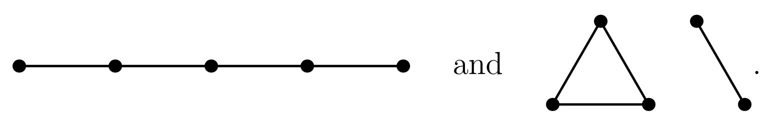
MAP363 Combinatorics Answers 1

Guidance on notation: graphs may have multiple edges, but may not have loops. A graph is simple if it has no multiple edges.

1. (a) *Show that if two graphs have the same degree sequence then they have the same number of vertices and the same number of edges. Find two non-isomorphic graphs with the degree sequence* $(2, 2, 2, 1, 1)$.

Suppose G and G' are graphs with degree sequence $(d_1, \ldots, d_n)$. Each entry records the degree of one vertex, so both G and G' have n vertices. By the Handshaking Theorem, $d_1 + \ldots + d_n$ is equal to twice the number of edges in either graph. So G and G' have the same number of edges.

Two non-isomorphic graphs with degree sequence $(2, 2, 1, 1, 1)$ are



(There is also a non-simple graph with this degree sequence.)

(b) *Find all simple graphs on* 4 *vertices, up to isomorphism.*

There are 11 simple graphs on 4 vertices (up to isomorphism). Any such graph has between 0 and 6 edges; this can be used to organise the hunt. (With more vertices, it might also be useful to first work out the possible degree seqences.) The table below show the number of graphs for edge possible number of edges.

edges	0	1	2	3	4	5	6
number of graphs	1	1	2	3	2	1	1

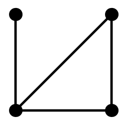
You might like to work out the reason for the symmetry in the table.

2. *For each of the sequences below, decide whether or not it is the degree sequence of a graph. (If it is, give an explicit example, if not, say why not.)*

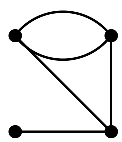
$$\text{(i) } (9,7,5,3,1), \text{ (ii) } (3,2,2,1), \text{ (iii) } (10,3,3,2), \text{ (iv) } (3,3,3,1).$$

Do any of your answers change if the graph has to be simple?

(i) No: by the Handshaking Theorem, the sum of the degrees of a graph is even.

(ii) Yes, for example  .

(iii) No: the vertex of greatest degree can have at most $3+3+2$ edges coming into it, so its degree is at most 8.

(iv) Yes, for example  .

Suppose G is a simple graph with this degree sequence. Then G has vertices x, y, z of degree 3, and one further vertex, w, of degree 1. Each of x, y, z is connected to all the other 3, so in particular to w. But this says w has degree 3, a contradiction.

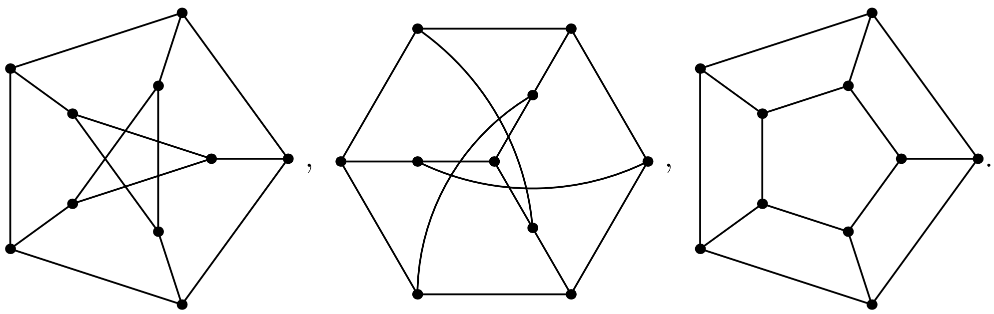
3. *Let G and G' be graphs. Suppose that $f : V(G) \to V(G')$ is an isomorphism. Let $x, y \in V(G)$. Use induction to show that that the distance between x and y in G is equal to the distance between $f(x)$ and $f(y)$ in G'.*

Note: I should have added the assumption that G is connected — without this there might be no path between x and y, and the distance between them would be undefined. (It follows from the proof below that if G is connected then so is G', so we only have to make this assumption for one of the graphs.)

Write $d(x, y)$ for the distance between x and y. If $d(x, y) = 0$ then $x = y$, so $f(x) = f(y)$ and so $d(f(x), f(y)) = 0$. This gives the base case in the induction. Now suppose that $d(x, y) = m \geq 1$. Let $x_1, \ldots, x_r$ be the neighbours of x in G. Pick a path of shortest length from x to y. It must go through one of $x_1, \ldots, x_r$; we may choose the labelling so that it passes through x_1. Thus $d(x_1, y) = m - 1$. By induction, $d(f(x_1), f(y)) = m - 1$. Since $f(x)$ and $f(x_1)$ are adjacent in G', this shows that $d(f(x), f(y)) \leq m$.

Suppose for some i we have $d(f(x_i), f(y)) < m-1$. By another application of the inductive hypothesis, $d(x_i, y) < m - 1$, so $d(x, y) < m$, a contradiction. Hence it takes a least $m-1$ steps to go from any neighbour of $f(x)$ to $f(y)$, and so at least m steps to go from $f(x)$ to $f(y)$. This shows that $d(f(x), f(y)) = m$. The inductive step is now complete.

