MT454 / MT5454 Combinatorics

Mark Wildon, mark.wildon@rhul.ac.uk

Please take:

- Introduction Notes
- Preliminary Problem Sheet
- Challenge Problems.

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(A) Enumeration

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(B) Generating Functions: Recurrences and applications to enumeration. Problem sheets will ask you to read the early sections of H. S. Wilf, *generatingfunctionology*.

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(C) **Ramsey Theory**: 'Complete disorder is impossible'.

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(A) **Enumeration**

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(C) **Ramsey Theory**: 'Complete disorder is impossible'.

(D) **Probabilistic Methods**: counting *via* discrete probability, lower bounds in Ramsey theory.

All course material is available from Moodle:
moodle.royalholloway.ac.uk/course/view.php?id=371.
Everyone has access to the page.

Recommended Reading

[1] *A First Course in Combinatorial Mathematics*. Ian Anderson, OUP 1989, second edition.

[2] *Discrete Mathematics.* N. L. Biggs, OUP 1989.

[3] *Combinatorics: Topics, Techniques, Algorithms.* Peter J. Cameron, CUP 1994.

[4] *Concrete Mathematics.* Ron Graham, Donald Knuth and Oren Patashnik, Addison-Wesley 1994.

[5] *Invitation to Discrete Mathematics.* Jiri Matoušek and Jaroslav Nešetřil, OUP 2009, second edition.

[6] *Probability and Computing: Randomized Algorithms and Probabilistic Analysis.* Michael Mitzenmacher and Eli Upfal, CUP 2005.

[7] *generatingfunctionology*. Herbert S. Wilf, A K Peters 1994, second / third edition. Second edition available from `http://www.math.upenn.edu/~wilf/DownldGF.html`.

Problem Sheets

- The preliminary problem sheet is designed to get you thinking about the basic counting ideas seen in the first three lectures.
- There will be nine marked problem sheets; the first will be due in on Tuesday 13th October.
  - **You are very welcome to discuss the problems with the lecturer.**
  - You do not have to wait until answers appear on Moodle. I will give you full marks if you discuss the question with me but write up your answer on your own.
  - MSc students: you must submit answers to a mini-project problem sheet (see Moodle in week 3) by the end of term.

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Warning: You will not pass this course by last minute cramming. You must attempt the compulsory questions on problem sheets and learn the techniques for yourself.

Permutations

Definition 2.1

A *permutation* of a set X is a bijective function

$$\sigma : X \to X.$$

A *fixed point* of a permutation σ of X is an element $x \in X$ such that $\sigma(x) = x$. A permutation is a *derangement* if it has no fixed points.

Exercise: For $n \in \mathbf{N}_0$, how many permutations are there of $\{1, 2, \ldots, n\}$? How many of these permutations have 1 as a fixed point?

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BCP1 Multiplying Choices. If an object can be specified uniquely by a sequence of r choices so that, when making the ith choice, we always have exactly c_i possibilities to choose from, then there are exactly $c_1 c_2 \ldots c_r$ objects.

Correction

On Tuesday I used generating functions to find a formula for the Fibonacci numbers. There was a sign error. Replace

$$\frac{1}{\sqrt{5}}\left(\frac{1}{1-\phi x}+\frac{1}{1-\psi x}\right)$$

with

$$\frac{1}{\sqrt{5}}\left(\frac{1}{1-\phi x}-\frac{1}{1-\psi x}\right)$$

and replace $\frac{1}{\sqrt{5}}(\phi^n+\psi^n)$ with $\frac{1}{\sqrt{5}}(\phi^n-\psi^n)$.

Problem 2.2 (Derangements)

How many permutations of $\{1, 2, \ldots, n\}$ *are derangements?*

Let d_n be the number of permutations of $\{1, 2, \ldots, n\}$ that are derangements. By definition, although you may regard this as a convention if you prefer, $d_0 = 1$.

Exercise: Check, by listing permutations, or some cleverer method, that $d_1 = 0$, $d_2 = 1$, $d_3 = 2$ and $d_4 = 9$.

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We used the following counting principle to get $d_4 = 9$.

BCP2: Adding Choices. If a finite set of objects can be partitioned into two disjoint sets A and B, then the total number of objects is $|A| + |B|$.

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> **BCP2: Adding Choices.** If a finite set of objects can be partitioned into two disjoint sets A and B, then the total number of objects is $|A| + |B|$.

In the proof of Lemma 2.3 we will also need:

> **BCP0: Bijections.** If there is a bijection between finite sets A and B then $|A| = |B|$.

Derangements: Exercise

Suppose we try to construct a derangement of $\{1, 2, 3, 4, 5\}$ such that $\sigma(1) = 2$. Show that there are

- two derangements such that $\sigma(1) = 2, \sigma(2) = 1$,
- three derangements such that $\sigma(1) = 2$, $\sigma(2) = 3$.

How many choices are there for $\sigma(3)$ in each case?

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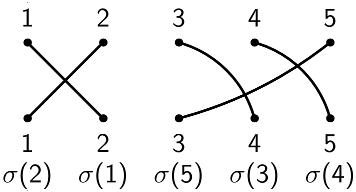
How many choices are there for $\sigma(3)$ in each case?

Here are the two derangements such that $\sigma(1) = 2$ and $\sigma(2) = 1$.



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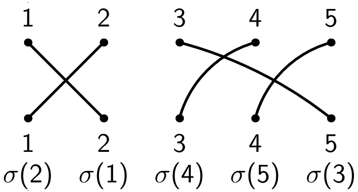

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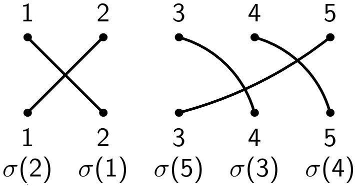

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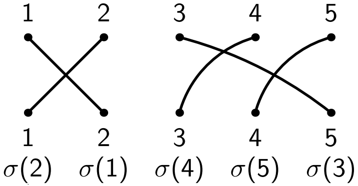
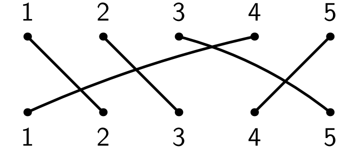


Two of the three derangements such that $\sigma(1) = 2$ and $\sigma(2) = 3$.



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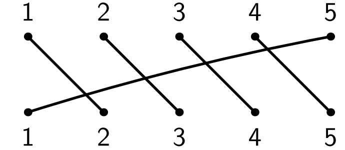


Moodle and Part A handout

- **Please take the Part A handout**
- **Everyone has access to the Moodle page**
  `moodle.royalholloway.ac.uk/course/view.php?id=371`.
  (You can get this link from my website.)

  If you are doing 354 or 5454, then you may not see Combinatorics in your list of Moodle courses. So make a bookmark and use that.

  Provided your Campus Connect record of courses is correct, there is nothing to worry about.

Derangements: An Ad-hoc Solution

Recall that d_n is the number of permutations of $\{1, 2, \ldots, n\}$ that are derangements. By definition, although you may regard this as a convention, if you prefer, $d_0 = 1$.

Overview: Using Lemma 2.3 we will prove Theorem 2.4, which gives a recurrence relation for d_n. We will then solve the recurrence in Corollary 2.5 to get a formula for d_n.

Lemma 2.3

If $n \geq 2$, there are $d_{n-2} + d_{n-1}$ derangements σ of $\{1, 2, \ldots, n\}$ such that $\sigma(1) = 2$.

Theorem 2.4

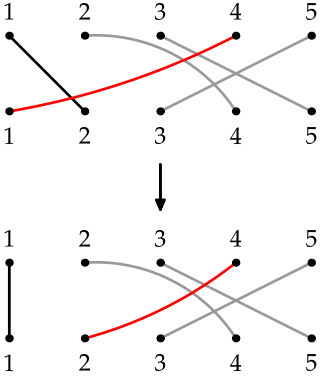
If $n \geq 2$ then $d_n = (n-1)(d_{n-2} + d_{n-1})$.

Example of Lemma 2.3

Take $n = 5$ and let D_2 be the set of permutations σ of $\{1, 2, 3, 4, 5\}$ such that $\sigma(1) = 2$.

In Case (i) we showed that $|\{\sigma \in X : \sigma(2) = 1\}| = d_{n-2}$.

In Case (ii) we showed that $|\{\sigma \in X : \sigma(2) \neq 1\}| = d_{n-1}$ by the 'swapping letters' bijection.



$$\left\{ \begin{array}{c} \text{permutations } \sigma \text{ of } \{1, 2, \ldots, n\} \text{ such that} \\ \sigma(1) = 2,\ \sigma(2) \neq 1,\ \sigma(3) \neq 3,\ \sigma(4) \neq 4,\ \sigma(5) \neq 5 \end{array} \right\}$$

$$\Big\downarrow$$

$$\left\{ \begin{array}{c} \text{permutations } \tau \text{ of } \{1, 2, \ldots, n\} \text{ such that} \\ \tau(1) = 1,\ \tau(2) \neq 2,\ \tau(3) \neq 3,\ \tau(4) \neq 4,\ \tau(5) \neq 5 \end{array} \right\}$$

Formula for d_n and Two Probabilistic Results

Theorem 2.4

If $n \geq 2$ then $d_n = (n-1)(d_{n-2} + d_{n-1})$.

Corollary 2.5

For all $n \in \mathbf{N}_0$,

$$d_n = n!\Big(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\Big).$$

Theorem 2.6

(i) *The probability that a permutation of $\{1, 2, \ldots, n\}$, chosen uniformly at random, is a derangement tends to $1/\mathrm{e}$ as $n \to \infty$.*

(ii) *The average number of fixed points of a permutation of $\{1, 2, \ldots, n\}$ is 1.*

We'll prove more results like these in Part D of the course.

Preliminary Problem Sheet

1. A menu has 3 starters, 4 main courses and 6 desserts.

(a) How many ways are there to order a starter, main course and dessert? [*Hint:* multiply choices.]

(b) How many ways are there to order a two course meal, including exactly one main course?

3. A deck consists of 52 cards. There are four Aces, four Kings, four Queens and four Jacks. How many hands of five cards are there that

(a) have at least one Ace, King, Queen and Jack? [*Hint:* first count hands of the form AKQJx, then hands of the form AAKQJ, and so on. **Note hands are unordered: AKQJ3 is the same hand as JQ3AK.**]

(b) have at least one Ace, King and Queen?

Part A: Enumeration

§3: Binomial Coefficients

Notation 3.1

If Y is a set of size k then we say that Y is a *k-set*. To emphasise that Y is a subset of some other set X then we may say that Y is a *k-subset of X*.

We shall define binomial coefficients combinatorially.

Definition 3.2

Let n, $k \in \mathbf{N}_0$. Let $X = \{1, 2, \ldots, n\}$. The *binomial coefficient* $\binom{n}{k}$ is the number of k-subsets of X.

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Quiz: which of the following are true?

(A) $(1,2) = (2,1)$ (B) $\{1,2\} = (1,2)$

(C) $\{1,2\} = \{2,1\}$ (D) $\{1,2,1\} = \{1,2\}$.

Bijective Proofs

We should prove that the combinatorial definition agrees with the usual one. This proof generalizes Question 2 on the Preliminary Problem Sheet (answers available from Moodle).

Lemma 3.3

If n, $k \in \mathbf{N}_0$ and $k \leq n$ then

$$\binom{n}{k} = \frac{n(n-1)\ldots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}.$$

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Exercise: give an alternative proof of Lemma 3.3 by double counting the set of pairs

$$\big(X, (x_1, \ldots, x_k)\big)$$

such that $X = \{x_1, \ldots, x_k\} \subseteq \{1, \ldots, n\}$.

Lemma 3.4

If n, $k \in \mathbf{N}_0$ then

$$\binom{n}{k} = \binom{n}{n-k}.$$

More Bijective Proofs

Lemma 3.5 (Fundamental Recurrence)

If $n,\ k \in \mathbf{N}$ then

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Exercise: Prove bijectively that $(n-r)\binom{n}{r} = (r+1)\binom{n}{r+1}$ if $0 \leq r \leq n$.

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The bijective proof used the bijection

$$\left\{(Y,z) : \begin{array}{c} Y \subseteq \{1,2,\ldots,n\}, |Y| = r, \\ z \in \{1,2,\ldots,n\}, z \notin Y \end{array}\right\} \to \left\{(Z,z) : \begin{array}{c} Z \subseteq \{1,2,\ldots,n\}, \\ |Z| = r+1, z \in Z \end{array}\right\}.$$

defined by $(Y,z) \mapsto (Y \cup \{z\}, z)$. You might prefer a more informal version.

More Bijective Proofs

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Binomial coefficients are so-named because of the famous binomial theorem. (A binomial is a term of the form $x^r y^s$.)

Theorem 3.6 (Binomial Theorem)

Let $x, y \in \mathbf{C}$. If $n \in \mathbf{N}_0$ then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Example of Bijection in Lemma 3.5

$\binom{5}{3}$ counts size of $\left\{ \begin{array}{l} \{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \dots \\ \quad \dots, \{2,3,5\}, \{2,4,5\}, \{3,4,5\} \end{array} \right\}$

CASE 1
contains 5

$\left\{ \begin{array}{l} \{1,2,5\}, \{1,3,5\}, \{1,4,5\} \\ \{2,3,5\}, \{2,4,5\}, \{3,4,5\} \end{array} \right\}$

Add 5 $\qquad$ Remove 5

$\left\{ \begin{array}{l} \{1,2\}, \{1,3\}, \{1,4\} \\ \{2,3\}, \{2,4\}, \{3,4\} \end{array} \right\}$

counted by $\binom{4}{2}$

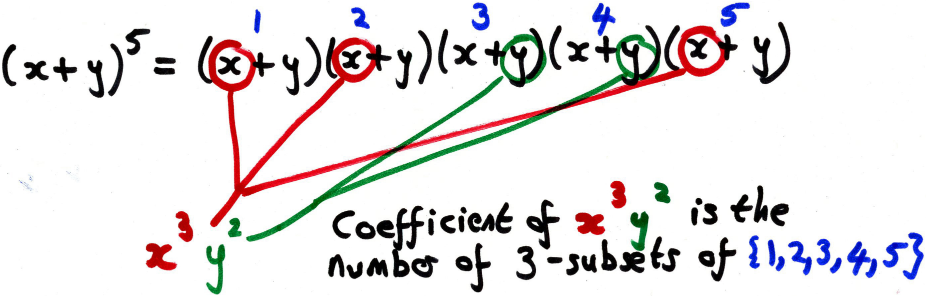
CASE 2

$\left\{ \begin{array}{l} \{1,2,3\}, \{1,2,4\} \\ \{1,3,4\}, \{2,3,4\} \end{array} \right\}$

counted by $\binom{4}{3}$

Example of Bijection in Theorem 3.6 (Binomial Theorem)

The 3-subset $\{1, 2, 5\}$ corresponds to expanding $(x + y)^5$ by choosing x from terms 1, 2 and 5, and y from the other terms, obtaining x^3y^2. Since there are $\binom{5}{3} = 10$ distinct 3-subsets of $\{1, 2, 3, 4, 5\}$, the coefficient of x^3y^2 is 10.



