CHARACTER DEFLATIONS AND A GENERALIZATION OF THE MURNAGHAN–NAKAYAMA RULE

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Abstract. Given natural numbers m and n, we define a deflation map from the characters of the symmetric group S_{mn} to the characters of S_n. This map is defined by first restricting a character of S_{mn} to the wreath product $S_m \wr S_n$, and then taking the sum of the irreducible constituents of the restricted character on which the base group $S_m \times \cdots \times S_m$ acts trivially. We prove a combinatorial rule which gives the values of the images of the irreducible characters of S_{mn} under this map. This rule is shown to generalize the Murnaghan–Nakayama rule. As a corollary we obtain a new combinatorial formula for the character multiplicities that are the subject of the long-standing Foulkes Conjecture. The proof uses a result on skew characters of symmetric groups which is of independent interest. We also prove a number of analogous results for more general deflation maps in which the base group in the wreath product is not required to act trivially.

1. Introduction

Tableaux combinatorics is a pivotal theme in the representation theory of the symmetric groups. Two of the fundamental results in this area are the Murnaghan–Nakayama rule for the values taken by irreducible characters of symmetric groups, and Young's rule (or its equivalent, Pieri's rule), which can be used to decompose the characters of Young permutation modules. The main object of this paper is to prove Theorem 1.1, which gives a combinatorial description of the restrictions of characters of symmetric groups to their maximal imprimitive subgroups. This result is a simultaneous generalization of the Murnaghan–Nakayama rule and Young's rule. As a corollary, we obtain a new combinatorial formula for the character multiplicities that are the subject of Foulkes' Conjecture, a long-standing problem which spans representation theory, invariant theory and algebraic combinatorics.

We first introduce the ideas needed to state Theorem 1.1 and its corollary for Foulkes' Conjecture, Corollary 1.3. By a construction originally

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due to Frobenius, the irreducible characters of the symmetric group S_r are canonically labelled by the partitions of r. As is usual, we write χ^λ for the irreducible character labelled by the partition λ, and $\chi^{\lambda/\mu}$ for the character labelled by the skew-partition λ/μ. We refer the reader to [4, Chapter 2] or [8, §7.18] for a construction of these characters and to [8, page 309] for background on skew-partitions.

For each $r \in \mathbf{N}$, it is well known (see, for example, [2, Exercise 5.2.8]) that the maximal imprimitive subgroups of S_r are precisely the imprimitive wreath products $S_m \wr S_n \leq S_r$ for $m, n \in \mathbf{N}$ such that $mn = r$. Let ϑ be a character of S_m, and let V be a representation of S_m affording ϑ. Then $V^{\otimes n}$ is a representation of the base group $S_m \times \cdots \times S_m$. The complement S_n of this base group acts on $V^{\otimes n}$ by permuting the factors:

$$g(v_1 \otimes \cdots \otimes v_n) = v_{g^{-1}(1)} \otimes \cdots \otimes v_{g^{-1}(n)}$$

for $g \in S_n$ and $v_1, \ldots, v_n \in V$. These two actions combine to give a representation of $S_m \wr S_n$ on $V^{\otimes n}$ (see [4, 4.3.6]). We shall denote by $\widetilde{\vartheta^{\times n}}$ the character of $S_m \wr S_n$ afforded by this representation. We also need the characters of $S_m \wr S_n$ whose kernel contains $S_m \times \cdots \times S_m$. These characters are precisely the inflations of the characters of S_n to $S_m \wr S_n$ along the canonical surjection $S_m \wr S_n \twoheadrightarrow S_n$. If ν is a partition of n, we denote by $\operatorname{Inf}_{S_n}^{S_m \wr S_n} \chi^\nu$ the irreducible character of $S_m \wr S_n$ constructed in this way. It is easily seen that the characters $\widetilde{\vartheta^{\times n}} \operatorname{Inf}_{S_n}^{S_m \wr S_n} \chi^\nu$ obtained by multiplying characters of these two types are irreducible. (By [4, Theorem 4.3.33], any irreducible character of $S_m \wr S_n$ is induced from a suitable product of characters of this form.)

Given a finite group G, we let $\mathcal{C}(G)$ denote the abelian group of virtual characters of G.

