proof.** There used to be a section here claiming an alternating proof by taking a sequence of polynomials converging uniformly to f on the interval $[a,b]$ and using uniform convergence to interchange the integral with the limit of the polynomials: this reduces to the case where $f(x)=x^r$. However it is no longer possible to reduce to the case where $a=0$ and $b=1$, and the main inductive step in the proof is not substantially simplified from the previous section.

It is however worth noting that if $m>r$ and we take $a=0$ and $b=N$ then the result comes out very easily. In this case Euler's formula says that

$$\sum_{0\le k<N} k^r = \frac{N^{r+1}}{r+1} + \sum_{k=1}^{r} \frac{B_k}{k!} r^{\underline{k-1}} N^{r-(k-1)}$$

Using the identity $\frac{1}{k}\binom{r}{k-1} = \frac{1}{r+1}\binom{r+1}{k}$, the right-hand side can be rewritten as

$$\frac{N^{r+1}}{r+1} + \sum_{k=1}^{r} \frac{B_k}{r+1}\binom{r+1}{k} N^{r+1-k}$$

which is equal to $\sum_{0\le k<N} k^r$ by (4).

2.4. **Estimates for error term.** We can state Euler's summation formula in the following form

$$\sum_{a\le k<b} f(k) = \int_a^b f(x)\mathrm{d}x - \tfrac{1}{2}(f(b)-f(a)) + \sum_{k=1}^{m} \frac{B_{2k}}{(2k)!} f^{(2k-1)}(x)\Big|_a^b + R_{2m}. \tag{6}$$

It can be shown that $|B_{2m}(\{x\})| \le B_{2m}$ so a rough estimate for the error term is given by §1.5:

$$|R_{2m}| \le \frac{B_{2m}}{(2m)!}\int_a^b |f^{(2m)}(x)|\mathrm{d}x.$$

If $f^{(2m)}$ is positive then this shows that the magnitude of the error term is at most the magnitude of the final term in the sum.

Note that very often the remainder term R_{2m} will not tend to 0 as the upper limit in the summation, b tends to ∞. However it will often have *some* non-zero limit. In this case we have

$$R_{2m} = R_{2m}(\infty) - \int_b^\infty \frac{B_{2m}(\{x\})}{(2m)!} f^{(2m)}(x)\mathrm{d}x$$

where the right-hand-side tends to $R_{2m}(\infty)$ as $b\to\infty$.

It is proved in [1] p475 that if $f^{(2m+2)}$ and $f^{(2m+4)}$ are positive for $x\in[a,b]$ then

$$R_{2m} = \theta_m \frac{B_{2m+2}}{(2m+2)!} f^{2m+1}(x)\Big|_a^b$$

where $0\le\theta_m\le 1$. So the remainder term lies somewhere between 0 and what would have been the next term in the sum.

3. Examples of Euler summation

3.1. **First example.** We shall attempt to use Euler summation to find

$$S_n = \sum_{k=0}^{n-1} t^k$$

where $0 \leq t < 1$. Of course we already know the answer, $S_n = (1-t^n)/(1-t)$, and in fact Euler summation turns out to be a very ineffective way of finding it! Still there are some points of interest.

If $f(x) = t^x$ then $f^{(r)}(x) = (\log t)^r t^x$ so by equation (6) we have

$$\begin{aligned}\sum_{k=0}^{n-1} t^k &= \int_0^n t^x \mathrm{d}x - \tfrac{1}{2}(t^n - 1) + \sum_{k=1}^{m} \frac{B_{2k}}{(2k)!} (\log t)^{2k-1} t^x \Big|_0^n + R_{2m} \\ &= (1-t^n)\left(\frac{1}{2} - \frac{1}{\log t} - \sum_{k=1}^{m} \frac{B_{2k}}{(2k)!}(\log t)^{2k-1}\right) + R_{2m}\end{aligned}$$

for $n \geq 1$ and $m \geq 1$. Now something a little suprising happens: if $t > \mathrm{e}^{-\pi}$ then $|\log t| < \pi$ and so we can use the exponential generating function for the Bernoulli numbers to obtain

$$\sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}(\log t)^{2k-1} = \frac{1}{t-1} - \frac{1}{\log t} + \frac{1}{2}$$

and so

$$\sum_{k=0}^{n-1} t^k = \frac{1-t^n}{1-t} + (1-t^n) \sum_{k=m+1}^{\infty} \frac{B_{2k}}{(2k)!}(\log t)^{2k-1} + R_{2m}$$