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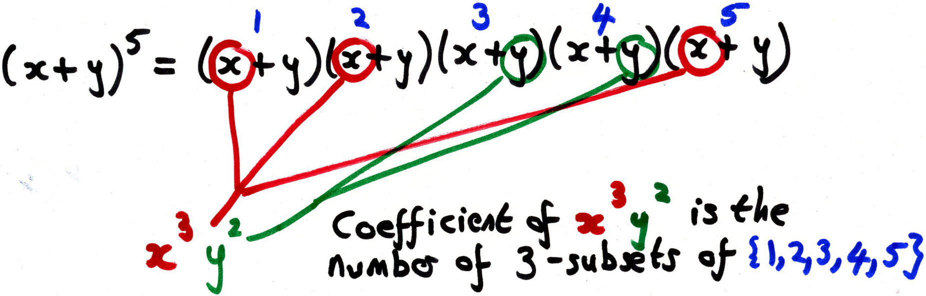
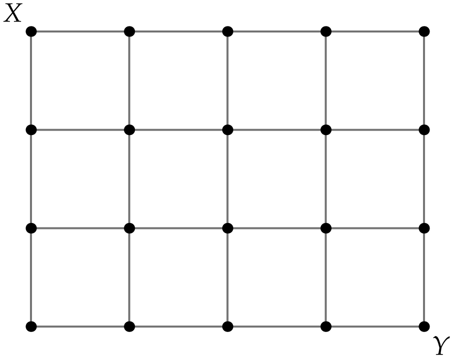


Exercise: Write out an alternative proof of the Binomial Theorem by induction on n, using Lemma 3.5 in the inductive step. Which proof do you find more convincing?

§4: Further Binomial Identities and Balls and Urns

Quiz: How many ways are there to walk from X to Y, taking only steps South and East?



(A) 7 (B) 21 (C) 35 (D) 128

§4: Further Binomial Identities and Balls and Urns

The entry in row n and column r of Pascal's Triangle is $\binom{n}{r}$. Pascal's Triangle can be computed by hand using $\binom{n}{0} = \binom{n}{n} = 1$ and the Fundamental Recurrence.

Lemma 4.1 (Alternating row sums)

If $n \in \mathbf{N}$, $r \in \mathbf{N}_0$ and $r \leq n$ then

$$\sum_{k=0}^{r} (-1)^k \binom{n}{k} = (-1)^r \binom{n-1}{r}.$$

Lemma 4.2 (Diagonal sums, a.k.a. parallel summation)

If $n \in \mathbf{N}_0$, $r \in \mathbf{N}_0$ then

$$\sum_{k=0}^{r} \binom{n+k}{k} = \binom{n+r+1}{r}$$

Pascal's Triangle: entry in row n column k is $\binom{n}{k}$

$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	1									
2	1	2	1								
3	1	3	3	1							
4	1	4	6	4	1						
5	1	5	10	10	5	1					
6	1	6	15	20	15	6	1				
7	1	7	21	35	35	21	7	1			
8	1	8	28	56	70	56	28	8	1		
9	1	9	36	84	126	126	84	36	9	1	
10	1	10	45	120	210	252	210	120	45	10	1

Lemma 4.1: Alternating Row Sums

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	1									
2	1	2	1								
3	1	3	3	1							
4	1	4	6	4	1						
5	1	5	10	10	5	1					
6	1	6	15	20	15	6	1				
7	1	7	21	35	35	21	7	1			
8	1	8	28	56	70	56	28	8	1		
9	**1** −	**9** +	**36** −	**84** +	**126**	126	84	36	9	1	
10	1	10	45	120	210	252	210	120	45	10	1

Lemma 4.2: Alternating Row Sums

$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	1									
2	1	2	1								
3	1	3	3	1							
4	1	4	6	4	1						
5	1	5	10	10	5	1					
6	1	6	15	20	15	6	1				
7	1	7	21	35	35	21	7	1			
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10	1	10	45	120	210	252	210	120	45	10	1

Lemma 4.2: Diagonal Sums a.k.a. Parallel Summation

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	1									
2	1	2	1								
3	**1**	3	3	1							
4	1	**4**	6	4	1						
5	1	5	**10**	10	5	1					
6	1	6	15	**20**	15	6	1				
7	1	7	21	35	**35**	21	7	1			
8	1	8	28	56	**70**	56	28	8	1		
9	1	9	36	84	126	126	84	36	9	1	
10	1	10	45	120	210	252	210	120	45	10	1

Column Sums (see Sheet 1, Question 3)

$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	1									
2	1	2	1								
3	**1**	3	3	**1**							
4	1	**4**	6	**4**	1						
5	1	5	**10**	**10**	5	1					
6	1	6	15	**20**	15	6	1				
7	1	7	21	**35**	**35**	21	7	1			
8	1	8	28	56	**70**	56	28	8	1		
9	1	9	36	84	126	126	84	36	9	1	
10	1	10	45	120	210	252	210	120	45	10	1

Quiz and Hint for Q3 on Sheet 1

Quiz: How many empty sets are there?

(A) 0 (B) 1

(C) infinitely many (D) it depends on your axioms

(3) Let n, $r \in \mathbf{N}$. Prove that

$$\binom{r}{r} + \binom{r+1}{r} + \binom{r+2}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1}$$

in two ways:

(a) by induction on n (where r is fixed in the inductive argument);

(b) bijectively, by reasoning with subsets of $\{1, 2, \ldots, n+1\}$. [*Hint: interpret each summand as counting the* $(r+1)$*-subsets with a particular maximum element.*]

If $n < r$ then both sides in (a) are 0. (The sum has no terms!) So you can take the base case of the induction to be $n = r$. Work by induction on n, keeping r fixed.

Arguments with subsets

Lemma 4.3 (Subset of a subset)

If $k, r, n \in \mathbf{N}_0$ and $k \leq r \leq n$ then

$$\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}.$$

Lemma 4.4 (Vandermonde's convolution)

If $a, b \in \mathbf{N}_0$ and $m \in \mathbf{N}_0$ then

$$\sum_{k=0}^{m} \binom{a}{k}\binom{b}{m-k} = \binom{a+b}{m}.$$

Corollaries of the Binomial Theorem

Corollary 4.5

(i) *If* $n \in \mathbf{N}_0$ *then* $\displaystyle\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

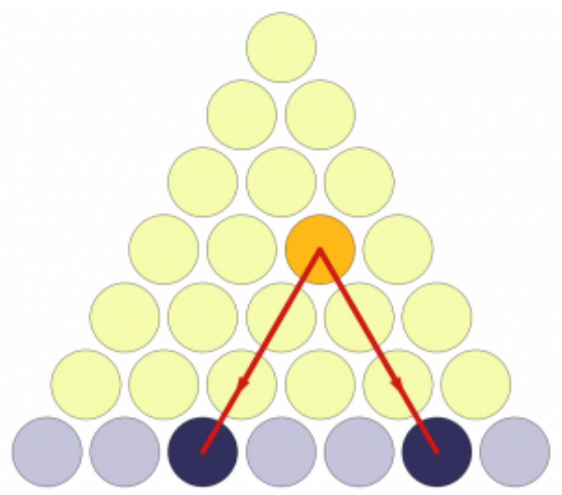
(ii) *If* $n \in \mathbf{N}$ *then* $\displaystyle\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$.

Exercise: Find a bijective proof of (i) and a bijective proof of (ii) when n is odd. *Harder exercise:* Is there a bijective proof of (ii) when n is even?

Corollary 4.6

For all $n \in \mathbf{N}$ *there are equally many subsets of* $\{1, 2, \ldots, n\}$ *of even size as there are of odd size.*

Bijective Proof that $\binom{n+1}{2}$ is the nth Triangle Number



Balls and Urns

How many ways are there to put k balls into n numbered urns?
The answer depends on whether the balls are distinguishable. We may consider urns of unlimited capacity, or urns that can only contain one ball.

	Numbered balls	Indistinguishable balls
≤ 1 ball per urn		
unlimited capacity		

Balls and Urns

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The answer depends on whether the balls are distinguishable. We may consider urns of unlimited capacity, or urns that can only contain one ball.

	Numbered balls	Indistinguishable balls
≤ 1 ball per urn	$n(n-1)\dots(n-k+1)$	
unlimited capacity		

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unlimited capacity		

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≤ 1 ball per urn	$n(n-1)\ldots(n-k+1)$	$\binom{n}{k}$
unlimited capacity	n^k	

Balls and Urns

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The answer depends on whether the balls are distinguishable. We may consider urns of unlimited capacity, or urns that can only contain one ball.

	Numbered balls	Indistinguishable balls
≤ 1 ball per urn	$n(n-1)\dots(n-k+1)$	$\binom{n}{k}$
unlimited capacity	n^k	$\binom{n+k-1}{k}$

Balls and Urns

How many ways are there to put k balls into n numbered urns? The answer depends on whether the balls are distinguishable. We may consider urns of unlimited capacity, or urns that can only contain one ball.

	Numbered balls	Indistinguishable balls
≤ 1 ball per urn	$n(n-1)\ldots(n-k+1)$	$\frac{n(n-1)\cdots(n-k+1)}{k!}$
unlimited capacity	n^k	a $\frac{(n+k-1)\cdots(n+1)n}{k!}$

Balls and Urns

How many ways are there to put k balls into n numbered urns? The answer depends on whether the balls are distinguishable. We may consider urns of unlimited capacity, or urns that can only contain one ball.

$k = 2,\ n = 3$	Numbered balls	Indistinguishable balls
≤ 1 ball per urn	$3 \times 2 = 6$	$\binom{3}{2} = \frac{3 \times 2}{2} = 3$
unlimited capacity	$3 \times 3 = 9$	$\binom{4}{2} = \frac{4 \times 3}{2} = 6$

Unnumbered Balls, Urns of Unlimited Capacity

Theorem 4.7

Let $n \in \mathbf{N}$ and let $k \in \mathbf{N}_0$. The number of ways to place k indistinguishable balls into n urns of unlimited capacity is $\binom{n+k-1}{k}$.

Unnumbered Balls, Urns of Unlimited Capacity

Theorem 4.7

Let $n \in \mathbf{N}$ and let $k \in \mathbf{N}_0$. The number of ways to place k indistinguishable balls into n urns of unlimited capacity is $\binom{n+k-1}{k}$.

$n \backslash k$	0	1	2	3	4	$\ldots$
1						
2						
3						
4						
5						
$\vdots$						

Unnumbered Balls, Urns of Unlimited Capacity

Theorem 4.7

Let $n \in \mathbf{N}$ and let $k \in \mathbf{N}_0$. The number of ways to place k indistinguishable balls into n urns of unlimited capacity is $\binom{n+k-1}{k}$.

$n \backslash k$	0	1	2	3	4	$\dots$
1	1	1	1	1	1	
2	1					
3	1					
4	1					
5	1					
$\vdots$						

Unnumbered Balls, Urns of Unlimited Capacity

Theorem 4.7

Let $n \in \mathbf{N}$ and let $k \in \mathbf{N}_0$. The number of ways to place k indistinguishable balls into n urns of unlimited capacity is $\binom{n+k-1}{k}$.

$n \backslash k$	0	1	2	3	4	$\ldots$
1	1	1	1	1	1	
2	1	2				
3	1					
4	1					
5	1					
$\vdots$						

Unnumbered Balls, Urns of Unlimited Capacity

Theorem 4.7

Let $n \in \mathbf{N}$ and let $k \in \mathbf{N}_0$. The number of ways to place k indistinguishable balls into n urns of unlimited capacity is $\binom{n+k-1}{k}$.

$n \backslash k$	0	1	2	3	4	$\ldots$
1	1	1	1	1	1	
2	1	2	3			
3	1	3				
4	1					
5	1					
$\vdots$						

Unnumbered Balls, Urns of Unlimited Capacity

Theorem 4.7

Let $n \in \mathbf{N}$ and let $k \in \mathbf{N}_0$. The number of ways to place k indistinguishable balls into n urns of unlimited capacity is $\binom{n+k-1}{k}$.

$n \backslash k$	0	1	2	3	4	...
1	1	1	1	1	1	
2	1	2	3	4		
3	1	3	6			
4	1	4				
5	1					
$\vdots$						

Unnumbered Balls, Urns of Unlimited Capacity

Theorem 4.7

Let $n \in \mathbf{N}$ and let $k \in \mathbf{N}_0$. The number of ways to place k indistinguishable balls into n urns of unlimited capacity is $\binom{n+k-1}{k}$.

$n \backslash k$	0	1	2	3	4	$\ldots$
1	1	1	1	1	1	
2	1	2	3	4	5	
3	1	3	6	10		
4	1	4	10			
5	1	5				
$\vdots$						

Unnumbered Balls, Urns of Unlimited Capacity

Theorem 4.7

Let $n \in \mathbf{N}$ and let $k \in \mathbf{N}_0$. The number of ways to place k indistinguishable balls into n urns of unlimited capacity is $\binom{n+k-1}{k}$.

$n \backslash k$	0	1	2	3	4	...
1	1	1	1	1	1	
2	1	2	3	4	5	
3	1	3	6	10	15	
4	1	4	10	20		
5	1	5	15			
$\vdots$						

Unnumbered Balls, Urns of Unlimited Capacity

Theorem 4.7

Let $n \in \mathbf{N}$ and let $k \in \mathbf{N}_0$. The number of ways to place k indistinguishable balls into n urns of unlimited capacity is $\binom{n+k-1}{k}$.

$n \backslash k$	0	1	2	3	4	…
1	1	1	1	1	1	
2	1	2	3	4	5	
3	1	3	6	10	15	
4	1	4	10	20	35	
5	1	5	15	35		
⋮						

Unnumbered Balls, Urns of Unlimited Capacity

Theorem 4.7

Let $n \in \mathbf{N}$ and let $k \in \mathbf{N}_0$. The number of ways to place k indistinguishable balls into n urns of unlimited capacity is $\binom{n+k-1}{k}$.

$n \backslash k$	0	1	2	3	4	$\ldots$
1	1	1	1	1	1	
2	1	2	3	4	5	
3	1	3	6	10	15	
4	1	4	10	20	35	
5	1	5	15	35	70	
$\vdots$						

Counting n-Tuples with Sum k

The following reinterpretation of Theorem 4.7 is often useful.

Corollary 4.8

Let $n \in \mathbf{N}$ and let $k \in \mathbf{N}_0$. The number of solutions to the equation

$$t_1 + t_2 + \cdots + t_n = k$$

with $t_1, t_2, \ldots, t_n \in \mathbf{N}_0$ is $\binom{n+k-1}{k}$.

Questions from Sheet 1

1. Prove that

$$r\binom{n}{r} = n\binom{n-1}{r-1}$$

for n, $r \in \mathbf{N}$ in two ways:

(a) using the formula for a binomial coefficient;

(b) by reasoning with subsets.