Definition 1.1. Let $m, n \in \mathbf{N}$ and let ϑ be an irreducible character of S_m. Let ξ be an irreducible character of $S_m \wr S_n$. We define

$$\operatorname{Def}_{S_n}^{\vartheta} \xi = \begin{cases} \chi^\nu & \text{if } \xi = \widetilde{\vartheta^{\times n}} \operatorname{Inf}_{S_n}^{S_m \wr S_n} \chi^\nu \text{ where } \nu \text{ is a partition of } n \\ 0 & \text{otherwise.} \end{cases}$$

Let $\operatorname{Def}_{S_n}^{\vartheta} : \mathcal{C}(S_m \wr S_n) \to \mathcal{C}(S_n)$ be the group homomorphism defined by linear extension of this definition. Given $\psi \in \mathcal{C}(S_m \wr S_n)$, we say that $\operatorname{Def}_{S_n}^{\vartheta} \psi$ is the *deflation of ψ with respect to ϑ*. Let $\operatorname{Defres}_{S_n}^{\vartheta} : \mathcal{C}(S_{mn}) \to \mathcal{C}(S_n)$ be the group homomorphism defined by

$$\operatorname{Defres}_{S_n}^{\vartheta} \chi = \operatorname{Def}_{S_n}^{\vartheta} \operatorname{Res}_{S_m \wr S_n}^{S_{mn}} \chi$$

for $\chi \in \mathcal{C}(S_{mn})$.

The most important case for us is when ϑ is the trivial character of S_m. In this case we shall omit the deflating character and simply write Def_{S_n} and Defres_{S_n}. If V is a complex representation of $S_m \wr S_n$ with character χ, then $\mathrm{Def}_{S_n} \chi$ is the character of the maximal subrepresentation of V on which the base group $S_m \times \cdots \times S_m$ acts trivially.

Our main theorem gives a combinatorial rule for the values of $\mathrm{Defres}_{S_n} \chi^{\lambda/\mu}$ where λ/μ is a skew-partition of mn. The combinatorial content comes from a generalization of the border-strip tableaux defined by Stanley in [8, §7.17].

Recall that a skew-partition σ/τ is said to be a *border strip* (or rim hook) if the Young diagram of σ/τ is connected and contains no 2×2 square. The *length* of the border strip σ/τ is $|\sigma/\tau|$ and its *height* is defined to be one less than its number of non-empty rows. If λ/μ is a skew-partition, then we define a *border-strip tableau* of shape λ/μ to be an assignment of the elements of a set $J \subseteq \mathbf{N}$ to the boxes of the Young diagram of λ/μ so that the rows and columns are non-decreasing, and for each $j \in J$, the boxes labelled j form a border strip; if $J = \{1, \ldots, k\}$, and for each $j \in J$ the border strip formed by the boxes labelled j has length α_j, then we say that the tableau has *type* $(\alpha_1, \ldots, \alpha_k)$. We need the following three further definitions: for an example see Figure 1 and Example 1.2 below.

Definition 1.2. Let T be a border-strip tableau. The *sign* of T is defined by $\mathrm{sgn}(T) = (-1)^h$, where h is the sum of the heights of the border strips forming T.

Definition 1.3. Let λ/τ be a border strip in a partition λ. If the lowest-numbered row of λ met by λ/τ is row k then we define the *row number* of λ/τ to be k, and write $N(\lambda/\tau) = k$.

In the next definition it is useful to note that if T is a border-strip tableau of shape λ/μ and type $(\alpha_1, \ldots, \alpha_k)$ then there are partitions

$$\mu = \lambda^0 \subset \lambda^1 \subset \cdots \subset \lambda^{k-1} \subset \lambda^k = \lambda$$

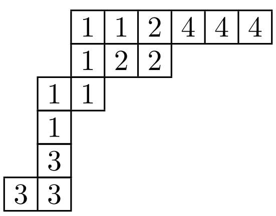


FIGURE 1. A border-strip tableau of shape $(8, 5, 3, 2, 2, 2)/(2, 2, 1, 1, 1)$ and type $(6, 3, 3, 3)$. The heights of the border strips labelled $1, 2, 3, 4$ are $3, 1, 1, 0$ respectively, and the sign of this border-strip tableau is thus -1.

such that for each $j \in \{1, \ldots, k\}$, the border strip in T labelled j is λ^j/λ^{j-1}.