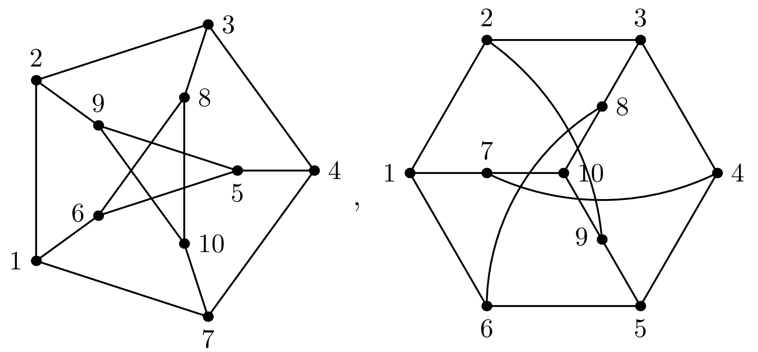
4. (a) *Show that the graphs below all have the same degree sequence.*



Inspection shows that all vertices have degree 3.

(b) *Show that the first two graphs are isomorphic.* Hint: *start by finding a closed path of length* 6 *in the first.*

Following the hint, we start by taking the closed path $1, 2, 3, 4, 5, 6$ in the first graph, and mapping this to the closed path forming the outside of the second graph. We then check that we can consistently complete the labelling.



(c) *Show that neither is isomorphic to the third.*

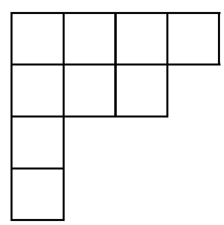
There are many possible arguments. For example, using Question 3, we could argue that the third graph has two vertices a distance 3 apart, but the first does not.

5. *Let G be a connected graph with k vertices of odd degree, where $k > 0$. Show that the minimum number of trails with mutually distinct edges needed to cover every edge of G is $k/2$.*

It follows from the Handshaking Theorem that k is even: say $k = 2r$. Say $z_1, \ldots, z_{2r}$ have odd degree. Introduce r further edges $\{z_1, z_2\}, \ldots, \{z_{2r-1}, z_{2r}\}$. This gives a new graph in which every vertex has even degree. By Theorem 3.4, this new graph has a closed Eulerian trail. If one cuts out from this graph the r edges not present in G, then we are left with $r = k/2$ trails covering G.

We must also show that no fewer trails suffice. If we remove from G all the edges involved in a single trail, then we can change the parity of the degree at most 2 vertices, namely the vertices at the start and end of the trail. So as G has $2r$ vertices of odd parity, we need at least r trails.

How many continuous pen-strokes are needed to draw the diagram below if it is forbidden to draw any line segment twice?



As there are 8 vertices of odd degree, 4 trails are the minimum. You can either use the algorithm above to find 4 suitable trails, or just do it by inspection.

6. (a) *Show that the complete graph on n vertices has $n(n-1)/2$ edges.*

There are n vertices, each connected to $n-1$ other vertices. This gives $n(n-1)$ edges, but double counting each one (as in the Handshaking Theorem).

Or argue that the edges are in bijection with 2-subsets of n, of which there are $\binom{n}{2} = n(n-1)/2$.

(b) *Show that if G is a simple graph on 6 vertices which is not connected then G has at most 10 edges. Can equality occur?*

Suppose G has a connected component with m vertices. This component has at most $m(m-1)/2$ edges, by the previous part, None of the remaining $6-m$ vertices are connected to these m, so they can contribute at most $(6-m)(6-m-1)/2$ edges. So the total number of edges is at most

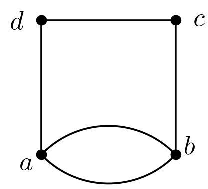
$$\frac{m(m-1)}{2} + \frac{(6-m)(6-m-1)}{2}$$

Now put in $m = 1, 2, 3, 4, 5$ and check that in all cases the number of edges is at most 10. Equality occurs if we have a copy of the complete graph on 5 vertices, with the 6th vertex not connected to any others.

MAP363 Combinatorics Answers 2

Guidance on notation: graphs may have multiple edges, but may not have loops. A graph is simple if it has no multiple edges.

1. *Which of the graphs shown below have (i) a closed Eulerian trail? (ii) a closed Hamiltonian path?*



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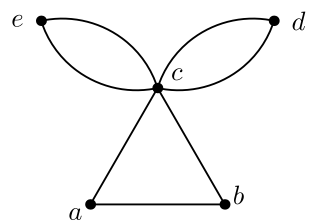

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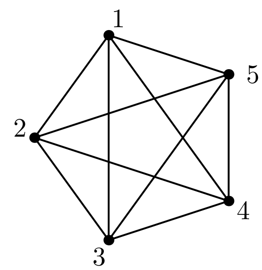
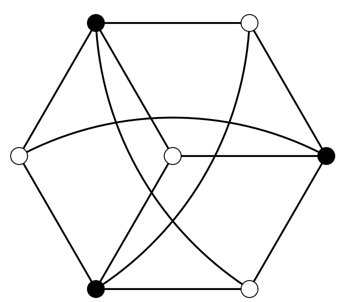


The first has (a, b, c, d, a) as a closed Hamiltonian path. It doesn't have a closed Eulerian trail as not all vertices have degree 2 (see Lemma 3.3 from lectures).

It's fairly obvious that any walk in the second graph visiting all the vertices must pass through vertex c twice. Hence it doesn't have a closed Hamiltonian path. It has the closed Eulerian trail (c, e, c, d, c, b, a, c).

The third has a closed Hamiltonian path: $(1, 2, 3, 4, 5, 1)$ and a closed Eulerian trail: $(1, 2, 3, 4, 5, 1, 4, 2, 5, 3, 1)$. *Further exercise:* count the number of such paths and trails.