3. Let n, $r \in \mathbf{N}$. Prove that

$$\binom{r}{r} + \binom{r+1}{r} + \binom{r+2}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1}$$

in two ways:

(a) by induction on n (where r is fixed in the inductive argument);

(b) by reasoning with subsets of $\{1, 2, \ldots, n+1\}$.

5. A lion tamer has n cages in a row. Let $g(n,k)$ be the number of ways is which she may accommodate k indistinguishable lions so that no cage contains more than one lion, and no two lions are housed in adjacent cages.

(a) Show that $g(n,k) = g(n-2,k-1) + g(n-1,k)$ if $n \geq 2$ and $k \geq 1$.

(b) Prove by induction that $g(n,k) = \binom{n-k+1}{k}$ for all $n \in \mathbf{N}$ and $k \in \mathbf{N}_0$ such that $k \leq n$.

In (b), we need to know the formula is true for $g(n-2,k-1)$ and $g(n-1,k)$ in the inductive step.

Induction on n, assuming (b) for all $k \leq n$

$n\backslash k$	0	1	2	3	4	...
1						
2						
3						
4						
5						
6						
$\vdots$						

5. A lion tamer has n cages in a row. Let $g(n,k)$ be the number of ways is which she may accommodate k indistinguishable lions so that no cage contains more than one lion, and no two lions are housed in adjacent cages.

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(b) Prove by induction that $g(n,k) = \binom{n-k+1}{k}$ for all $n \in \mathbf{N}$ and $k \in \mathbf{N}_0$ such that $k \leq n$.

In (b), we need to know the formula is true for $g(n-2,k-1)$ and $g(n-1,k)$ in the inductive step.

Induction on n, assuming (b) for all $k \leq n$

$n\backslash k$	0	1	2	3	4	...
1	1	1				
2	1					
3	1					
4	1					
5	1					
6	1					
$\vdots$						

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In (b), we need to know the formula is true for $g(n-2,k-1)$ and $g(n-1,k)$ in the inductive step.

Induction on n, assuming (b) for all $k \leq n$

$n \backslash k$	0	1	2	3	4	...
1	1	1				
2	1	2	0			
3	1					
4	1					
5	1					
6	1					
$\vdots$						

5. A lion tamer has n cages in a row. Let $g(n,k)$ be the number of ways is which she may accommodate k indistinguishable lions so that no cage contains more than one lion, and no two lions are housed in adjacent cages.

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Induction on n, assuming (b) for all $k \leq n$

$n \backslash k$	0	1	2	3	4	...
1	1	1				
2	1	2	0			
3	1	3	1	0		
4	1					
5	1					
6	1					
$\vdots$						

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Induction on n, assuming (b) for all $k \leq n$

$n \backslash k$	0	1	2	3	4	...
1	1	1				
2	1	2	0			
3	1	3	1	0		
4	1	4	2	0	0	
5	1					
6	1					
$\vdots$						

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Induction on n, assuming (b) for all $k \leq n$

$n \backslash k$	0	1	2	3	4	...
1	1	1				
2	1	2	0			
3	1	3	1	0		
4	1	4	2	0	0	
5	1	5	6	0	0	
6	1					
$\vdots$						

5. A lion tamer has n cages in a row. Let $g(n,k)$ be the number of ways is which she may accommodate k indistinguishable lions so that no cage contains more than one lion, and no two lions are housed in adjacent cages.

(a) Show that $g(n,k) = g(n-2,k-1) + g(n-1,k)$ if $n \geq 2$ and $k \geq 1$.

(b) Prove by induction that $g(n,k) = \binom{n-k+1}{k}$ for all $n \in \mathbf{N}$ and $k \in \mathbf{N}_0$ such that $k \leq n$.

In (b), we need to know the formula is true for $g(n-2,k-1)$ and $g(n-1,k)$ in the inductive step.

Induction on n, assuming (b) for all $k \leq n$

$n \backslash k$	0	1	2	3	4	...
1	1	1				
2	1	2	0			
3	1	3	1	0		
4	1	4	2	0	0	
5	1	5	6	0	0	
6	1	6	10	1	0	
$\vdots$						

§5: Principle of Inclusion and Exclusion

Example 5.1

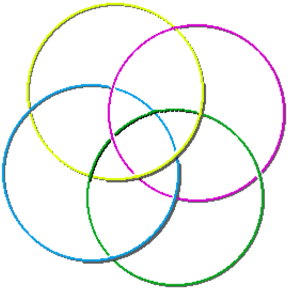
If A, B, C are subsets of a finite set X then

$$\begin{aligned} |A \cup B| &= |A| + |B| - |A \cap B| \\ |\overline{A \cup B}| &= |X| - |A| - |B| + |A \cap B| \end{aligned}$$

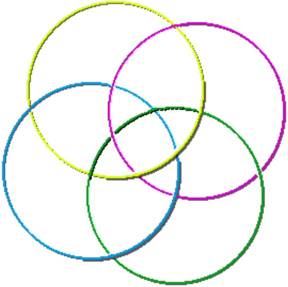
and

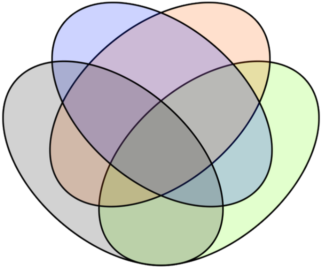
$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| \\ &\qquad - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C| \\ |\overline{A \cup B \cup C}| &= |X| - |A| - |B| - |C| \\ &\qquad + |A \cap B| + |B \cap C| + |C \cap A| - |A \cap B \cap C| \end{aligned}$$

Venn Diagrams for Four Sets

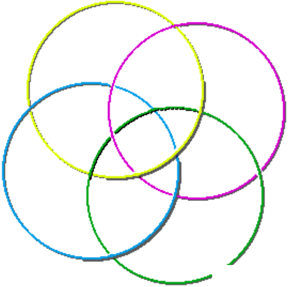


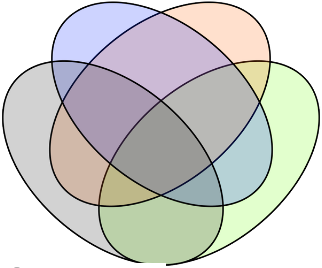
Venn Diagrams for Four Sets

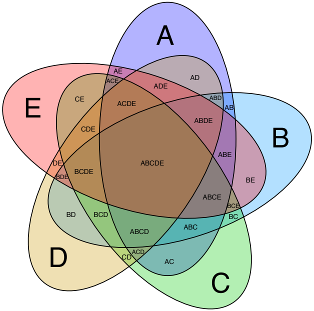




Venn Diagrams for Four Sets



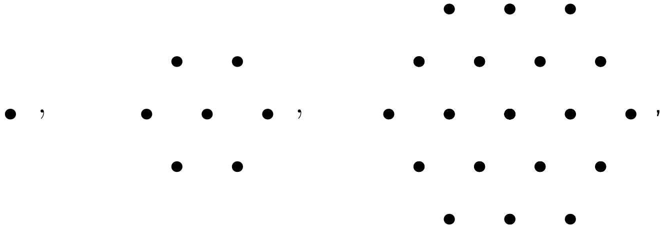




Hexagonal Numbers

Example 5.2

The formula for $|A \cup B \cup C|$ gives a nice way to find a formula for the (centred) hexagonal numbers.



It is easier to find the sizes of the intersections of the three rhombi making up each hexagon than it is to find the sizes of their unions. Whenever intersections are easier to think about than unions, the PIE is likely to work well.

Principle of Inclusion and Exclusion

In general we have finite universe set X and subsets $A_1, A_2, \ldots, A_n \subseteq X$. For each non-empty subset $I \subseteq \{1, 2, \ldots, n\}$ we define

$$A_I = \bigcap_{i \in I} A_i.$$

By convention we set $A_\emptyset = X$.

Theorem 5.3 (Principle of Inclusion and Exclusion)

If $A_1, A_2, \ldots, A_n$ are subsets of a finite set X then

$$|\overline{A_1 \cup A_2 \cup \cdots \cup A_n}| = \sum_{I \subseteq \{1,2,\ldots,n\}} (-1)^{|I|} |A_I|.$$

Exercise: Check that Theorem 5.3 holds when $n = 1$ and check that it agrees with Example 5.1 when $n = 2$ and $n = 3$.

Principle of Inclusion and Exclusion

In general we have finite universe set X and subsets $A_1, A_2, \ldots, A_n \subseteq X$. For each non-empty subset $I \subseteq \{1, 2, \ldots, n\}$ we define

$$A_I = \bigcap_{i \in I} A_i.$$

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Theorem 5.3 (Principle of Inclusion and Exclusion)

If $A_1, A_2, \ldots, A_n$ are subsets of a finite set X then

$$|\overline{A_1 \cup A_2 \cup \cdots \cup A_n}| = \sum_{I \subseteq \{1,2,\ldots,n\}} (-1)^{|I|} |A_I|.$$

Think of A_i as the set of objects in the universe U having property i. Then the PIE counts all those objects having *none* of the properties $P_1, \ldots, P_n$. If $I = \{i_1, i_2, \ldots, i_k\}$ then A_I consists of those objects having (at least) properties i_1, i_2, $\ldots$, i_k.

Application: Counting Prime Numbers

Example 5.4

Let $X = \{1, 2, \ldots, 48\}$. We define three subsets of X:

$$\begin{aligned} B(2) &= \{m \in X,\ m \text{ is divisible by } 2\} \\ B(3) &= \{m \in X,\ m \text{ is divisible by } 3\} \\ B(5) &= \{m \in X,\ m \text{ is divisible by } 5\} \end{aligned}$$

Any composite number ≤ 48 is divisible by either 2, 3 or 5. So

$$\overline{B(2) \cup B(3) \cup B(5)} = \{1\} \cup \{p : 5 < p \leq 48,\ \ p \text{ is prime}\}.$$

Counting Prime numbers

Lemma 5.5

Let r, $M \in \mathbf{N}$. There are exactly $\lfloor M/r \rfloor$ numbers in $\{1, 2, \ldots, M\}$ that are divisible by r.

Theorem 5.6

Let p_1, ..., p_n be distinct prime numbers and let $M \in \mathbf{N}$. The number of natural numbers $\leq M$ that are not divisible by any of primes $p_1, \ldots, p_n$ is

$$\sum_{I \subseteq \{1,2,\ldots,n\}} (-1)^{|I|} \left\lfloor \frac{M}{\prod_{i \in I} p_i} \right\rfloor .$$

Example 5.7

Let $M = pq$ where p, q are distinct prime numbers. The numbers of natural numbers $\leq pq$ that are coprime to M is

$$M\left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{q}\right)$$

Application: Counting Derangements

Let $n \in \mathbf{N}$. Let X be the set of all permutations of $\{1, 2, \ldots, n\}$ and let

$$A_i = \{\sigma \in X : \sigma(i) = i\}.$$

To apply the PIE to count derangements we need this lemma.

Lemma 5.8

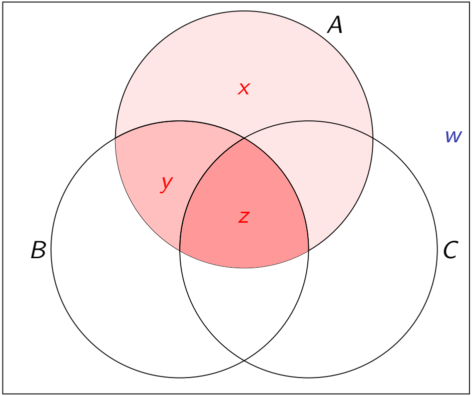
(i) *A permutation $\sigma \in X$ is a derangement if and only if*

$$\sigma \in \overline{A_1 \cup A_2 \cup \cdots \cup A_n}.$$

(ii) *If $I \subseteq \{1, 2, \ldots, n\}$ then A_I consists of all permutations of $\{1, 2, \ldots, n\}$ which fix the elements of I. If $|I| = k$ then*

$$|A_I| = (n-k)!.$$

$$|\overline{A \cup B \cup C}| = |X| - |A| - |B| - |C| + |A \cap B| + \\ + |B \cap C| + |C \cap A| - |A \cap B \cap C|$$



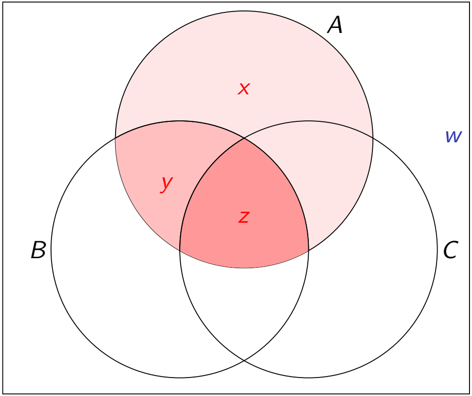
$$f(x) = 1 - 1 - 0 - 0 + 0 + 0 + 0 - 0 = 0$$

$$f(y) = 1 - 1 - 1 - 0 + 1 + 0 + 0 - 0 = 0$$

$$f(z) = 1 - 1 - 1 - 1 + 1 + 1 + 1 - 0 = 0$$

$$f(w) = 1 - 0 - 0 - 0 + 0 + 0 + 0 - 0 = 1$$

$$|\overline{A \cup B \cup C}| = |X| - |A| - |B| - |C| + |A \cap B| + \\ + |B \cap C| + |C \cap A| - |A \cap B \cap C|$$



$$f(x) = 1 - 1 - 0 - 0 + 0 + 0 + 0 - 0 = \binom{1}{0} - \binom{1}{1} = 0$$

$$f(y) = 1 - 1 - 1 - 0 + 1 + 0 + 0 - 0 = \binom{2}{0} - \binom{2}{1} + \binom{2}{2} = 0$$

$$f(z) = 1 - 1 - 1 - 1 + 1 + 1 + 1 - 1 = \binom{3}{0} - \binom{3}{1} + \binom{3}{2} - \binom{3}{3} = 0$$

$$f(w) = 1 - 0 - 0 - 0 + 0 + 0 + 0 - 0 = 1$$

§6: Rook Polynomials

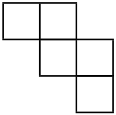
Definition 6.1

A *board* is a subset of the squares of an $n \times n$ grid. Given a board B, we let $r_k(B)$ denote the number of ways to place k rooks on B, so that no two rooks are in the same row or column. Such rooks are said to be *non-attacking*. The *rook polynomial* of B is defined to be

$$f_B(x) = r_0(B) + r_1(B)x + r_2(B)x^2 + \cdots + r_n(B)x^n.$$

Example 6.2

The rook polynomial of the board B below is $1 + 5x + 6x^2 + x^3$.



Examples

Exercise: Let B be a board. Check that $r_0(B) = 1$ and that $r_1(B)$ is the number of squares in B.

Example 6.3

After the recent spate of cutbacks, only four professors remain at the University of Erewhon. Prof. W can lecture courses 1 or 4; Prof. X is an all-rounder and can lecture 2, 3 or 4; Prof. Y refuses to lecture anything except 3; Prof. Z can lecture 1 or 2. If each professor must lecture exactly one course, how many ways are there to assign professors to courses?