As we know the answer, the last equation implies that $R_{2m} \to 0$ as $m \to \infty$. Less artificially, we can attempt to prove this directly. We find

$$\begin{aligned}R_{2m} &= (\log t)^{2m-1} \int_0^n \frac{B_{2m}(\{x\})}{(2m)!} t^x \mathrm{d}x \\ &\leq |(\log t)|^{2m-2} \frac{B_{2m}}{(2m)!}(1-t^n) \\ &\leq |(\log t)|^{2m-2} \frac{4}{(2\pi)^{2m}}(1-t^n)\end{aligned}$$

So the error term $R_{2m} \to 0$ as $m \to \infty$, provided $\mathrm{e}^{-2\pi} < t < \mathrm{e}^{2\pi}$, which holds as we have already assumed that $\mathrm{e}^{-\pi} < t < 1$.

The limiting behaviour with respect to n is also of interest. Knowing the limit of S_n allows us to see that

$$\lim_{n\to\infty} R_{2m} = -\sum_{k=m+1}^{\infty} \frac{B_{2k}}{(2k)!}(\log t)^{2k-1}$$

This is an example of the point made earlier, that while the remainder term will not usually tend to 0 as n tends to ∞, it may well have some limiting value.

3.2. **Summing square roots.** Let $f(x) = \sqrt{x}$. Note that $\int_0^n f(x)\mathrm{d}x = \frac{2}{3}(n^{\frac{3}{2}} - 1)$ and that

$$f^{(r)}(x) = \binom{\frac{1}{2}}{r} r!\, x^{\frac{1}{2}-r}.$$

Euler's summation formula gives

$$\sum_{k=1}^{n} \sqrt{k} = \tfrac{2}{3}(n^{\frac{3}{2}} - 1) + \tfrac{1}{2}\sqrt{n} + \tfrac{1}{2} + \sum_{k=1}^{m} \frac{B_{2k}}{2k} \binom{\frac{1}{2}}{2k-1} (n^{\frac{3}{2}-2k} - 1) + R_{2m}.$$

Setting $m = 1$ we get

$$\sum_{k=1}^{n} \sqrt{k} = \tfrac{2}{3}(n^{\frac{3}{2}} - 1) + \tfrac{1}{2}\sqrt{n} + \tfrac{1}{2} + \tfrac{1}{12}(n^{-\frac{1}{2}} - 1) + R_2.$$

We can estimate R_2 by using the bound $|B_2(\{x\})| < B_2 = \frac{1}{6}$:

$$|R_2| \leq \int_1^n \tfrac{1}{6.8} x^{-\frac{3}{2}} = \tfrac{1}{24}(1 - n^{-\frac{1}{2}})$$

So we have

$$\sum_{k=1}^{n} \sqrt{k} = \tfrac{2}{3}(n^{\frac{3}{2}} - 1) + \tfrac{1}{2}\sqrt{n} + C + \mathrm{O}(n^{-\frac{1}{2}})$$

for some constant C. In general the remark on the previous page shows that the error is given by

$$R_{2m} = \theta_m \frac{B_{2m+2}}{2m+2} \binom{\frac{1}{2}}{2m-1} (n^{\frac{1}{2}-2m} - 1) = C(m) + \mathrm{O}(n^{\frac{1}{2}-2m})$$

where $\theta_m \in [0, 1]$ and $C(m)$ does not depend on n, This gives

$$\sum_{k=1}^{n} \sqrt{k} = \tfrac{2}{3}(n^{\frac{3}{2}} - 1) + \tfrac{1}{2}\sqrt{n} + C(m) + \sum_{k=1}^{m} \frac{B_{2k}}{2k} \binom{\frac{1}{2}}{2k-1} n^{\frac{3}{2}-2k} + \mathrm{O}(n^{\frac{1}{2}-2m}).$$

There is a small simplification: $C(m)$ can be determined by taking a limit with respect to n, so

$$C = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \sqrt{k} - \tfrac{2}{3}(n^{\frac{3}{2}} - 1) - \tfrac{1}{2}\sqrt{n} \right)$$

and so does not depend on m. (Exercise 9.27 in [1] reveals that $C = \zeta(-\frac{1}{2})$; in fact the definition of $\zeta(\alpha)$ for $\alpha > -1$.)