2. *Show that the graph below is isomorphic to the complete bipartite graph $K_{4,3}$.*



For which $n \in \mathbf{N}_0$ does $K_{4,3}$ have (i) a closed path of length n; (ii) a closed trail of length n?

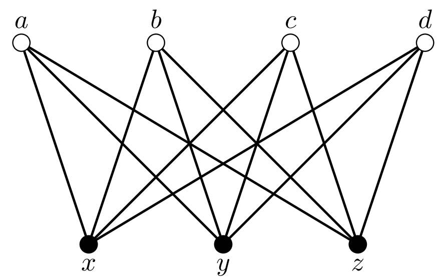
The isomorphism is indicated by the colouring: 4 white vertices are connected to each of the 3 black vertices.

We don't allow closed paths (or walks or trails) of length 0. As there are no multiple edges, there are no closed paths or closed trails of length 2. (Any such path looks like (x, y, x) for two vertices x and y.)

Clearly there are closed paths of lengths 4 and 6. There are only 7 vertices, so no path can have length > 7.

Turn over

For trails it it useful to use the alternative representation of $K_{4,3}$ shown below.



The paths already found give closed trails of lengths 4 and 6. A closed trail of length 8 is $(a, x, b, y, c, z, d, y, a)$. A closed trail of length 12 would be an Eulerian trail (there are $3 \times 4 = 12$ edges), but $K_{4,3}$ has vertices of degree 3, so we know no such trail can exist.

It remains to show that there is no closed trail of length 10. Any such trail visits 9 vertices, alternately visiting vertices from the sets $\{x, y, z\}$ and $\{a, b, c, d\}$. It must therefore visit vertices in the set $\{x, y, z\}$ at least 4 times. Without loss of generality we may assume that x is visited twice. This requires 4 edges attached to x, but x only has degree 3 (contradiction).

3. *Use Euler's formula to give an alternative proof that $K_{3,3}$ is not planar.* Hint: *adapt the proof given in lectures for K_5, noting that each face in a planar drawing of $K_{3,3}$ must be bounded by at least 4 edges.*

Suppose for a contradiction that $K_{3,3}$ may be drawn in the plane, with f faces. This graph has 6 vertices and $3 \times 3 = 9$ edges, so by Euler's formula we have $6 - 9 + f = 2$. Hence $f = 5$. Suppose that the faces have $e_1, e_2, \ldots, e_5$ edges respectively. Counting up edges by walking round each face in turn gives $e_1 + \ldots + e_5$ edges; this counts every edge twice as each edge bounds exactly 2 faces. Thus

$$e_1 + e_2 + e_3 + e_4 + e_5 = 2 \times 9 = 18.$$

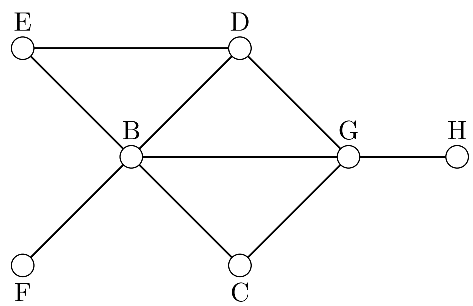
On the other hand, a bipartite graph can't have any triangles (see Q6(a) for a more general result), so each face is bounded by at least 4 edges. Hence

$$e_1 + \ldots + e_5 \geq 5 \times 4 = 20$$

which contradicts the previous equation.

4. *Professors Beeblebrox, Catalan, Descartes, Euler, Frobenius, Gauss and Hamilton go punting. Gauss considers it beneath his dignity to be in the same punt as anyone except Euler or Frobenius. Euler will tolerate Gauss, but dislikes Descartes. No-one except Hamilton is willing to share a punt with Beeblebrox.*

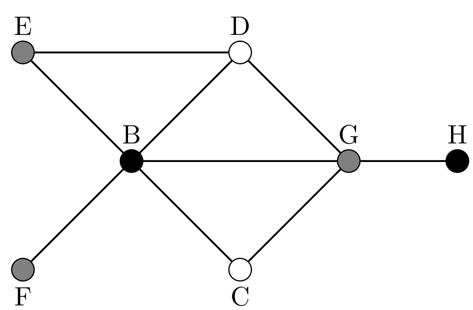
(a) *Explain the relevance of the graph shown below.*



With the given labelling, person X is connected to person Y if and only if they *cannot* be put in the same punt. Thus if the graph is coloured so that no two adjacent vertices have the same colour, then mathematicians with the same colour may safely be put in the same punt.

(You could perhaps make use of the graph in some other way, but I think this is the only way that makes a genuine connection with the theory of graph colourings.)

(b) *By colouring this graph, or otherwise, determine the minimum number of punts required. How many possible seating plans are there using this number of punts?*



It's quite easy to find the 3-colouring shown. Apart from permuting the colours, the only possible change which does not increase the number of colours is to make Frobenius white instead of grey. So there are just 2 different seating plans using 3 punts:

$$\big\{\{E, F, G\}, \{C, D\}, \{B, H\}\big\}$$
$$\big\{\{E, G\}, \{C, D, F\}, \{B, H\}\big\}.$$

5. *Let G be a planar graph.*

(a) *Show that it follows from Euler's formula that G has a vertex of degree* 5 *or less.* Hint: *suppose G has n vertices, e edges, and f faces, and that all vertices of G have degree* 6 *or more. Show that $e \geq 3n$ and $f \leq 2e/3$, and then use Euler's formula to get a contradiction.*

Each face is bounded by at least 3 edges, so by the usual argument (counting up edges by going round faces), $3f \leq 2e$. On the other hand, the counting argument used in the Hand-shaking theorem shows that $6n \leq 2e$. (Each vertex has at least 6 edges coming out of it, so counting up edges by vertices gives at least $6n$ edges. This double-counts each edge, so we have $6n \leq 2e$.)