Examples

Exercise: Let B be a board. Check that $r_0(B) = 1$ and that $r_1(B)$ is the number of squares in B.

Example 6.3

After the recent spate of cutbacks, only four professors remain at the University of Erewhon. Prof. W can lecture courses 1 or 4; Prof. X is an all-rounder and can lecture 2, 3 or 4; Prof. Y refuses to lecture anything except 3; Prof. Z can lecture 1 or 2. If each professor must lecture exactly one course, how many ways are there to assign professors to courses?

Example 6.4

How many derangements σ of $\{1,2,3,4,5\}$ have the property that $\sigma(i) \neq i + 1$ for $1 \leq i \leq 4$?

Square Boards

Lemma 6.5

The rook polynomial of the $n \times n$-board is

$$\sum_{k=0}^{n} k! \binom{n}{k}^2 x^k.$$

Peer marking

Question 2 on Sheet 3 will be used for a peer-marking exercise.

- Write answers to Question 2 on a separate sheet and hand them in next Friday 30th.
- A detailed marking scheme will be issued.
- Peer-markers should return marked work to me on Tuesday 3rd November.
- I will give it a quick check and return it to you on Thursday 5th November.
- Try to write clearly! One aim of the exercise is to encourage you to think about mathematical writing.

Answers to Sheet 2 are available from Moodle. There are some remarks on common errors. I added an example at the end of the proof of the PIE.

Lemmas for Calculating Rook Polynomials

The two following lemmas are very useful when calculating rook polynomials.

Lemma 6.6

Let C be a board. Suppose that the squares in C can be partitioned into sets A and B so that no square in A lies in the same row or column as a square of B. Then

$$f_C(x) = f_A(x)f_B(x).$$

Lemma 6.7

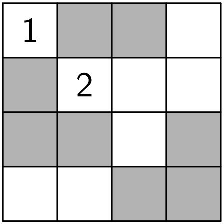
Let B be a board and let s be a square in B. Let D be the board obtained from B by deleting s and let E be the board obtained from B by deleting the entire row and column containing s. Then

$$f_B(x) = f_D(x) + xf_E(x).$$

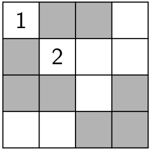
Example of Lemma 6.7

Example 6.8

The rook-polynomial of the boards in Examples 6.3 and 6.4 can be found using Lemma 6.7. For the board in Example 6.3 it works well to apply the lemma first to the square marked 1, then to the square marked 2 (in the new boards).

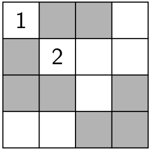


Example 6.8

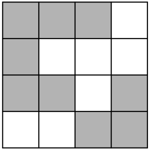


B

Example 6.8

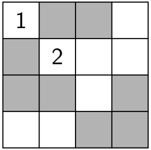


B

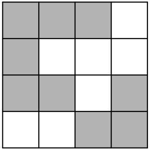


D

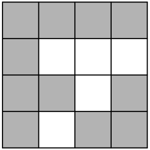
Example 6.8



B

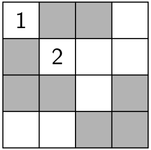


D

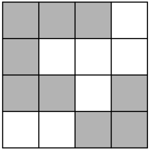


E

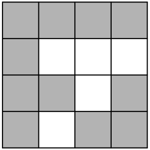
Example 6.8



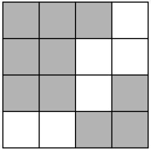
B



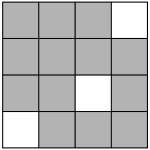
D



E

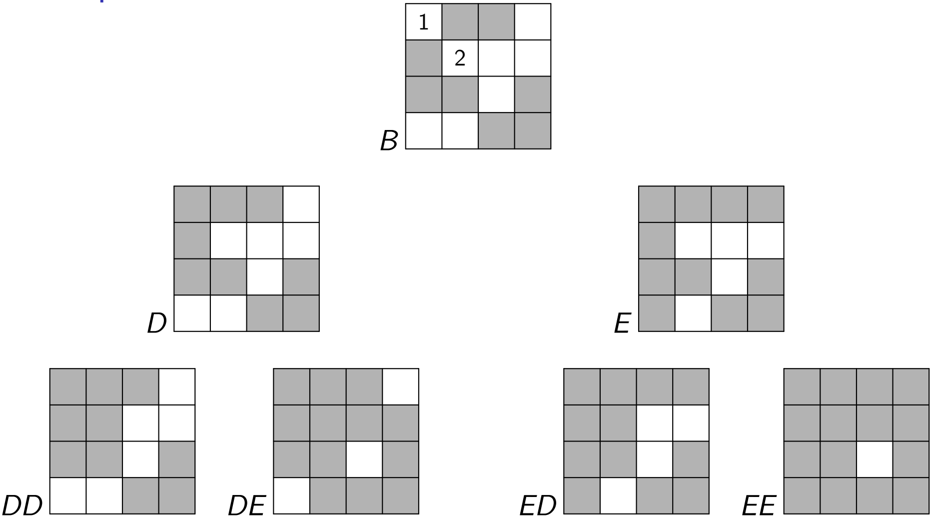


DD



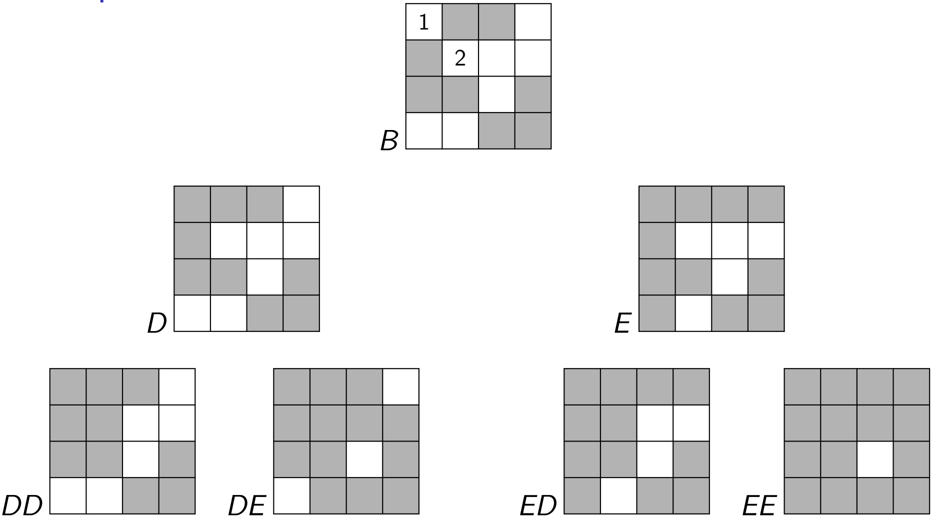
DE

Example 6.8



$(1+2x)(1+4x+3x^2)$ $\quad$ $(1+x)^3$ $\quad$ $(1+x)(1+3x+x^2)$ $\quad$ $1+x$

Example 6.8



$$(1+2x)(1+4x+3x^2) \qquad (1+x)^3 \qquad (1+x)(1+3x+x^2) \qquad 1+x$$

$$(1+2x)(1+4x+3x^2) \qquad x(1+x)^3 \qquad x(1+x)(1+3x+x^2) \qquad x^2(1+x)$$

Placements on the Complement

Lemma 6.9

Let B be a board contained in an $n \times n$ grid and let $0 \leq k \leq n$ The number of ways to place k red rooks on B and $n-k$ blue rooks anywhere on the grid, so that the n rooks are non-attacking, is $r_k(B)(n-k)!$.

Theorem 6.10

Let B be a board contained in an $n \times n$ grid. Let $\bar{B}$ denote the board formed by all the squares in the grid that are not in B. The number of ways to place n non-attacking rooks on $\bar{B}$ is

$$n! - (n-1)!\, r_1(B) + (n-2)!\, r_2(B) - \cdots + (-1)^n r_n(B).$$

Part B: Generating Functions

§7: Introduction to Generating Functions

Definition 7.1

The *ordinary generating function* associated to the sequence $a_0, a_1, a_2, \ldots$ is the power series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots .$$

Usually we shall drop the word 'ordinary' and just write 'generating function'.

The sequences we deal with usually have integer entries, and so the coefficients in generating functions will usually be integers.

Sums and Products of Power Series

Let $F(x) = \sum_{n=0}^{\infty} a_n x^n$ and $G(x) = \sum_{n=0}^{\infty} b_n x^n$. Then

- $F(x) + G(x) = \sum_{n=0}^{\infty} (a_n + b_n) x^n$
- $F(x)G(x) = \sum_{n=0}^{\infty} c_n x^n$ where $c_n = \sum_{m=0}^{n} a_m b_{n-m}$.
- $F'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}$.

It is also possible to define the reciprocal $1/F(x)$ whenever $a_0 \neq 0$. By far the most important case is the case $F(x) = 1 - x$, when

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

is the usual formula for the sum of a geometric progression.

Analytic and Formal Interpretations.

We can think of a generating function $\sum_{n=0}^{\infty} a_n x^n$ in two ways. Either:

- As a formal power series with x acting as a place-holder. This is the 'clothes-line' interpretation (see Wilf *generatingfunctionology*, page 4), in which we regard the power-series merely as a convenient way to display the terms in our sequence.

- As a function of a real or complex variable x convergent when $|x| < r$, where r is the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$.

Quiz

(1) What is a closed form for

$$\sum_{k=0}^{\infty}(-1)^k 2^k x^k = 1 - 2x + 4x^2 - 8x^3 + \cdots ?$$

(A) $\dfrac{1}{1-x}$ (B) $\dfrac{1}{1-2x}$ (C) $\dfrac{1}{1+2x}$ (D) $\exp(-2x)$.

(2) What is a closed form for $\displaystyle\sum_{n=1}^{\infty} nx^n = x + 2x^2 + 3x^3 + 4x^4 + \cdots ?$

(A) $\dfrac{1}{(1-x)^2}$ (B) $\dfrac{x}{(1+x)^2}$ (C) $\dfrac{x}{1+2x}$ (D) $x\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{1}{1-x}$

Quiz

(1) What is a closed form for

$$\sum_{k=0}^{\infty}(-1)^k 2^k x^k = 1 - 2x + 4x^2 - 8x^3 + \cdots ?$$

(A) $\dfrac{1}{1-x}$ (B) $\dfrac{1}{1-2x}$ (C) $\dfrac{1}{1+2x}$ (D) $\exp(-2x)$.

(2) What is a closed form for $\displaystyle\sum_{n=1}^{\infty} n x^n = x + 2x^2 + 3x^3 + 4x^4 + \cdots ?$

(A) $\dfrac{1}{(1-x)^2}$ (B) $\dfrac{x}{(1+x)^2}$ (C) $\dfrac{x}{1+2x}$ (D) $x\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{1}{1-x}$

Quiz

(1) What is a closed form for

$$\sum_{k=0}^{\infty}(-1)^k 2^k x^k = 1 - 2x + 4x^2 - 8x^3 + \cdots ?$$

(A) $\dfrac{1}{1-x}$ (B) $\dfrac{1}{1-2x}$ (C) $\dfrac{1}{1+2x}$ (D) $\exp(-2x)$.

(2) What is a closed form for $\displaystyle\sum_{n=1}^{\infty} nx^n = x + 2x^2 + 3x^3 + 4x^4 + \cdots ?$

(A) $\dfrac{1}{(1-x)^2}$ (B) $\dfrac{x}{(1+x)^2}$ (C) $\dfrac{x}{1+2x}$ (D) $x\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{1}{1-x}$

(3) What is the coefficient of x^3 in $\dfrac{x}{(1-2x)^4}$?

(A) 10 (B) 40 (C) 16 (D) -24

(4) True or false: $\sum_{n=0}^{\infty} nx^n = \sum_{k=1}^{\infty} kx^k$?

(5) Let $F(x) = \sum_{n=0}^{\infty}(n+1)x^n$. Which of the following equals $xF(x)$?

(A) $\displaystyle\sum_{n=0}^{\infty}(n+1)x^{n+1}$ (B) $\displaystyle\sum_{n=1}^{\infty}(n+1)x^n$

(C) $\displaystyle\sum_{n=1}^{\infty} nx^n$ (D) $\displaystyle\sum_{n=1}^{\infty} n(n+1)x^n$

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(C) $\displaystyle\sum_{n=1}^{\infty} nx^n$ (D) $\displaystyle\sum_{n=1}^{\infty} n(n+1)x^n$

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(5) Let $F(x) = \sum_{n=0}^{\infty}(n+1)x^n$. Which of the following equals $xF(x)$?

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(A) $\displaystyle\sum_{n=0}^{\infty}(n+1)x^{n+1}$ (B) $\displaystyle\sum_{n=1}^{\infty}(n+1)x^n$

(C) $\displaystyle\sum_{n=1}^{\infty} nx^n$ (D) $\displaystyle\sum_{n=1}^{\infty} n(n+1)x^n$

Administration

- Please return marked answers to Sheet 3, Question 2. (You will get your own work back on Thursday.)
- Please take your work on the other questions.
- The Coulter McDowell Lecture is tomorrow (Wednesday):
  - Gareth Griffiths
  - *Chess at 200 mph: The Game of Formula 1 Strategy*
  - 6.15pm Windsor Auditorium, refreshments from 5.30pm

Examples

Example 7.2

(Variation) How many ways are there to tile a $2 \times n$ path using 1×2 red bricks and 2×2 yellow and red bricks?

Example 7.3

Let $k \in \mathbf{N}$. Let b_k be the number of 3-tuples (t_1, t_2, t_3) such that $t_1, t_2, t_3 \in \mathbf{N}_0$ and $t_1 + t_2 + t_3 = k$. Will find b_k using generating functions.

To complete the example we needed the following theorem, proved as Question 4 of Sheet 3.

Theorem 7.4

If $n \in \mathbf{N}$ then

$$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k$$

General Binomial Theorem

Theorem 7.5

If $\alpha \in \mathbf{R}$ then

$$(1+y)^\alpha = \sum_{k=0}^{\infty} \frac{\alpha(\alpha-1)\ldots(\alpha-(k-1))}{k!} y^k$$

for all y such that $|y| < 1$.

Exercise: Let $\alpha \in \mathbf{Z}$.

(i) Show that if $\alpha \geq 0$ then Theorem 7.4 agrees with the Binomial Theorem for integer exponents, proved in Theorem 3.6, and with Theorem 7.5.

(ii) Show that if $\alpha < 0$ then Theorem 7.4 agrees with Question 4 on Sheet 3. (Substitute $-x$ for y.)

§8: Recurrence Relations and Asymptotics

Three step programme for solving recurrences:

(a) Use the recurrence to write down an equation satisfied by the generating function $F(x) = \sum_{n=0}^{\infty} a_n x^n$;

(b) Solve the equation to get a closed form for the generating function;

(c) Use the closed form for the generating function to find a formula for the coefficients.