3.3. **An estimate for $P(x)$.**

$$\begin{aligned}
\log P(\mathrm{e}^{-t}) &= \sum_{k=1}^{\infty} -\log(1 - \mathrm{e}^{-kt}) \\
&= \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \frac{\mathrm{e}^{-tmk}}{m} \\
&= \sum_{m=1}^{\infty} \frac{1}{e^{mt} - 1}.
\end{aligned}$$

This suggests 2 ways we might apply Euler summation. (But they turn out to be equivalent.)

3.3.1. *Taken from Knuth fascicle Exercise* 25. Let

$$f(x) = -\log(1-\mathrm{e}^{-tx}) = \sum_{m=1}^{\infty} \frac{e^{-mtx}}{m}.$$

Then we have $\log P(\mathrm{e}^{-t}) = \sum_{k=1}^{\infty} f(k)$. The integral of f is given by the Li_2 function:

$$\int_1^x f(u)\mathrm{d}u = \sum_{m=1}^{\infty} \frac{e^{-mtu}}{tm^2}\bigg|_x^1 = \sum_{m=1}^{\infty} \frac{\mathrm{e}^{-mt}}{tm^2} - \sum_{m=1}^{\infty} \frac{\mathrm{e}^{-mtx}}{tm^2} = \frac{\mathrm{Li}_2(\mathrm{e}^{-t})}{t} - \frac{\mathrm{Li}_2(\mathrm{e}^{-tx})}{t}.$$

The derivatives of f are connected with the Eulerian numbers: let $\left\langle {n \atop m} \right\rangle$ denote the number of permutations in S_n with exactly m ascents (or descents). Then we claim

Lemma 3.1. *For $n \geq 1$ we have*

$$f^{(n)}(x) = \frac{\mathrm{e}^{-tx}(-t)^n}{(1-\mathrm{e}^{-tx})^n} \sum_k \left\langle {n-1 \atop k} \right\rangle \mathrm{e}^{-ktx}.$$

Proof. By induction on m. If $m = 1$ then the formula is readily verified. Suppose true for m. Then

$$\begin{aligned}
f^{(m+1)}(x) &= \frac{-m\mathrm{e}^{-2tx}(-t)^m t}{(1-\mathrm{e}^{-tx})^{m+1}} \sum_k \left\langle {n-1 \atop k} \right\rangle \mathrm{e}^{-ktx} \\
&\quad + \frac{\mathrm{e}^{-tx}(-t)^n(-t)}{(1-\mathrm{e}^{-tx})^n} \sum_k k \left\langle {n-1 \atop k} \right\rangle \mathrm{e}^{-ktx} \\
&= \frac{\mathrm{e}^{-tx}(-t)^{n+1}}{(1-\mathrm{e}^{-tx})^{n+1}} \left(\sum_k n \left\langle {n-1 \atop k-1} \right\rangle + (1-\mathrm{e}^{-tx}) \sum_k k \left\langle {n-1 \atop k} \right\rangle \mathrm{e}^{-ktx} \right) \\
&= \frac{\mathrm{e}^{-tx}(-t)^{n+1}}{(1-\mathrm{e}^{-tx})^{n+1}} \left((n-k) \left\langle {n-1 \atop k-1} \right\rangle + k \left\langle {n-1 \atop k} \right\rangle \right) \mathrm{e}^{-ktx}
\end{aligned}$$

To finish we apply the identity[1] $(n-m)\left\langle {n-1 \atop m-1} \right\rangle + (m+1)\left\langle {n-1 \atop m} \right\rangle = \left\langle {n \atop m} \right\rangle$. $\square$

We are now ready to apply Euler summation. Taking $m = 1$ in (6) we get

$$\sum_{k=1}^{n} f(k) = \frac{\mathrm{Li}_2(\mathrm{e}^{-t})}{t} - \frac{\mathrm{Li}_2(\mathrm{e}^{-nt})}{t} - \tfrac{1}{2}(\log(1-\mathrm{e}^{-nt}) + \log(1-\mathrm{e}^{-t})) \\ + \tfrac{1}{12} \frac{\mathrm{e}^{-tx}(-t)}{1-\mathrm{e}^{-tx}}\bigg|_1^n + R_2.$$