Substituting in $n - e + f$ gives

$$n - e + f \leq e/3 - e + 2e/3 = 0$$

which contradicts Euler's formula.

(b) *Show by induction on the number of vertices of G that G may be coloured using* 6 *colours.* Hint: *in the inductive step delete from G the vertex given by (a), along with all the edges to which it belongs.*

Let x be a vertex of G of degree 5 or less. By induction we can colour the graph obtained from G by deleting x with 6-colours. As x has at most 5 neighbours in G, there is at least one colour which isn't used to colour any neighbour of x. If this colour is used to colour x then we get a 6-colouring of G.

Remark: the same ideas can be pushed a bit harder to show that any planar graph is 5-colourable: see Wilson, Theorem 17.4.

6. (a) *Show that if G is a bipartite graph then every closed walk in G has even length.*

Let G have bipartition $\{A, B\}$. Suppose we have a closed walk in G starting in A. As there are no edges between vertices in A, the next vertex visited in the walk must lie in B. Similarly, the vertex after that must lie in A. So the walk has the form

$$(a_1, b_1, a_2, b_2, \ldots, a_m, b_m, a_1)$$

for some $a_i \in A, b_i \in B$. In particular, it has even length $2m$. The argument is similar if the walk starts in B.

(b) *Now suppose that G is a connected graph and that every closed walk in G has even length. Fix $x \in G$. Set*

$$\begin{aligned} A &= \{y \in V(G) : \textit{there is a walk from } x \textit{ to } y \textit{ of even length}\}, \\ B &= \{y \in V(G) : \textit{there is a walk from } x \textit{ to } y \textit{ of odd length}\}. \end{aligned}$$

Show that $A \cup B = V(G)$, $A \cap B = \varnothing$ and that if $\{u, v\}$ is an edge of G then either $u \in A$ and $v \in B$ or $u \in B$ and $v \in A$. Deduce that G is bipartite.

Let $y \in V(G)$. Since G is connected, there is a walk from x to y. Either this walk has even length, so $y \in A$, or it has odd length, so $y \in B$. Hence $A \cup B = V(G)$.

Suppose $y \in A \cap B$. Then there is a walk from x to y of even length, say $(z_0, z_1, \ldots, z_{2r})$ where $z_0 = x$,$z_{2r} = y$, and another, inevitably different, walk from x to y of odd length, say $(w_0, w_1, \ldots, w_{2s-1})$ where $w_0 = x$, $w_{2s-1} = y$. Now

$$(z_0, \ldots, z_{2r} = w_{2s-1}, w_{2s-2}, \ldots, w_0)$$

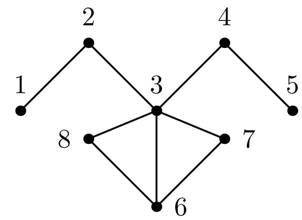
is a closed walk starting and ending at x of odd length. This contradicts our assumption that all closed walks in G have even length. Hence $A \cap B = \varnothing$.

Now suppose that (a, a') is an edge between two vertices in A. We can walk from x to a in an even number of steps. By taking this walk, and then taking one final step to a', we can walk from x to a' in an odd number of steps. Therefore $a' \in A \cap B = \varnothing$, another contradiction. Similarly there are no edges between any two vertices in B.

Hence G is bipartite with bipartition $\{A, B\}$.

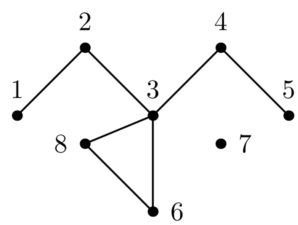
MAP363 Combinatorics Answers 3

Guidance on notation: graphs may have multiple edges, but may not have loops. A graph is simple if it has no multiple edges. A *tree* is a connected simple graph with no closed paths. ($\star$) indicates an optional question, included for interest only.

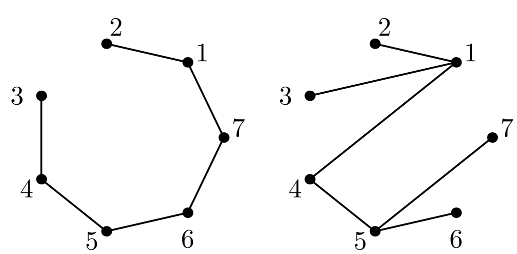
1. *Find all spanning trees in the following graph:* .

You should find that there are 8 different spanning trees.

Clearly any spanning tree will have the edges $\{1,2\}$, $\{2,3\}$, $\{3,4\}$ and $\{4,5\}$. A tree on 8 vertices has 7 edges, so we must take 3 edges from $\{3,6\}$, $\{3,7\}$, $\{3,8\}$, $\{6,7\}$, $\{6,8\}$. As $\binom{5}{3} = 10$, this gives 10 candidates: two of them have a closed path of length 3 so aren't trees. One of the rejected subgraphs is shown below. The other is its mirror image.



2. *Find the Prüfer codes of the spanning trees in K_7 shown below.*



Which spanning trees in K_7 have Prüfer codes $(1,1,1,1,1)$ and $(3,3,4,4,6)$? Which spanning trees in K_7 have Prüfer codes of the form (i,j,k,l,m) where i,j,k,l,m are mutually distinct?

The first has Prüfer code $(1,7,4,5,6)$.