Example 8.1

Will solve $a_{n+2} = 5a_{n+1} - 6a_n$ for $n \in \mathbf{N}_0$ subject to the initial conditions $a_0 = A$ and $a_1 = B$, using the three-step programme.

Partial Fractions

Theorem 8.2

Let $f(x)$ *and* $g(x)$ *be polynomials with* $\deg f < \deg g$. *If*

$$g(x) = \alpha(x - 1/\beta_1)^{d_1} \dots (x - 1/\beta_k)^{d_k}$$

where $\alpha, \beta_1, \beta_2, \dots, \beta_k$ *are distinct non-zero complex numbers and* $d_1, d_2, \dots, d_k \in \mathbf{N}$, *then there exist polynomials* $P_1, \dots, P_k$ *such that* $\deg P_i < d_i$ *and*

$$\frac{f(x)}{g(x)} = \frac{P_1(x)}{(1 - \beta_1 x)^{d_1}} + \cdots + \frac{P_k(x)}{(1 - \beta_k x)^{d_k}}.$$

Quiz

- What is a possible partial fractions form for $\dfrac{4}{1-x^4}$?.

  (1) $\dfrac{1}{1-x}+\dfrac{1}{1+x}+\dfrac{1}{1+2x}+\dfrac{1}{1-2x}$,

  (2) $\dfrac{1}{1-x}+\dfrac{1}{1-ix}+\dfrac{1}{1+x}+\dfrac{1}{1+ix}$,

  (3) $\dfrac{2}{1+x^2}+\dfrac{2}{1-x^2}$,

  (4) $\dfrac{2}{(1-x)^2}+\dfrac{2}{(1+x)^2}$.

- What is the coefficient of x^3 in

$$\left(1+\frac{1}{1!}x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\cdots\right)\left(1+\frac{1}{1!}x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\cdots\right)?$$

(A) 8/3 (B) 1/6 (C) 4/3 (D) 1/2

Quiz

- What is a possible partial fractions form for $\dfrac{4}{1-x^4}$?.

  (1) $\dfrac{1}{1-x} + \dfrac{1}{1+x} + \dfrac{1}{1+2x} + \dfrac{1}{1-2x}$,

  (2) $\dfrac{1}{1-x} + \dfrac{1}{1-ix} + \dfrac{1}{1+x} + \dfrac{1}{1+ix}$,

  (3) $\dfrac{2}{1+x^2} + \dfrac{2}{1-x^2}$,

  (4) $\dfrac{2}{(1-x)^2} + \dfrac{2}{(1+x)^2}$.

- What is the coefficient of x^3 in

$$\left(1+\frac{1}{1!}x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\cdots\right)\left(1+\frac{1}{1!}x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\cdots\right)?$$

(A) 8/3 (B) 1/6 (C) 4/3 (D) 1/2

Quiz

- What is a possible partial fractions form for $\dfrac{4}{1-x^4}$?.

  (1) $\dfrac{1}{1-x}+\dfrac{1}{1+x}+\dfrac{1}{1+2x}+\dfrac{1}{1-2x}$,

  (2) $\dfrac{1}{1-x}+\dfrac{1}{1-ix}+\dfrac{1}{1+x}+\dfrac{1}{1+ix}$,

  (3) $\dfrac{2}{1+x^2}+\dfrac{2}{1-x^2}$,

  (4) $\dfrac{2}{(1-x)^2}+\dfrac{2}{(1+x)^2}$.

- What is the coefficient of x^3 in

$$\left(1+\frac{1}{1!}x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\cdots\right)\left(1+\frac{1}{1!}x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\cdots\right)?$$

(A) 8/3 (B) 1/6 (C) 4/3 (D) 1/2

More Examples and Derangements **[Missing Work!]**

Example 8.3

Will solve $b_n = 3b_{n-1} - 4b_{n-3}$ for $n \geq 3$.

Endgame: we got $\sum_{n=0}^{\infty} b_n x^n = \dfrac{Q(x)}{(1+x)(1-2x)^2}$ for some $Q(x)$ of degree ≤ 2. Then Theorem 8.2 gives a constant A and a polynomial $P(x)$ of degree ≤ 1 such that

$$\frac{Q(x)}{(1+x)(1-2x)^2} = \frac{A}{1+x} + \frac{P(x)}{(1-2x)^2}.$$

Theorem 8.4

Let $p_n = d_n/n!$ be the probability that a permutation of $\{1, 2, \ldots, n\}$, chosen uniformly at random, is a derangement. Then

$$np_n = (n-1)p_{n-1} + p_{n-2}$$

for all $n \geq 2$ and

$$p_n = 1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{(-1)^n}{n!}.$$

Singularities

Let G be a complex valued function. A *singularity* of G is a point where G is undefined. For example, if $G(z) = 1/(1 - z^2)$ then G has singularities at $z = 1$ and $z = -1$ and G has no singularities w such that $|w| < 1$.

Theorem 8.5

Let $F(z) = \sum_{n=0}^{\infty} a_n z^n$ be the generating function for the sequence $a_0, a_1, a_2, \ldots$. Fix $R \in \mathbf{R}$. Suppose that F has no singularities w such that $|w| < R$. Then for any $\epsilon > 0$ we have

$$|a_n| \leq \left(\frac{1}{R} + \epsilon\right)^n$$

for all sufficiently large $n \in \mathbf{N}$. Moreover, if F has a singularity w such that $|w| = R$ then there exist infinitely many n such that

$$|a_n| \geq \left(\frac{1}{R} - \epsilon\right)^n.$$

Example of Theorem 8.5

Example 8.6

Consider the recurrence relation $a_{n+3} = a_n + a_{n+1} + a_{n+2}$. Step (1) of the three-step programme shows that the generating function for a_n is $F(z) = P(z)/(1 - z - z^2 - z^3)$ for some polynomial $P(z)$. The roots of $1 - z - z^2 - z^3 = 0$ are, to five decimal places,

0.543689 **[not** 0.543790**]**, $0.7718445 + 1.115143i$, $0.7718445 - 1.115143i$.

So the singularity of $F(z)$ of smallest modulus is at $0.543689\ldots$. By Theorem 8.5, $a_n \leq \frac{1}{0.543789^n} \leq 2^n$ for all sufficiently large n. **[Correction: The printed notes have** 0.543791^n**, going in the wrong direction.]** (Note that the initial values a_0, a_1 and a_2 were not needed.)

Theorem 8.5

Let $F(z) = \sum_{n=0}^{\infty} a_n z^n$ be the generating function for the sequence $a_0, a_1, a_2, \ldots$. Fix $R \in \mathbf{R}$. Suppose that F has no singularities w such that $|w| < R$. Then for any $\epsilon > 0$ we have

$$|a_n| \leq \left(\frac{1}{R} + \epsilon\right)^n$$

for all sufficiently large $n \in \mathbf{N}$. [Final part omitted]

Let (a_n) be the sequence of numbers with generating function

$$\frac{1}{1 - x - x^3} = 1 + x + x^2 + 2x^3 + 3x^4 + 4x^5 + 6x^6 + 9x^7 + 13x^8 + \cdots$$

Given that $x^3 - x - 1$ has roots at, approximately, $-0.341164 - 1.16154i$, $-0.341164 + 1.16154i$, 0.682328, which of the following is an upper bound for a_n, for all n sufficiently large?

(A) 0.69^n (B) $(3/2)^n$ (C) 1.2^n (D) n

More Interesting Example of Theorem 8.5

Example 8.7

Let $G(z) = \sum_{n=0}^{\infty} p_n z^n$ be the generating function for the proportion of permutations of $\{1, 2, \ldots, n\}$ that are derangements. You should have found that

$$G(z) = \frac{\exp(-z)}{1 - z}.$$

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$$G(z) = \frac{\exp(-z)}{1 - z}.$$

Take out the part of $G(z)$ responsible for the singularity at $z = 1$

$$G(z) = \frac{\mathrm{e}^{-1}}{1 - z} - \frac{\mathrm{e}^{-1} - \mathrm{e}^{-z}}{1 - z}.$$

We now apply Theorem 8.5 to the non-singular part.

§9: Convolutions and the Catalan Numbers

The problems in this section fit into the following pattern: suppose that $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are classes of combinatorial objects and that each object has a *size* in $\mathbf{N}_0$. Write $\mathrm{size}(X)$ for the size of X. Suppose that there are finitely many objects of any given size.

Let a_n, b_n and c_n denote the number of objects of size n in $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, respectively.

Theorem 9.1

Suppose there is a bijection

$$\Big\{Z \in \mathcal{C} : \mathrm{size}(Z) = n\Big\} \leftrightarrow \Big\{(X,Y) : \begin{array}{c} X \in \mathcal{A}, Y \in \mathcal{B} \\ \mathrm{size}(X) + \mathrm{size}(Y) = n \end{array}\Big\}$$

Then

$$\sum_{n=0}^{\infty} c_n x^n = \Big(\sum_{n=0}^{\infty} a_n x^n\Big)\Big(\sum_{n=0}^{\infty} b_n x^n\Big)$$

Convolutions and First Example

The critical step in the proof is to show that

$$c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_{n-1} b_1 + a_n b_0 = \sum_{m=0}^{n} a_m b_{n-m}.$$

If sequences (a_n), (b_n) and (c_n) satisfy this relation then we say that (c_n) is the *convolution* of (a_n) and (b_n).

Example 9.2

The grocer sells indistinguishable apples and bananas in unlimited quantities. Bananas are only sold in bunches of three.

(a) What is the generating function for the number of ways to buy n pieces of fruit?

(b) How would your answer to (a) change if dates are also sold?

(c) Let c_n be the number of ways to buy n pieces of fruit and then to donate some number of the apples just purchased to the Students' Union. Find the generating function $\sum_{n=0}^{\infty} c_n x^n$.

Feedback on Sheet 4

- Multiplication of power series.
  - Quiz: what is $\left(\sum_{n=1}^{\infty} x^n\right)\left(\sum_{n=1}^{\infty} x^n\right)$?

    (A) $\sum_{n=2}^{\infty} x^{2n}$ (B) $\sum_{n=2}^{\infty} nx^n$

    (C) $\sum_{n=2}^{\infty} (n-1)x^n$ (D) $x^n \sum_{n=2}^{\infty} x^n$
- Check your answers! E.g. is $p_n = (-1)^n/n!$ a plausible answer for a derangement probability?

You are very welcome to email me with queries. Office hours are Monday 4pm, Thursday 2pm, Friday 4pm.

Feedback on Sheet 4

- Multiplication of power series.
  - Quiz: what is $\left(\sum_{n=1}^{\infty} x^n\right)\left(\sum_{n=1}^{\infty} x^n\right)$?

$$\text{(A)} \sum_{n=2}^{\infty} x^{2n} \qquad \text{(B)} \sum_{n=2}^{\infty} n x^n$$
$$\text{(C)} \sum_{n=2}^{\infty} (n-1) x^n \qquad \text{(D)} \; x^n \sum_{n=2}^{\infty} x^n$$

- Check your answers! E.g. is $p_n = (-1)^n / n!$ a plausible answer for a derangement probability?

You are very welcome to email me with queries. Office hours are Monday 4pm, Thursday 2pm, Friday 4pm.

Feedback on Sheet 4: Question 4

Let $n \in \mathbf{N}$ be given. Let b_k be the number of n-tuples $(t_1, \ldots, t_n)$ such that $t_i \in \mathbf{N}$ for each i and $t_1 + \cdots + t_n = k$. Such a tuple is called a *composition* of k into n parts.

(a) Show that $b_k = 0$ if $k < n$ and give formulae for b_n and b_{n+1}.

(b) Let $F(x) = \sum_{k=0}^{\infty} b_k x^k$. Show that

$$F(x) = \left(\frac{x}{1-x}\right)^n.$$

[*Hint:* Compare Example 7.3: now $t_i \in \mathbf{N}$ for each i.]

(c) Deduce from Theorem 7.4 (or Question 4 on Sheet 3) that

$$F(x) = \sum_{r=0}^{\infty} \binom{n+r-1}{r} x^{r+n}.$$

Find the coefficient of x^k in the right-hand side and show that $b_k = \binom{k-1}{n-1}$.

(d) Hence, or otherwise, show that the number of compositions of $k \in \mathbf{N}$ into any number of parts is 2^{k-1}. **[Update: changed notation from n to k to make consistent.]**

Example 9.3

Lemma 6.6 on rook placements states that if C is a board that A and B where no square in A lies in the same row or column as a square in B has a very short proof using Theorem 9.1.

Exercise: Show that splitting a non-attacking placement of rooks on C into the placements on the sub-boards A and B gives a bijection satisfying the hypotheses of Theorem 9.1. (Define the size of a rook placement and the sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$.) Hence prove Lemma 6.6.

Lemma 6.6

Let C be a board. Suppose that the squares in C can be partitioned into sets A and B so that no square in A lies in the same row or column as a square of B. Then

$$f_C(x) = f_A(x) f_B(x).$$

Rooted Binary Trees

Definition 9.4

A rooted binary tree is either empty, or consists of a *root vertex* together with a pair of rooted binary trees: a *left subtree* and a *right subtree*. The *Catalan number* C_n is the number of rooted binary trees on n vertices.

Theorem 9.5

If $n \in \mathbf{N}_0$ *then* $C_n = \dfrac{1}{n+1}\dbinom{2n}{n}$.

Derangements by Convolution: Warm-up Quiz

Let $n \in \mathbf{N}$. How many permutations of $\{1, 2, \ldots, n\}$ fix 1?

(A) $n!$ (B) 2^n (C) $(n-1)!$ (D) $\binom{n-1}{2}$

Let $n \geq 3$. How many permutations of $\{1, 2, \ldots, n\}$ fix 1 and 2 and 3?

(A) $(n-3)!$ (B) $(n-2)!$ (C) $(n-1)!$ (D) $n!$

How many permutations of $\{1, 2, 3, 4, 5, 6\}$ have exactly 2 fixed points? [*Hint:* $d_4 = 9$.]

(A) 360 (B) 135 (C) 108 (D) 160

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How many permutations of $\{1, 2, 3, 4, 5, 6\}$ have exactly 2 fixed points? [*Hint:* $d_4 = 9$.]

(A) 360 (B) 135 (C) 108 (D) 160

Derangements by Convolution

Lemma 9.6

If $n \in \mathbf{N}_0$ then

$$\sum_{k=0}^{n} \binom{n}{k} d_{n-k} = n!.$$

The sum in the lemma becomes a convolution after a small amount of rearranging.

Theorem 9.7

If $G(x) = \sum_{n=0}^{\infty} d_n x^n / n!$ then

$$\exp(x) G(x) = \frac{1}{1-x}.$$

It is now easy to deduce the formula for d_n; the argument needed is the same as the final step in the proof of Theorem 8.4. The generating function G used above is an example of an *exponential generating function*.

Pure Mathematics Seminar this Tuesday (Today)

- **Sean Eberhard**, *Permutations fixing a k-set*.