[1] This identity can be proved as follows. Let $g \in S_n$ have exactly m ascents. If n appears in a position $\ldots bna \ldots$ or $\ldots n$ then removing n gives a permutation in S_{n-1} with $m-1$ ascents. Conversely, given such a permutation, there are $n-m-1$ *descents*, and putting n between any 2 numbers involved in a descent, or at the end, gives a permutation in S_n with m ascents. Otherwise we have $\ldots anb \ldots$ or $n \ldots$ in which case removing n does not change the number of ascents. Conversely given a permutation in S_{n-1} with m ascents, putting n between the two 2 numbers involved in an ascent, or at the start, does not change the number of ascents.

where as the even derivatives are positive

$$R_2 \leq \tfrac{1}{12} \frac{\mathrm{e}^{-tx}(-t)}{1-\mathrm{e}^{-tx}}\Big|_1^n \leq \frac{t\mathrm{e}^{-t}}{1-\mathrm{e}^{-t}} = \frac{t}{\mathrm{e}^t - 1} \leq 1.$$

So letting $n \to \infty$ we get

$$\log P(\mathrm{e}^{-t}) = \frac{\mathrm{Li}_2(\mathrm{e}^{-t})}{t} - \tfrac{1}{2}\log(1-\mathrm{e}^{-t}) + \mathrm{O}(1)$$

where $O(1)$ stands for a quantity always lying in $[0,1]$. If we use the identity suggested by Knuth, namely

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) = \zeta(2) - \log x \log(1-x)$$

or, equivalently,

$$\frac{\mathrm{Li}_2(\mathrm{e}^{-t})}{t} = -\frac{\mathrm{Li}_2(1-\mathrm{e}^{-t})}{t} + \frac{\zeta(2)}{t} + t\log(1-\mathrm{e}^{-t}$$

we get

$$\begin{aligned}\log P(\mathrm{e}^{-t}) &= \frac{\zeta(2)}{t} - \frac{\mathrm{Li}_2(1-\mathrm{e}^{-t})}{t} - \tfrac{3}{2}\log(1-\mathrm{e}^{-t}) + \mathrm{O}(1)\\ &= \frac{\zeta(2)}{t} - \tfrac{1}{2}\log(1-\mathrm{e}^{-t}) + O(1) \quad ; \text{where } O(1) \in [-1,1]\\ &= \frac{\zeta(2)}{t} - \frac{\log t}{2} + \mathrm{O}(1) \quad ; \text{where } O(1) \in [-1-\log 2, 1-\log 2].\end{aligned}$$

As $\mathrm{Li}_2(y) \leq -\log(1-y)$ gives $\mathrm{Li}_2(1-\mathrm{e}^{-t}) \leq t$ and $\log(1-\mathrm{e}^{-t}) = \log t + \log(1-t/2+\ldots) = \log t + \mathrm{O}(1)$. In particular by being a little more careful with the $\mathrm{O}(1)$ errors we can get a lower bound for $\log P(\mathrm{e}^{-t})$:

$$\frac{\pi^2}{6t} + \frac{\log t}{2} - 1 - \log 2 \leq \log P(\mathrm{e}^{-t}) \leq \frac{\pi^2}{6t} + \frac{\log t}{2} + 1 - \log 2.$$

In fact the 'right' constant is $-\log(\sqrt{2\pi}) \approx -0.9189$; this follows from

$$\log P(e^{-t}) = \frac{\pi^2}{6t} + \frac{\log t}{2} - \log\sqrt{2\pi} + O(t)$$

where the $O(t)$ term is (by the functional equation), $-\frac{t}{24} + \log P(e^{-4\pi^2/t})$. Note that

$$-1.6931 < -0.9189 < .6931.$$

3.3.2. *Alternative.* We might also try to use Euler summation to sum

$$\sum_{k=1}^{\infty} -\log(1-\mathrm{e}^{-kt}).$$

Perhaps surprisingly this turns out to be equivalent to the previous approach, since

$$\frac{\mathrm{d}}{\mathrm{d}x}\log(1-\mathrm{e}^{-xt}) = \frac{t\mathrm{e}^{-xt}}{1-\mathrm{e}^{-xt}} = \frac{t}{e^{xt}-1}$$

which is essentially the function we summed earlier.

References

[1] Ronald L. Graham, Donald E. Knuth, and Oren Patashnik. *Concrete Mathematics.* Addison Wesley, 1994.