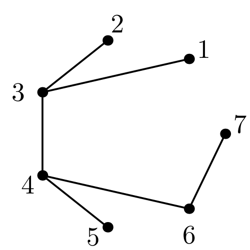
The second has Prüfer code $(1,1,4,5,5)$.

Turn over

The 'star' with edges $\{1,2\}$, $\{1,3\}$, ..., $\{1,7\}$ has Prüfer code $(1,1,1,1,1)$. To find a tree with Prüfer code $(3,3,4,4,6)$ we use the algorithm given in lectures. The initial set of candidate leaves is $\{1,2,3,4,5,6,7\}$. The smallest number not appearing in the code is 1, so the first edge is $\{1,3\}$. This uses up the leaf 1, so we remove 1 from the set of candidate leaves, leaving $\{2,3,4,5,6,7\}$, and then carry on with the remaining part of the code, $(3,4,4,6)$. The table below shows all the steps.

edge added	candidate leaves for next step
$\{1,3\}$	$\{2,3,4,5,6,7\}$
$\{2,3\}$	$\{3,4,5,6,7\}$
$\{3,4\}$	$\{4,5,6,7\}$
$\{5,4\}$	$\{4,6,7\}$
$\{4,6\}$	$\{6,7\}$

The last edge $\{6,7\}$. It is now probably wise to draw the tree and check directly that it has the Prüfer code we want.



3. *The purpose of this question is to give an alternative proof of Theorem 4.2, that the number of edges in a tree on n vertices is $n-1$.*

(a) *Let T be a tree. Show that if $(x_0, x_1, \ldots, x_{m-1}, x_m)$ is a path in T of greatest possible length then x_0 and x_m are leaves.*

Suppose, for a contradiction, that x_0 is not a leaf. Then there must be some vertex y, other than x_1, which is joined to x_0.

We can't have $y \in \{x_2, \ldots, x_m\}$ as if $\{x_k, y\}$ is an edge then $(x_0, x_1, \ldots, x_k, y, x_0)$ is a closed path in T. Hence $y \notin \{x_1, \ldots, x_m\}$, and so $(y, x_0, x_1, \ldots, x_m)$ is a path of length $m+1$. We had assumed that $(x_0, x_1, \ldots, x_m)$ had the greatest possible length of any path in T, so this is a contradiction.

(b) *Deduce that every non-empty tree has a leaf.*

Take a path of greatest possible length in the tree. By (a) either of its end vertices is a leaf.

(c) *Use the previous part and induction to prove that a tree on n vertices has exactly $n-1$ edges.*

If our tree has just 1 vertex then it has no edges, so the results holds in this case.

Now suppose our tree has $n+1$ vertices. Choose any leaf and remove it and its associated edge. The remaining graph is still a tree (clearly it still has no closed paths, and removing a leaf can't disconnect a graph). By induction it has $n-1$ edges. We removed one edge to get it, so the original tree has n edges, as required.

4. *Show that a connected simple graph is a tree if and only if there is a unique path between any two of its vertices.*

'if': Let G be a connected graph satisfying the hypothesis that there is a unique path between any two of its vertices. If there is a closed path starting and finishing at the vertex x_0, say $(x_0, x_1, \ldots, x_m, x_0)$ then (x_0, x_1) and $(x_0, x_m, \ldots, x_1)$ are two different paths from x_0 to x_1. So G has no closed paths. By assumption G is connected. Hence G is a tree.

'only if': Left to you.

5. *Let $(s_1, \ldots, s_{n-2})$ be the Prüfer code of a tree on $\{1, 2, \ldots, n\}$. Show that if vertex j has degree d then j appears exactly $d-1$ times in the sequence $(s_1, \ldots, s_{n-2})$. Deduce that the tree has leaves $\{1, 2, \ldots, n\} \backslash \{s_1, \ldots, s_{n-2}\}$.*

We shall prove the following claim by induction on n. (As in lectures, it is necessary to prove something very slightly more general, in order for the induction to go through.)

Claim: Let T be a tree on a set $X \subset \mathbf{N}_0$ of size n with Prüfer code $(s_1, \ldots, s_{n-2})$. If vertex $j \in X$ has degree d then j appears exactly $d-1$ times in the sequence $(s_1, \ldots, s_{n-2})$.

Proof: If $n = 2$ then both vertices of T are leaves and the Prüfer code is $()$. Hence the claim holds in this case.

Suppose we know the claim holds for all trees with $n-1$ vertices. Let the smallest numbered leaf of T be b_1. By definition of the Prüfer code, $\{s_1, b_1\}$ is an edge of T. Let T^- be the tree obtained from T by removing b_1 and the edge $\{s_1, b_1\}$.

Now T^- is a tree on $X \backslash \{b_1\}$ with Prüfer code $(s_2, \ldots, s_{n-2})$. Let $j \in X$ and suppose that j has degree d *as a vertex of* T. If $j \neq s_1, b_1$ then j also has degree d as a vertex of T^-. By induction we know that j appears exactly $d-1$ times in $(s_2, \ldots, s_n)$. Hence j appears exactly $d-1$ times in $(s_1, \ldots, s_n)$, as required.