  Tuesday 2pm in C219.

§10: Partitions

Definition 10.1

A *partition* of a number $n \in \mathbf{N}_0$ is a sequence of natural numbers $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ such that

(i) $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1$.

(ii) $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$.

The entries in a partition λ are called the *parts* of λ. Let $p(n)$ be the number of partitions of n.

Example 10.2

Let a_n be the number of ways to pay for an item costing n pence using only 2p and 5p coins. Equivalently, a_n is the number of partitions of n into parts of size 2 and size 5. Will find the generating function for a_n.

Quiz

What is the coefficient of x^{20} in $\dfrac{1}{(1-x^2)(1-x^5)}$?

(A) 1 (B) 2 (C) 3 (D) 4.

Quiz

What is the coefficient of x^{20} in $\dfrac{1}{(1-x^2)(1-x^5)}$?

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Generating Function for Partitions

Theorem 10.3

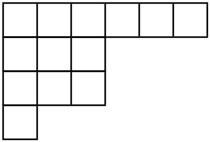
The generating function for $p(n)$ is

$$\sum_{n=0}^{\infty} p(n)x^n = \frac{1}{(1-x)(1-x^2)(1-x^3)\ldots}.$$

Young Diagrams

It is often useful to represent partitions by *Young diagrams*. The Young diagram of $(\lambda_1, \ldots, \lambda_k)$ is an array of boxes such that there are exactly λ_i boxes in row i for each $i \in \{1, \ldots, k\}$.

For example, the Young diagram of $(6, 3, 3, 1)$ is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Theorem 10.4

Let $n \in \mathbf{N}$ and let $k \leq n$. The number of partitions of n into parts of size $\leq k$ is equal to the number of partitions of n with at most k parts.

A Result from Generating Functions

While there are bijective proofs of the next theorem, it is much easier to prove it using generating functions.

Theorem 10.5

Let $n \in \mathbf{N}$. The number of partitions of n with at most one part of any given size is equal to the number of partitions of n into odd parts.

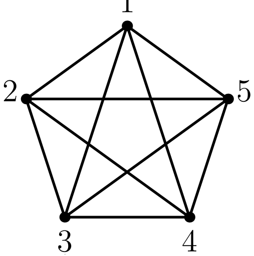
Part C: Ramsey Theory

§11: Introduction to Ramsey Theory

Definition 11.1

A *graph* consists of a set V of vertices together with a set E of 2-subsets of V called *edges*. The *complete graph* with vertex set V is the graph whose edge set is all 2-subsets of V.

The complete graph on $V = \{1, 2, 3, 4, 5\}$ is:



MSc students: I emailed you with a correction to the miniproject. The Stirling number $\left\{ {4 \atop 2} \right\}$ is 7, not 3. Please take a corrected copy.

Colourings

Definition 11.2

Let c, $n \in \mathbf{N}$. A *c-colouring* of the complete graph K_n is a function from the edge set of K_n to $\{1, 2, \ldots, c\}$. If S is an s-subset of the vertices of K_n such that all the edges between vertices in S have the same colour, then we say that S is a *monochromatic* K_s

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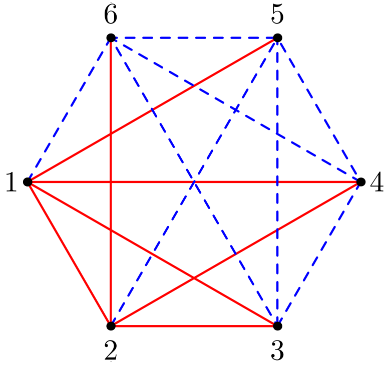
Exercise: find all red K_3s and blue K_4s in this colouring of K_6:



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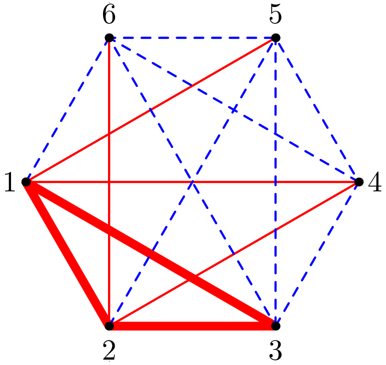

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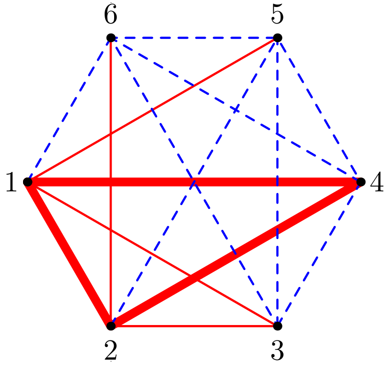

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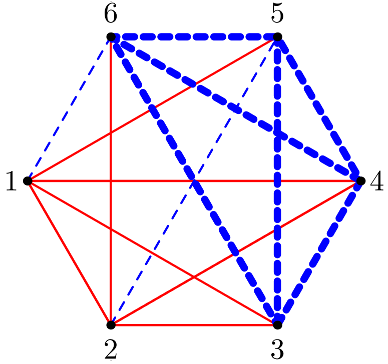

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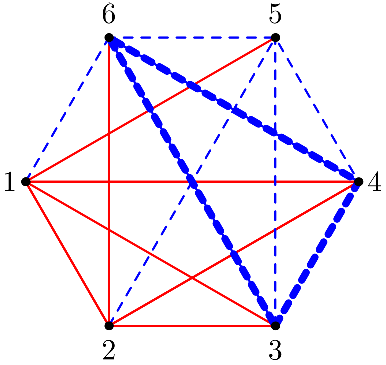

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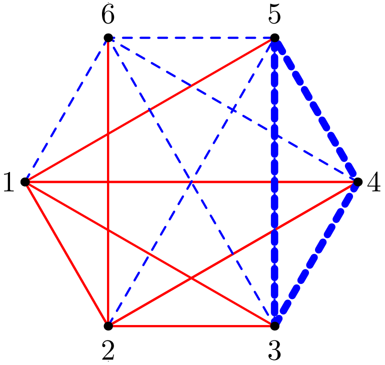


In any Room with Six People . . .

Example 11.3

In any red-blue colouring of the edges of K_6 there is either a red triangle or a blue triangle.

In any Room with Six People ...

Example 11.3

In any red-blue colouring of the edges of K_6 there is either a red triangle or a blue triangle.

Definition 11.4

Given s, $t \in \mathbf{N}$, with $s, t \geq 2$, we define the Ramsey number $R(s,t)$ to be the smallest n (if one exists) such that in any red-blue colouring of the complete graph on n vertices, there is either a red K_s or a blue K_t.

Definition 11.4

Given s, $t \in \mathbf{N}$, with $s, t \geq 2$, we define the Ramsey number $R(s, t)$ to be the smallest n (if one exists) such that in any red-blue colouring of the complete graph on n vertices, there is either a red K_s or a blue K_t.

Lemma 11.5

Let $s, t \in \mathbf{N}$ with $s, t \geq 2$. Let $N \in \mathbf{N}$. Assume that $R(s, t)$ exists.

(i) *If $N < R(s, t)$ there exist colourings of K_N with no red K_s or blue K_t.*

(ii) *If $N \geq R(s, t)$ then in any red-blue colouring of K_N there is either a red K_s or a blue K_t.*

$R(3,4) \leq 10$

Lemma 11.6

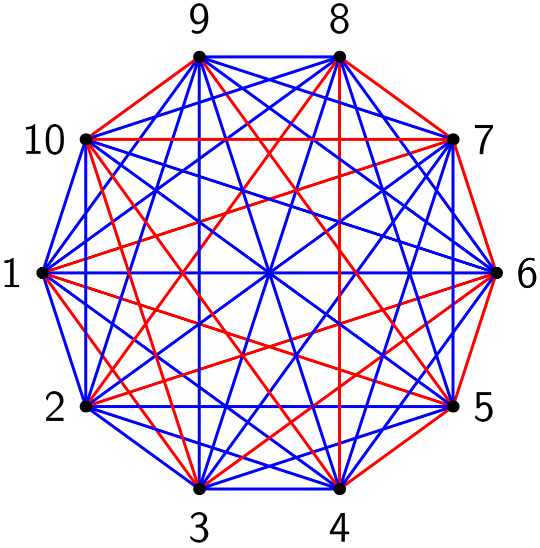
For any $s \in \mathbf{N}$ we have $R(s,2) = R(2,s) = s$.

The main idea need to prove the existence of all the Ramsey Numbers $R(s,t)$ appears in the next example.

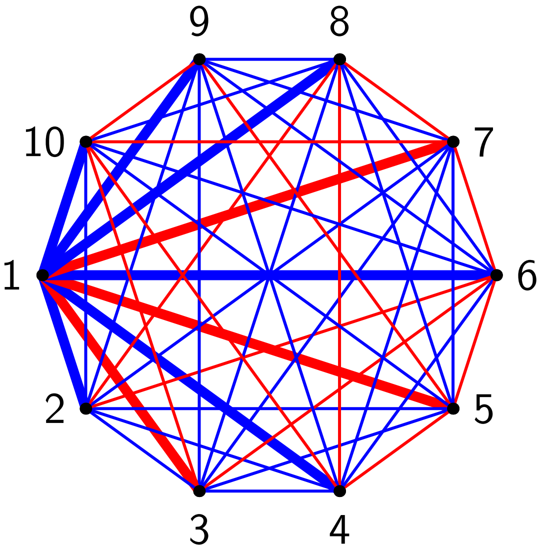
Example 11.7

In any two-colouring of K_{10} there is either a red K_3 or a blue K_4. Hence $R(3,4) \leq 10$.

Example $R(3,4) \leq 10$



Example $R(3,4) \leq 10$



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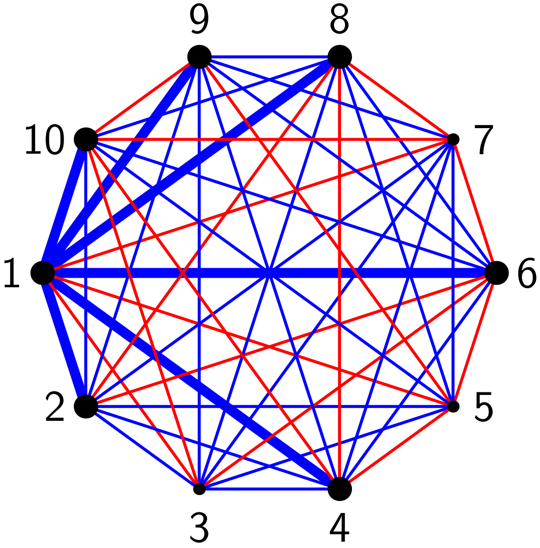
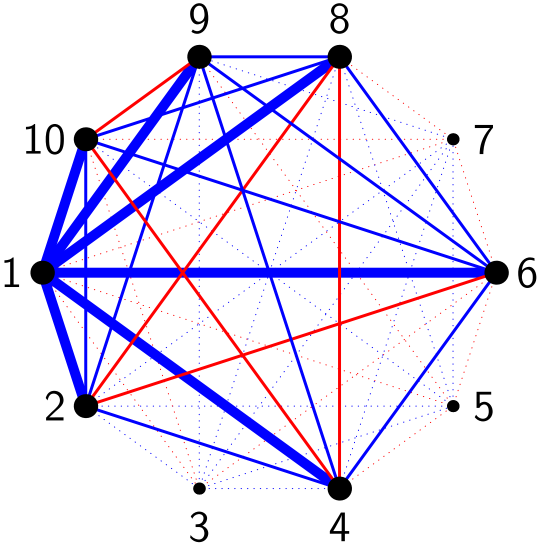


Example $R(3, 4) \leq 10$



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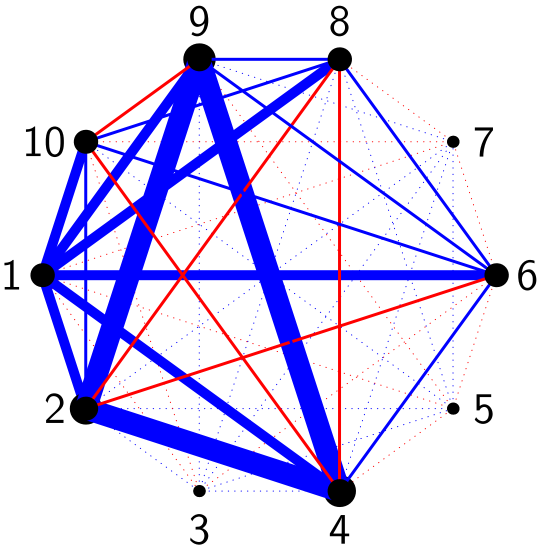

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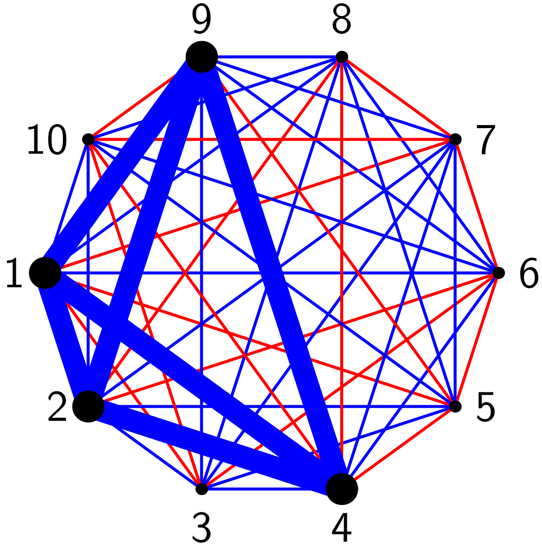

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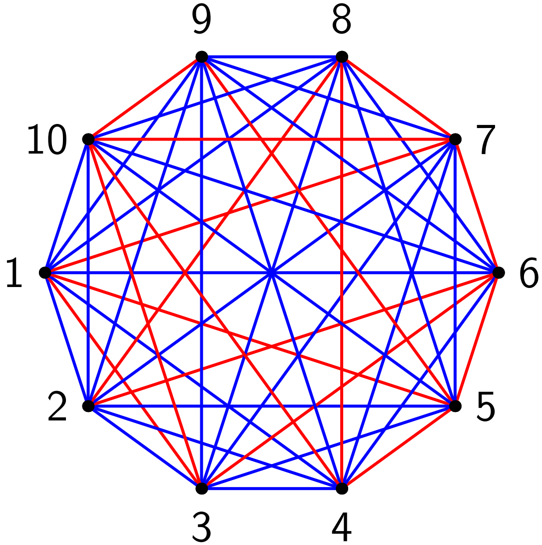

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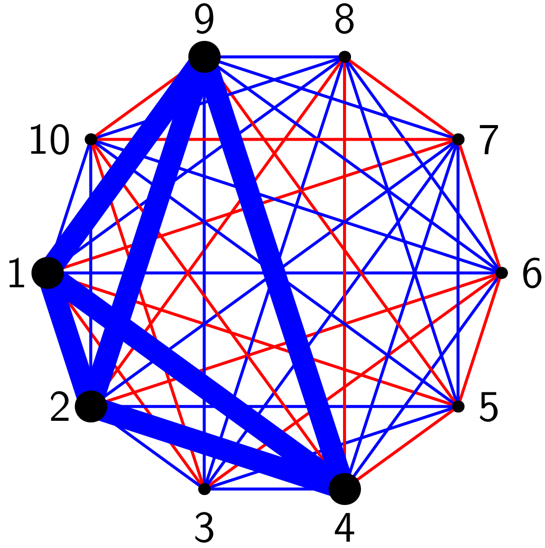


$R(3, 4) = 9$

Lemma 11.8 (Hand-Shaking Lemma)

Let G be a graph with vertex set $\{1, 2, ..., n\}$ and exactly e edges. If d_x is the degree of vertex x then

$$2e = d_1 + d_2 + \cdots + d_n.$$

In particular, the number of vertices of odd degree is even.

Theorem 11.9

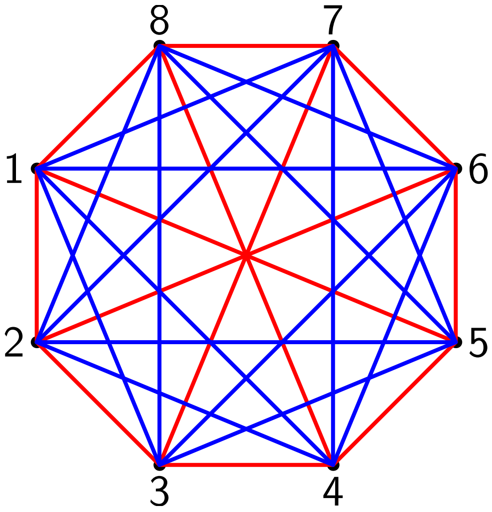
$R(3, 4) = 9$.

The red-blue colouring of K_8 used to show that $R(3, 4) > 8$ is a special case of a more general construction: see Question 3 on Sheet 7.

Theorem 11.10 (See Question 1 on Sheet 7)

$R(4, 4) \leq 18$.

$R(3,4) > 8$



§12: Ramsey's Theorem

We shall prove by induction on $s + t$ that $R(s, t)$ exists. To make the induction go through we must prove a stronger result giving an upper bound on $R(s, t)$.

Lemma 12.1

Let $s, t \in \mathbf{N}$ with $s, t \geq 3$. If $R(s-1, t)$ and $R(s, t-1)$ exist then $R(s, t)$ exists and

$$R(s, t) \leq R(s-1, t) + R(s, t-1).$$

Theorem 12.2

For any $s, t \in \mathbf{N}$ with $s, t \geq 2$, the Ramsey number $R(s, t)$ exists and

$$R(s, t) \leq \binom{s+t-2}{s-1}.$$

Inductive Proof of Ramsey's Theorem

$s \backslash t$	2	3	4	5	6	...
2						
3						
4						
5						
6						
$\vdots$						

Inductive Proof of Ramsey's Theorem

$s \backslash t$	2	3	4	5	6	...
2	2	3	4	5	6	...
3	3					
4	4					
5	5					
6	6					
$\vdots$	$\vdots$					

Base case: $R(2, s) = R(s, 2) = s$ for all $s \geq 2$.

Inductive Proof of Ramsey's Theorem

$s \backslash t$	2	3	4	5	6	...
2	2	3	4	5	6	...
3	3	6				
4	4					
5	5					
6	6					
$\vdots$	$\vdots$					

Base case: $R(2, s) = R(s, 2) = s$ for all $s \geq 2$.
Inductive step by Lemma 13.1

Inductive Proof of Ramsey's Theorem

$s\backslash t$	2	3	4	5	6	$\dots$
2	2	3	4	5	6	$\dots$
3	3	6	10			
4	4	10				
5	5					
6	6					
$\vdots$	$\vdots$					

Base case: $R(2, s) = R(s, 2) = s$ for all $s \geq 2$.
Inductive step by Lemma 13.1

Inductive Proof of Ramsey's Theorem

$s \backslash t$	2	3	4	5	6	$\dots$
2	2	3	4	5	6	$\dots$
3	3	6	10	15		
4	4	10	20			
5	5	15				
6	6					
$\vdots$	$\vdots$					

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Inductive step by Lemma 13.1

Inductive Proof of Ramsey's Theorem

$s \backslash t$	2	3	4	5	6	...
2	2	3	4	5	6	...
3	3	6	10	15	21	
4	4	10	20	35		
5	5	15	35			
6	6	21				
$\vdots$	$\vdots$					

Base case: $R(2, s) = R(s, 2) = s$ for all $s \geq 2$.
Inductive step by Lemma 13.1

Inductive Proof of Ramsey's Theorem

$s \backslash t$	2	3	4	5	6	$\ldots$
2	2	3	4	5	6	$\ldots$
3	3	6	10	15	21	
4	4	10	20	35	56	
5	5	15	35	70		
6	6	21	56			
$\vdots$	$\vdots$					

Base case: $R(2, s) = R(s, 2) = s$ for all $s \geq 2$.
Inductive step by Lemma 13.1

Inductive Proof of Ramsey's Theorem

$s \backslash t$	2	3	4	5	6	...
2	2	3	4	5	6	...
3	3	6	10	15	21	
4	4	10	20	35	56	
5	5	15	35	70	126	
6	6	21	56	126		
$\vdots$	$\vdots$					

Base case: $R(2, s) = R(s, 2) = s$ for all $s \geq 2$.
Inductive step by Lemma 13.1

Inductive Proof of Ramsey's Theorem

$s \backslash t$	2	3	4	5	6	$\dots$
2	2	3	4	5	6	$\dots$
3	3	6	10	15	21	
4	4	10	20	35	56	
5	5	15	35	70	126	
6	6	21	56	126	252	
$\vdots$	$\vdots$					

Base case: $R(2, s) = R(s, 2) = s$ for all $s \geq 2$.
Inductive step by Lemma 13.1

Inductive Proof of Ramsey's Theorem

$s \backslash t$	2	3	4	5	6	$\dots$
2	2	3	4	5	6	$\dots$
3	3	6	9	14	18	
4	4	9	18	25	41	
5	5	14	25	49	87	
6	6	18	41	87	143	
$\vdots$	$\vdots$					

Base case: $R(2, s) = R(s, 2) = s$ for all $s \geq 2$.
Inductive step by Lemma 13.1
Best known upper bounds and lower bounds (black if Ramsey number known exactly)

Inductive Proof of Ramsey's Theorem

$s \backslash t$	2	3	4	5	6	$\ldots$
2	2	3	4	5	6	$\ldots$
3	3	6	9	14	18	
4	4	9	18	25	35	
5	5	14	25	43	58	
6	6	18	35	58	102	
$\vdots$	$\vdots$					

Base case: $R(2, s) = R(s, 2) = s$ for all $s \geq 2$.
Inductive step by Lemma 13.1
Best known upper bounds and lower bounds (black if Ramsey number known exactly)

Diagonal Ramsey Numbers

Corollary 12.3

If $s \in \mathbf{N}$ *and* $s \geq 2$ *then*

$$R(s,s) \leq \binom{2s-2}{s-1} \leq 4^{s-1}.$$

Games and Multiple Colours

Red and Blue play a game. Red starts by drawing a red line between two corners of a hexagon, then Blue draws a blue line and so on. A player *loses* if they makes a triangle of their colour.

Exercise: can the game end in a draw?

Games and Multiple Colours

Red and Blue play a game. Red starts by drawing a red line between two corners of a hexagon, then Blue draws a blue line and so on. A player *loses* if they makes a triangle of their colour.

Exercise: can the game end in a draw?

Theorem 12.4

There exists $n \in \mathbf{N}$ such that if the edges of K_n are coloured red, blue and yellow then there exists a monochromatic triangle.

Sheet 6

1. Let a_n be the number of partitions of $n \in \mathbf{N}$ into parts of size 3 and 5.

(a) Show that $a_{15} = 2$ and find a_{14} and a_{16}.

(b) Explain why

$$\sum_{n=0}^{\infty} a_n x^n = \frac{1}{(1-x^3)(1-x^5)}.$$

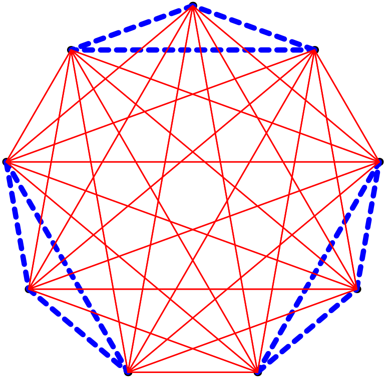
(c) Let c_n be the number of partitions with parts of sizes 3 and 5 whose sum of parts is *at most* n. Find the generating function of c_n.

2. Let b_n be the number of partitions of n that have at most one part of each odd size. For example, $b_6 = 5$: the relevant partitions are $(6), (5,1), (4,2), (3,2,1), (2,2,2)$. Express the generating function $\sum_{n=0}^{\infty} b_n x^n$ as an infinite product.

5. Let $s, t \geq 2$. By constructing a suitable red-blue colouring of $K_{(s-1)(t-1)}$ prove that $R(s,t) > (s-1)(t-1)$. [*Hint: start by partitioning the vertices into* $s-1$ *blocks each of size* $t-1$. *Colour edges within each block with one colour* $\ldots$]

5. Let $s, t \geq 2$. By constructing a suitable red-blue colouring of $K_{(s-1)(t-1)}$ prove that $R(s,t) > (s-1)(t-1)$. [*Hint: start by partitioning the vertices into* $s-1$ *blocks each of size* $t-1$. *Colour edges within each block with one colour* ...]

Example for $s = t = 4$.



Part D: Probabilistic Methods

§13: Revision of Discrete Probability

Definition 13.1

- A *probability measure* p on a finite set Ω assigns a real number p_ω to each $\omega \in \Omega$ so that $0 \leq p_\omega \leq 1$ for each ω and
$$\sum_{\omega \in \Omega} p_\omega = 1.$$
We say that p_ω is the *probability of* ω.
- A *probability space* is a finite set Ω equipped with a probability measure. The elements of a probability space are sometimes called *outcomes*.
- An *event* is a subset of Ω.
- The *probability* of an event $A \subseteq \Omega$, denoted $\mathbf{P}[A]$ is the sum of the probability of the outcomes in A; that is $\mathbf{P}[A] = \sum_{\omega \in A} p_\omega$.

Example 13.2: Probability Spaces

(1) To model a throw of a single unbiased die, we take

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

and put $p_\omega = 1/6$ for each outcome $\omega \in \Omega$. The event that we throw an even number is $A = \{2, 4, 6\}$ and as expected, $\mathbf{P}[A] = p_2 + p_4 + p_6 = 1/6 + 1/6 + 1/6 = 1/2$.

(2) To model a throw of a pair of dice we could take

$$\Omega = \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\}$$

and give each element of Ω probability $1/36$, so $p_{(i,j)} = 1/36$ for all $(i, j) \in \Omega$. Alternatively, if we know we only care about the sum of the two dice, we could take $\Omega = \{2, 3, \ldots, 12\}$ with

n	2	3	...	6	7	8	...	12
p_n	1/36	2/36	...	5/36	6/36	5/36	...	1/36

The former is natural and more flexible.

Example 13.2: Probability Spaces

(3) A suitable probability space for three flips of a coin is

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

where H stands for heads and T for tails, and each outcome has probability $1/8$. To allow for a biased coin we fix $0 \leq q \leq 1$ and instead give an outcome with exactly k heads probability $q^k(1-q)^{3-k}$.

Exercise: Let A be the event that there is at least one head, and let B the the event that there is at least one tail. Find $\mathbf{P}[A]$, $\mathbf{P}[B]$, $\mathbf{P}[A \cap B]$, $\mathbf{P}[A \cup B]$.

(4) Let $n \in \mathbf{N}$ and let Ω be the set of all permutations of $\{1, 2, \ldots, n\}$. Set $p_\sigma = 1/n!$ for each permutation $\sigma \in \Omega$. This gives a suitable setup for Theorem 2.6.

Conditional Probability

Definition 13.3

Let Ω be a probability space, and let A, $B \subseteq \Omega$ be events.

- If $\mathbf{P}[B] \neq 0$ then we define the *conditional probability of A given B* by
$$\mathbf{P}[A|B] = \frac{\mathbf{P}[A \cap B]}{\mathbf{P}[B]}.$$
- The events A, B are said to be *independent* if
$$\mathbf{P}[A \cap B] = \mathbf{P}[A]\mathbf{P}[B].$$

Exercise: Let $\Omega = \{HH, HT, TH, TT\}$ be the probability space for two flips of a fair coin. Let A be the event that both flips are heads, and let B be the event that at least one flip is a head. Write A and B as subsets of Ω and show that $\mathbf{P}[A|B] = 1/3$.

The Most Misunderstood Problem Ever?

Example 13.4 (The Monty Hall Problem)

On a game show you are offered the choice of three doors. Behind one door is a car, and behind the other two are goats. You pick a door and then the host, *who knows where the car is*, opens another door to reveal a goat. You may then either open your original door, or change to the remaining unopened door. Assuming you want the car, should you change?

Further Probability Examples

Example 13.5 (Sleeping Beauty)

Beauty is told that if a coin lands heads she will be woken on Monday and Tuesday mornings, but after being woken on Monday she will be given an amnesia inducing drug, so that she will have no memory of what happened that day. If the coin lands tails she will only be woken on Tuesday morning. At no point in the experiment will Beauty be told what day it is. Imagine that you are Beauty and are awoken as part of the experiment and asked for your credence that the coin landed heads. What is your answer?

Example 13.6

Suppose that one in every 1000 people has disease X. There is a new test for X that will always identify the disease in anyone who has it. There is, unfortunately, a tiny probability of $1/250$ that the test will falsely report that a healthy person has the disease. What is the probability that a person who tests positive for X actually has the disease?

Random Variables

Definition 13.7

Let Ω be a probability space. A *random variable* on Ω is a function $X : \Omega \to \mathbf{R}$.

Definition 13.8

If $X, Y : \Omega \to \mathbf{R}$ are random variables then we say that X and Y are *independent* if for all $x, y \in \mathbf{R}$ the events

$$\begin{aligned} A &= \{\omega \in \Omega : X(\omega) = x\} \quad \text{and} \\ B &= \{\omega \in \Omega : Y(\omega) = y\} \end{aligned}$$

are independent.

If $X : \Omega \to \mathbf{R}$ is a random variable, then '$X = x$' is the event $\{\omega \in \Omega : X(\omega) = x\}$. We mainly use this shorthand in probabilities, so for instance

$$\mathbf{P}[X = x] = \mathbf{P}\big[\{\omega \in \Omega : X(\omega) = x\}\big].$$

Example of Independence of Random Variables

Example 13.9

Let $\Omega = \{HH, HT, TH, TT\}$ be the probability space for two flips of a fair coin. Define $X : \Omega \to \mathbf{R}$ to be 1 if the first coin is heads, and zero otherwise. So

$$X(HH) = X(HT) = 1 \quad \text{and} \quad X(TH) = X(TT) = 0.$$

Define $Y : \Omega \to \mathbf{R}$ similarly for the second coin.