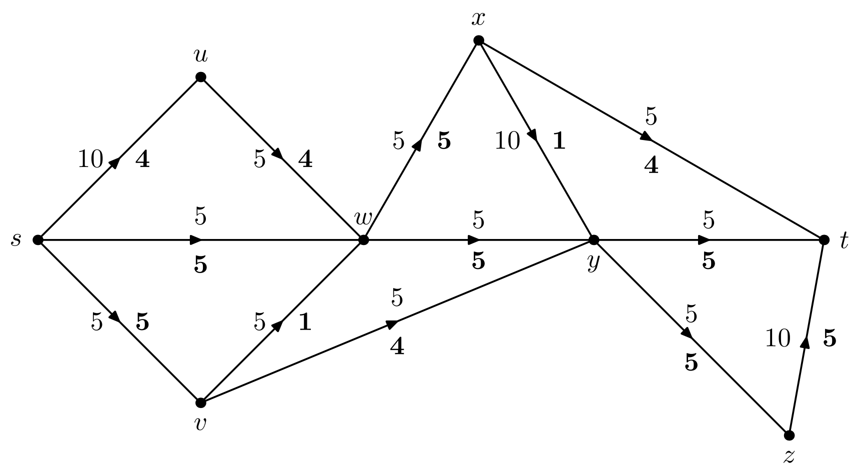
If $j = s_1$ then j has degree $d-1$ as a vertex of T. By induction, j appears exactly $d-2$ times in $(s_2, \ldots, s_n)$, so j appears exactly $d-1$ times in $(s_1, \ldots, s_n)$.

Finally, if $j = b_1$ then j doesn't appear at all in $(s_1, \ldots, s_n)$, and correspondingly $d = 1$, as j is a leaf.

The claim therefore holds for the tree T. This completes the inductive step.

Turn over

6. *The diagram below shows a network with source s and sink t. Plain numbers give the capacities, bold numbers one possible flow.*



(a) *Check that the bold numbers give a valid flow from s to t.*

Check that in each edge the flow is at most the capacity, and that flow is conserved at all vertices (except the source and the target).

(b) *Apply one iteration of the Ford–Fulkerson algorithm to this flow.*

We start by setting $S = \{s\}$.

(1) Add u as $f(s, u) < c(s, u)$; now $S = \{s, u\}$.

(2) Add w as $f(u, w) < c(u, w)$; now $S = \{s, u, w\}$.

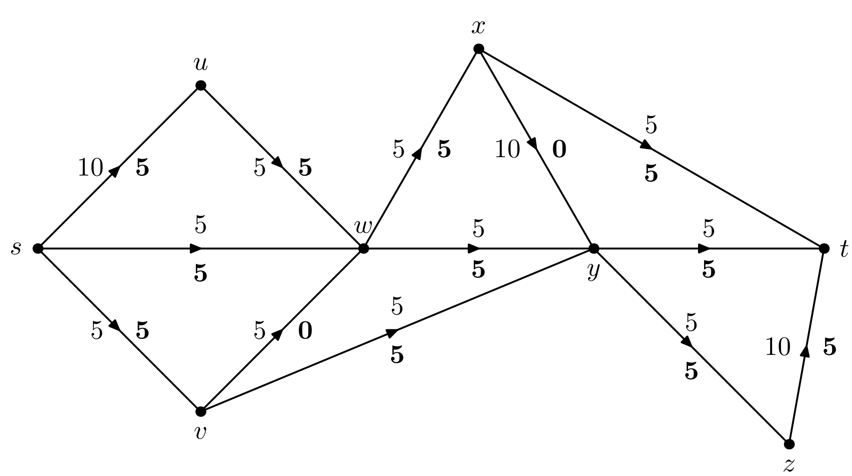
(3) Add v as $f(v, w) > 0$; now $S = \{s, u, v, w\}$

(4) Add y as $f(v, y) < c(v, y)$; now $S = \{s, u, v, w, y\}$

(5) Add x as $f(x, y) > 0$; now $S = \{s, u, v, w, y, x\}$

(6) Add t as $f(x, t) < c(x, t)$; now $S = \{s, u, v, w, y, x, t\}$

We can now stop, as we've reached t. The route to t is (s, u, w, v, y, x, t). We increase the flow in edges (s, u), (u, w), (v, y) amd (x, t), and decrease the flow in edges (v, w), (x, y) to get the new flow below.



(c) *Find, with proof, a maximal flow in this network.*

In fact the new flow is maximal. Applying one more iteration of the Ford–Fulkerson algorithm gives the cut $(S,T) = (\{s,u\},\{v,w,x,y,z,t\})$ which has capacity 15. No flow can have value greater than the capacity of this cut, and the new flow has value 15, so it must be maximal.

7. *Let f be a flow in a network with source s, target t and edge list E. By summing $f(x,y)$ over all edges $(x,y) \in E$ and using conservation of flow show that*

$$\sum_{y:(s,y)\in E} f(s,y) = \sum_{z:(z,t)\in E} f(z,t).$$

The conservation of flow equation is

$$\sum_{y:(x,y)\in E} f(x,y) = \sum_{z:(z,x)\in E} f(z,x)$$

for all vertices $x \neq s,t$. Sum this over all allowed vertices to get

$$\sum_{x\neq s,t}\sum_{y:(x,y)\in E} f(x,y) = \sum_{x\neq s,t}\sum_{z:(z,x)\in E} f(z,x).$$

Now notice that if (u,v) is an edge with $u \neq s$ and $v \neq t$ then $f(u,v)$ appears on the left-hand-side (when $x = u, y = v$) and on the right-hand-side (when $x = v, z = u$). After cancelling all these terms, we are left with

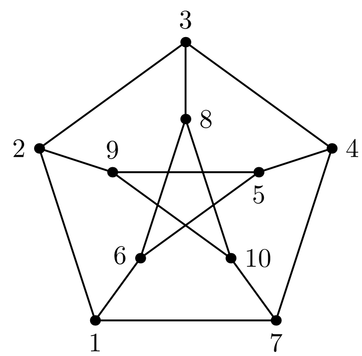
$$\sum_{x:(x,t)\in E} f(x,t) = \sum_{x:(s,x)\in E} f(s,x)$$

which is the required result (up to labelling of the dummy variables).

MAP363 Combinatorics Answers 4

Guidance on notation: A partition of a natural number n is a sequence of natural numbers $(\lambda_1, \ldots, \lambda_k)$ such that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$ and $\lambda_1 + \ldots + \lambda_k = n$. $(\star)$ indicates an optional question, included for interest only.

1. *Let G be the Petersen graph (shown below).*



Let H be the subgroup of $\mathrm{Sym}\{1, 2, \ldots, 10\}$ *consisting of the permutations h such that*

$$\{i, j\} \text{ is an edge of } G \iff \{h(i), h(j)\} \text{ is an edge of } G.$$

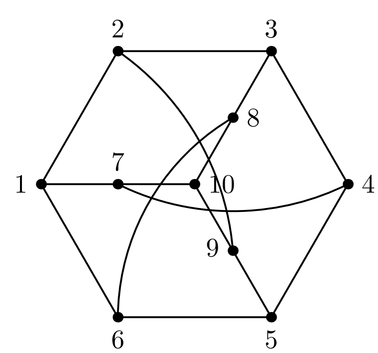
(a) *Find an element of H of order* 5*.*

The rotational symmetry of the graph gives one such element:

$$h = (1\,2\,3\,4\,7)(5\,10\,6\,9\,8).$$

(b) *Find an element of H of order* 6*. (*Hint: *The labelling above is consistent with Example 6.3 from lectures.)*

Another way to draw G (used in Example 6.3) is



From this drawing it is easy to see that

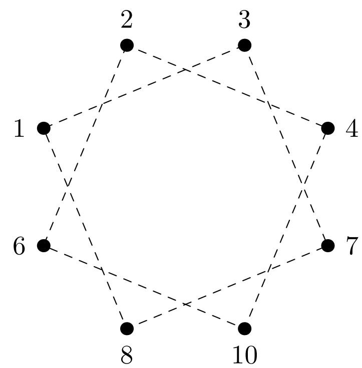
$$k = (1\,2\,3\,4\,5\,6)(7\,9\,8) \in H.$$

(c) ($\star$) *Find an element of H of order 4. Hence, or otherwise, draw the Petersen graph in a way that exhibits 4-fold symmetry.*

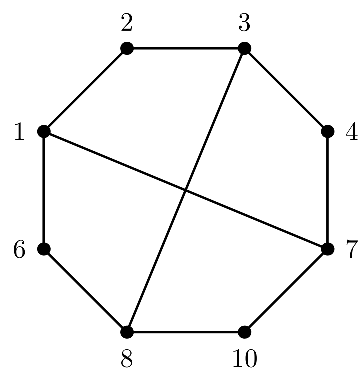
The product hk has order 4:

$$hk = (1\,3\,7\,8)(2\,4\,10\,6)(5\,9).$$

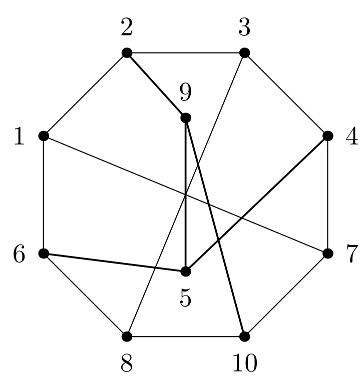
(And since H is a group, $hk \in H$.) This suggests that we might try putting vertices $1, 3, 7, 9$ and $2, 4, 10, 6$ at the corners of two squares.



Putting in the edges of G which meet these vertices gives a drawing of a subgraph of G with rotation symmetry of order 4.

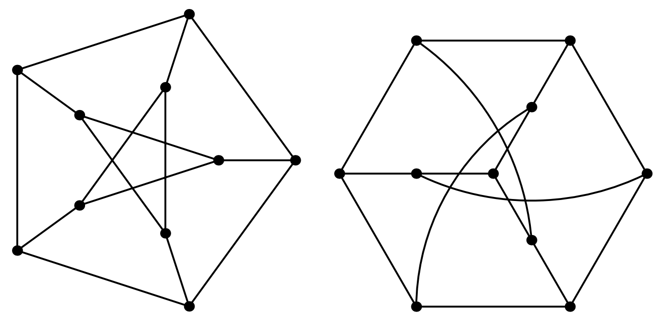


We now add vertices 5 and 9 (and their incident edges) above and below the plane of the octagon. This is rather fiddly to draw, but looks something like:



The resulting 3-dimensional figure has a symmetry of order 4, given by rotating by 90° and then reflecting in the plane of the octagon.

Remark: The 3 graphs shown below are isomorphic, being 3 different ways to draw the Petersen graph.



(Definition 2.5 defines graph isomorphisms. For other examples of isomorphic graphs see Example 2.6(a) from lectures and Question 4(b) on Sheet 1.)

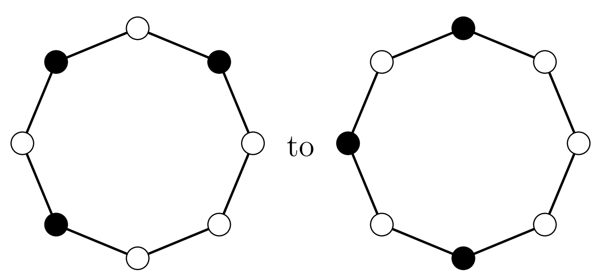
2. *A 5-bead necklace is made using red and blue beads. Two necklaces should be regarded as the same if one is a rotation of the other. Show that 8 different necklaces can be made. How does this answer change if reflections are also considered?*

This is a simpler version of the next question. You should find that in this case it doesn't make any difference to allow reflections, i.e. there are 8 necklaces in both cases.