(i) The random variables X and Y are independent.

(ii) Let Z be 1 if exactly one flip is heads, and zero otherwise. Then X and Z are independent, and Y and Z are independent.

(iii) There exist $x, y, z \in \{0, 1\}$ such that

$$\mathbf{P}[X = x, Y = y, Z = z] \neq \mathbf{P}[X = x]\mathbf{P}[Y = y]\mathbf{P}[Z = z].$$

Expectation

Definition 13.10

Let Ω be a probability space with probability measure p. The *expectation* $\mathbf{E}[X]$ of a random variable $X : \Omega \to \mathbf{R}$ is defined to be

$$\mathbf{E}[X] = \sum_{\omega \in \Omega} X(\omega) p_w.$$

Lemma 13.11

Let Ω be a probability space. If $X_1, X_2, \ldots, X_k : \Omega \to \mathbf{R}$ are random variables then

$$\mathbf{E}[a_1 X_1 + a_2 X_2 + \cdots + a_k X_k] = a_1 \mathbf{E}[X_1] + a_2 \mathbf{E}[X_2] + \cdots + a_k \mathbf{E}[X_k]$$

for any $a_1, a_2, \ldots, a_k \in \mathbf{R}$.

Lemma 13.12

If $X, Y : \Omega \to \mathbf{R}$ are independent random variables then $\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y]$.

Example of Linearity of Expectation (Question 11, Sheet 7)

11. Let $0 \leq p \leq 1$ and let $n \in \mathbf{N}$. Suppose that a coin biased to land heads with probability p is tossed n times. Let X be the number of times the coin lands heads.

(a) Describe a suitable probability space Ω and probability measure $p : \Omega \to \mathbf{R}$ and define X as a random variable $\Omega \to \mathbf{R}$.

(b) Find $\mathbf{E}[X]$ and $\mathbf{Var}[X]$. [*Hint: write X as a sum of n independent random variables and use linearity of expectation and Lemma 13.14(ii).*]

(c) Find a simple closed form for the generating function $\sum_{k=0}^{\infty} \mathbf{P}[X = k]x^k$. (Such power series are called *probability generating functions*.)

Variance

Definition 13.13

Let Ω be a probability space. The *variance* $\mathbf{Var}[X]$ of a random variable $X : \Omega \to \mathbf{R}$ is defined to be

$$\mathbf{Var}[X] = \mathbf{E}\big[(X - \mathbf{E}[X])^2\big].$$

Lemma 13.14

Let Ω be a probability space.

(i) *If $X : \Omega \to \mathbf{R}$ is a random variable then*

$$\mathbf{Var}[X] = \mathbf{E}[X^2] - (\mathbf{E}[X])^2.$$

(ii) *If $X, Y : \Omega \to \mathbf{R}$ are independent random variables then*

$$\mathbf{Var}[X + Y] = \mathbf{Var}[X] + \mathbf{Var}[Y].$$

§14: Introduction to Probabilistic Methods

Throughout this section we fix $n \in \mathbf{N}$ and let Ω be the set of all permutations of the set $\{1, 2, \ldots, n\}$. Define a probability measure on Ω so that permutations are chosen uniformly at random.

Exercise: Let $x \in \{1, 2, \ldots, n\}$ and let $A_x = \{\sigma \in \Omega : \sigma(x) = x\}$. Then A_x is the event that a permutation fixes x. What is the probability of A_x?

Theorem 14.1

Let $F : \Omega \to \mathbf{N}_0$ be defined so that $F(\sigma)$ is the number of fixed points of the permutation $\sigma \in \Omega$. Then $\mathbf{E}[F] = 1$.

Cycles

Definition 14.2

A permutation σ of $\{1, 2, \ldots, n\}$ *acts as a k-cycle* on a k-subset $S \subseteq \{1, 2, \ldots, n\}$ if S has distinct elements $x_1, x_2, \ldots, x_k$ such that

$$\sigma(x_1) = x_2, \ \sigma(x_2) = x_3, \ \ldots, \ \sigma(x_k) = x_1.$$

If $\sigma(y) = y$ for all $y \in \{1, 2, \ldots, n\}$ such that $y \notin S$ then we say that σ is a *k-cycle*, and write

$$\sigma = (x_1, x_2, \ldots, x_k).$$

Definition 14.3

We say that cycles $(x_1, x_2, \ldots, x_k)$ and $(y_1, y_2, \ldots, y_\ell)$ are *disjoint* if

$$\{x_1, x_2, \ldots, x_k\} \cap \{y_1, y_2, \ldots, y_\ell\} = \varnothing.$$

Cycle Decomposition of a Permutation

Lemma 14.4

A permutation σ of $\{1, 2, \ldots, n\}$ can be written as a composition of disjoint cycles. The cycles in this composition are uniquely determined by σ.

Exercise: Write the permutation of $\{1, 2, 3, 4, 5, 6\}$ defined by $\sigma(1) = 3$, $\sigma(2) = 4$, $\sigma(3) = 1$, $\sigma(4) = 6$, $\sigma(5) = 5$, $\sigma(6) = 2$ as a composition of disjoint cycles.

Quiz (relevant to Question 2 on Sheet 8): Let

$$\tau = (4, 5), \quad \sigma = (1, 4, 7)(2, 5)(6, 8).$$

Which is of the following is $\tau \circ \sigma$?

(A) $(1, 5, 2, 4, 7, 6, 8)$ (B) $(1, 4, 7)(2, 5)(6, 8)$

(C) $(4, 7, 1, 5, 2)(6, 8)$ (D) $(1, 5, 2, 4, 7)$.

Cycle Decomposition of a Permutation

Lemma 14.4

A permutation σ of $\{1, 2, \ldots, n\}$ can be written as a composition of disjoint cycles. The cycles in this composition are uniquely determined by σ.

Exercise: Write the permutation of $\{1, 2, 3, 4, 5, 6\}$ defined by $\sigma(1) = 3$, $\sigma(2) = 4$, $\sigma(3) = 1$, $\sigma(4) = 6$, $\sigma(5) = 5$, $\sigma(6) = 2$ as a composition of disjoint cycles.

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(C) $(4, 7, 1, 5, 2)(6, 8)$ (D) $(1, 5, 2, 4, 7)$.

Counting k-Cycles

Theorem 14.5

Let $1 \leq k \leq n$ and let $x \in \{1, 2, \ldots, n\}$. The probability that x lies in a k-cycle of a permutation of $\{1, 2, \ldots, n\}$ chosen uniformly at random is $1/n$.

Application to Derangements

Theorem 14.6

Let p_n be the probability that a permutation of $\{1, 2, \ldots, n\}$ chosen uniformly at random is a derangement. If $n \in \mathbf{N}$ then

$$p_n = \frac{p_{n-2}}{n} + \frac{p_{n-3}}{n} + \cdots + \frac{p_1}{n} + \frac{p_0}{n}.$$

Corollary 14.7

For all $n \in \mathbf{N}_0$,

$$p_n = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}.$$

It may be helpful to compare this result with Lemma 9.6: there we get a recurrence by considering fixed points; here we get a recurrence by considering cycles.

Counting Cycles

We can also generalize Theorem 14.1.

Theorem 14.8

Let $C_k : \Omega \to \mathbf{R}$ be the random variable defined so that $C_k(\sigma)$ is the number of k-cycles in the permutation σ of $\{1, 2, \ldots, n\}$. Then $\mathbf{E}[C_k] = 1/k$ for all k such that $1 \leq k \leq n$.

Counting Cycles

We can also generalize Theorem 14.1.

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Let $C_k : \Omega \to \mathbf{R}$ be the random variable defined so that $C_k(\sigma)$ is the number of k-cycles in the permutation σ of $\{1, 2, \ldots, n\}$. Then $\mathbf{E}[C_k] = 1/k$ for all k such that $1 \leq k \leq n$.

Feedback Forms: Please make sure to take a form with the correct code: 354 (3rd years), 454 (4th year MSci), 5454 (MSc).

- 'agree strongly' is at the far left,
- 'disagree strongly' is at the far right,
- 'tutor' means the lecturer.
- For question 2.4, please do not include time spent in lectures.

Comments are very welcome and will be taken seriously.

§15: Ramsey Numbers and the First Moment Method

Lemma 15.1 (First Moment Method)

Let Ω be a probability space and let $M : \Omega \to \mathbf{N}_0$ be a random variable taking values in $\mathbf{N}_0$. If $\mathbf{E}[M] = x$ then

(i) $\mathbf{P}[M \geq x] > 0$, *so there exists $\omega \in \Omega$ such that $M(\omega) \geq x$.*

(ii) $\mathbf{P}[M \leq x] > 0$, *so there exists $\omega' \in \Omega$ such that $M(\omega') \leq x$.*

Exercise: Check that the lemma holds in the case when

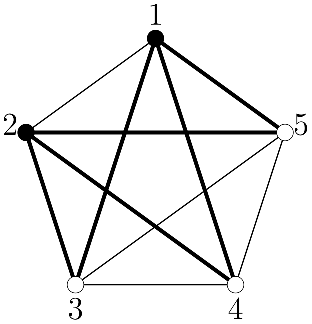
$$\Omega = \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\}$$

models the throw of two fair dice and $M(x, y) = x + y$.

Cut Sets in Graphs

Definition 15.2

Let G be a graph with vertex set V. A *cut* (A, B) of G is a partition of V into two subsets A and B. The *capacity* of a cut (A, B) is the number of edges of G that meet both A and B.



({1,2}, {3,4,5})

Theorem 15.3

Let G be a graph with vertex set $\{1, 2, \ldots, n\}$ and m edges. There is a cut of G with capacity $\geq m/2$.

Application to Ramsey Theory

Lemma 15.4

Let $n \in \mathbf{N}$ and let Ω be the set of all red-blue colourings of the complete graph K_n. Let $p_\omega = 1/|\Omega|$ for each $\omega \in \Omega$. Then

(i) *each colouring in Ω has probability $1/2^{\binom{n}{2}}$;*

(ii) *given any m edges in G, the probability that all m of these edges have the same colour is 2^{1-m}.*

Theorem 15.5

Let $n, s \in \mathbf{N}$. If

$$\binom{n}{s} 2^{1-\binom{s}{2}} < 1$$

then there is a red-blue colouring of the complete graph on $\{1, 2, \ldots, n\}$ with no red K_s or blue K_s.

Lower bound on $R(s, s)$

Corollary 15.6

For any $s \in \mathbf{N}$ with $s \geq 2$ we have

$$R(s, s) \geq 2^{(s-1)/2}.$$

This result can be strengthened slightly using the Lovász Local Lemma. See the printed lecture notes for an outline. (The contents of §16 are non-examinable.)

§16: Lovász Local Lemma

Let Ω be a probability space and let $A_1, \ldots, A_n$ be events in A.

Definition 16.1

Let $T \subseteq \{1, 2, \ldots, n\}$. We say that A_i is *mutually independent* of the events $\{A_j : j \in T\}$ if for all $U, U' \subseteq T$ such that $U \cap U' = \varnothing$ we have

$$\mathbf{P}\Big[A_i \,\Big|\, \Big(\bigcap_{k \in U} A_k\Big) \cap \Big(\bigcap_{\ell \in U'} \bar{A}_\ell\Big)\Big] = \mathbf{P}[A_i],$$

provided the event conditioned on has non-zero probability.

Definition 16.2

Let G be a digraph with edge set

$$E \subseteq \{(i, j) : 1 \leq i, j \leq n, i \neq j\}.$$

If A_i is mutually independent of $\{A_j : (i, j) \notin E\}$ for all $i \in \{1, 2, \ldots, n\}$ then we say that G is a *dependency digraph* for the events $A_1, A_2, \ldots, A_n$.

Lemma 16.3 (Asymmetric Lovász Local Lemma)

Let G be a dependency digraph with edge set E for the events $A_1, \ldots, A_n$. Suppose there exist $x_i \in \mathbf{R}$ such that $0 \leq x_i < 1$ and

$$P[A_i] \leq x_i \prod_{j:(i,j)\in E} (1 - x_j)$$

for all i. Then

$$\mathbf{P}\Big[\bigcap_{i=1}^{n} \bar{A}_i\Big] \geq \prod_{i=1}^{n}(1 - x_i).$$

Claim (to be proved by induction on $|S|$). Let $S \subseteq \{1, 2, \ldots, n\}$ and suppose $i \notin S$. Then

$$\mathbf{P}\Big[\bigcap_{j\in S} \bar{A}_j\Big] \neq 0$$

and

$$\mathbf{P}\Big[A_i \,\Big|\, \bigcap_{j\in S} \bar{A}_j\Big] \leq x_i.$$

Corollary 16.4 (Symmetric Lovász Local Lemma)

Suppose that the maximum degree in a dependency digraph for the events $A_1, \ldots, A_n$ is d. If $\mathbf{P}[A_i] \leq p$ for all i and $\mathrm{e}p(d+1) \leq 1$ then

$$\mathbf{P}\Big[\bigcap_{i=1}^{n} \bar{A}_i] \geq 0.$$

Application to diagonal Ramsey numbers

Theorem 16.5

Let $n, s \in \mathbf{N}$ with $s \geq 3$. If

$$\mathrm{e}\binom{s}{2}\binom{n-2}{s-2}2^{1-\binom{s}{2}} < 1$$

then there is a red-blue colouring of the complete graph K_n with no red K_s or blue K_s.

Example 16.6

When $s = 15$, the largest n such that $\binom{n}{15}2^{1-\binom{15}{2}} < 1$ is $n = 792$. So Theorem 15.5 tells us that $R(15, 15) > 792$. But

$$\mathrm{e}\binom{15}{2}\binom{n-2}{15-2}2^{1-\binom{15}{2}} < 1$$

provided $n \leq 947$. So Theorem 16.5 implies that $R(15, 15) > 947$.

Proposition 16.7

Let $s \geq 9$. Then $R(s, s) \geq 2^{(s-1)/2}s/\mathrm{e}$.

Application to edge disjoint paths

Suppose that G is a graph and that $\mathcal{F}_1, \ldots, \mathcal{F}_n$ are sets of paths in G of length at least m such that if $i \neq j$ and $P \in \mathcal{F}_i$ and $Q \in \mathcal{F}_j$ then P and Q have at most k edges in common.

Proposition 16.8

If

$$n \leq \frac{m}{2k\mathrm{e}}$$

then there are paths $P_1 \in \mathcal{F}_1$, *...*, $P_n \in \mathcal{F}_n$ *such that* $P_1, \ldots, P_n$ *are edge disjoint.*

For an application of the asymmetric Lovász Local Lemma to the Ramsey number $R(3, t)$ see the printed notes (Section 16 is updated and extended on Moodle).