3. *An 8-bead necklace is made using c different colours of beads. Two necklaces should be regarded as the same if one is a rotation of the other. (Do not consider reflections.)*

(a) *Find the number of different necklaces as a polynomial in c.*

Let X be the set of all c^8 ways to colour the vertices of an octagon with c different colours. Let h be the permutation of X given by rotating the octagon by $45°$. Thus h sends



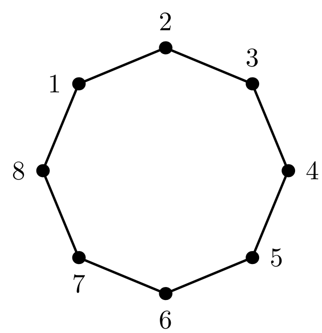
and so on. Let H be the subgroup of $\mathrm{Sym}(X)$ generated by h. The number of different necklaces is equal to the number of orbits of H on X. By the Orbit-Counting Theorem, the number of orbits is given by

$$\frac{1}{8}\sum_{k\in H} |\operatorname{Fix} k| = \frac{1}{8}\left(|\operatorname{Fix} e| + |\operatorname{Fix} h| + |\operatorname{Fix} h^2| + \ldots + |\operatorname{Fix} h^7|\right).$$

The table below shows the number of fixed points of each element of H. We use the relation $\operatorname{Fix} k = \operatorname{Fix} k^{-1}$ to reduce the number of cases.

element	# fixed points
e	c^8
h, h^7	c
h^2, h^6	c^2
h^3, h^5	c
h^4	c^4

For example, h^2 has two orbits on the vertices of the octagon: $\{1, 3, 5, 7\}$ and $\{2, 4, 6, 8\}$ in the diagram below.



If a colouring is fixed by h^2, vertices $1, 3, 5, 7$ must all be the same colour, and so must vertices $2, 4, 6, 8$. This gives 2 independent choices, hence there are c^2 colourings fixed by h^2.

The number of different necklaces is therefore

$$\frac{1}{8}\left(c^8 + c^4 + 2c^2 + 4c\right).$$

(b) *When $c = 2$, how many necklaces have exactly* 4 *beads of each colour?*

Let Y be the set of vertex colourings of the octagon in which 4 vertices are coloured white and 4 vertices are black. As $\binom{8}{4} = 70$ there are 70 colourings in Y. The orbits of H on Y correspond to necklaces with exactly 4 beads of each colour.

element	# fixed points in Y
e	80
h, h^7	0
h^2, h^6	2
h^3, h^5	0
h^4	6

The hardest case is h^4. If a vertex colouring in Y is fixed by h^4 then opposite vertices must have the same colour. There are $\binom{4}{2} = 6$ ways to choose 2 pairs of opposite vertices to be coloured white. The remaining vertices must then be coloured black. Hence the number of fixed point of h^4 in its action on Y is 6.

The number of necklaces with 4 beads of each colour is therefore

$$\frac{1}{8}(70 + 2 + 2 + 6) = 10.$$

(c) (Extra). Suppose that we want to consider two necklaces of length 8 on c colours to be the same if one is a reflection of the other. There are 8 reflections of the octagon: 4 reflections through pairs of opposite vertices, and 4 reflections through opposite edge midpoints.

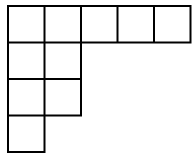
Element	# fixed points
reflection through opposite vertices	c^5
reflection through opposite edge midpoints	c^4

Hence the number of different necklaces up to reflection is

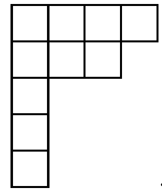
$$\frac{1}{16}\left(c^8 + 4c^5 + 5c^4 + 2c^2 + 2c\right).$$

For example, if $c = 2$, there are 36 necklaces if reflections are not considered, and 30 if they are.

4. *The* conjugate *of a partition is obtained by reflecting its Young diagram in its major diagonal. For example* $(5, 2, 2, 1)$ *has conjugate* $(4, 3, 1, 1, 1)$ *since*



reflects to



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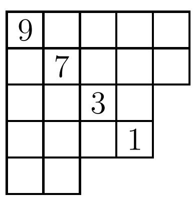
It is usual to write λ' *for the conjugate of* λ.

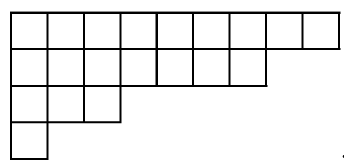
(a) *Show that* λ *has exactly* k *parts if and only if* k *is the the largest part of* λ'.

The partition λ has exactly k parts if and only if its Young diagram has exactly k rows. This holds if and only if the reflected Young diagram has exactly k columns, i.e. if and only if λ' has largest part of size k.

(b) *Show that the number of self-conjugate partitions of* n *is equal to the number of partitions of* n *into distinct odd parts.* [*Hint:* There is a bijective proof based on straightening 'hooks':



$\longleftrightarrow$



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This is just a matter of convincing yourself that this map is bijective.

(c) *Find a closed form for the generating function of self-conjugate partitions.*

The generating function for partitions into odd distinct parts is

$$(1 + x)(1 + x^3)(1 + x^5)(1 + x^7)(1 + x^9) \ldots .$$

(For instance, the partition $(9, 7, 3, 1)$ corresponds to expanding the product by choosing the x^9, x^7, x^3 and x terms.) By (b) this is also the generating function for self-conjugate partitions.