Definition 1.4. Let $m, n \in \mathbf{N}$ and let λ/μ be a skew-partition of mn. Given a composition $\gamma = (\gamma_1, \ldots, \gamma_d)$ of n, let $\gamma^{\star m} = (\gamma_1, \ldots, \gamma_1, \ldots, \gamma_d, \ldots, \gamma_d)$ denote the composition of mn obtained from γ by repeating each part m times. An *m-border-strip tableau* of *shape* λ/μ and *type* γ is a border-strip tableau of shape λ/μ and type $\gamma^{\star m}$ such that for each $j \in \{1, 2, \ldots, d\}$, the row numbers of the border strips

$$\lambda^{(j-1)m+1}/\lambda^{(j-1)m}, \ldots, \lambda^{jm}/\lambda^{jm-1}$$

corresponding to the m parts in $\gamma^{\star m}$ of length α_j satisfy

$$N(\lambda^{(j-1)m+1}/\lambda^{jm}) \geq \cdots \geq N(\lambda^{jm}/\lambda^{jm-1}). \tag{1}$$

Let

$$a_{\lambda/\mu,\gamma} = \sum_T \operatorname{sgn}(T)$$

where the sum is over all m-border-strip tableaux T of shape λ/μ and type γ.

We can now state our main theorem.

Theorem 1.1. *Let $m, n \in \mathbf{N}$ and let λ/μ be a skew-partition of mn. If γ is a composition of n and $g \in S_n$ has cycle type γ then*

$$(\operatorname{Defres}_{S_n} \chi^{\lambda/\mu})(g) = a_{\lambda/\mu,\gamma}.$$

The proof of Theorem 1.1 occupies most of this paper. We give an outline of the proof in §2 below, where we also state an intermediate result on character values of skew-characters (Theorem 2.1) which is of significant interest in its own right. First though we give an illustrative example and discuss the connection with earlier work.

Example 1.2. Let $\lambda = (6, 5, 3, 2)$ and let $\mu = (3, 1)$. The three different 2-border-strip tableaux of shape λ/μ and type $\gamma = (1, 2, 3)$ are shown below.

			2	6	6
	1	5	5	6	
3	4	5			
3	4				

			4	6	6
	1	2	4	6	
3	5	5			
3	5				

			2	6	6
	1	4	4	6	
3	5	5			
3	5				

Observe that for each $j \in \{1, 2, 3\}$, the row number of the border strip labelled $2j - 1$ in each tableau is at least the row number of the border strip labelled $2j$. Thus the first border strip corresponding to each part of γ is added no higher up in each partition diagram than the second. The sums of the heights of the border strips forming these tableaux are 4, 4 and 3 and so their signs are $+1$, $+1$ and -1, respectively. By Definition 1.4 we have $a_{\lambda/\mu,\gamma} = 1$. Hence if $g \in S_6$ has cycle type γ, then Theorem 1.1 implies that $(\mathrm{Defres}_{S_6} \chi^{(6,5,3,2)/(3,1)})(g) = 1$.

It is clear than if $m = 1$ then $\mathrm{Defres}_{S_n} \chi = \chi$ for any character χ of S_n, and so the special case $m = 1$ of Theorem 1.1 asserts that $\chi^{\lambda/\mu}(g) = a_{\lambda/\mu,\gamma}$ for any skew-partition λ/μ of n and any element $g \in S_n$ of cycle type γ. Equivalently,

$$\chi^{\lambda/\mu}(g) = \sum_T \mathrm{sgn}(T)$$

where the sum is over all border-strip tableaux of shape λ/μ and type γ. This is the Murnaghan–Nakayama rule, as stated in [8, Equation (7.75)]. It should be noted that we require the Murnaghan–Nakayama rule in §4.3 below, and so our work does not provide a new proof of this result. In practice the Murnaghan–Nakayama rule is most frequently used as a recursive formula for the values of characters or skew characters. Equation (10) at the end of §5 formulates Theorem 1.1 in this way.

As Stanley observes in [8, page 348], it is far from obvious that the character values given by the Murnaghan–Nakayama rule applied to a skew-partition λ/μ and a composition γ are independent of the order of the parts of γ. This remark applies even more strongly to Theorem 1.1. For example, the reader may check that if $\lambda/\mu = (6,5,3,2)/(3,1)$, as in Example 1.2, and $\gamma' = (2,1,3)$, then there is a unique 2-border-strip tableau of shape $(6,5,3,2)/(3,1)$ and type γ'. Thus $a_{\lambda/\mu,\gamma'} = 1$, but the sums defining $a_{\lambda/\mu,\gamma}$ and $a_{\lambda/\mu,\gamma'}$ are different.

Another special case of Theorem 1.1 worth noting occurs when g is the identity element of S_n. If ξ is an irreducible character of $S_m \wr S_n$ then either the base group $B = S_m \times \cdots \times S_m$ is contained in the kernel of ξ and $\langle \mathrm{Res}_B \, \xi, 1_B \rangle = \xi(1)$, or $\langle \mathrm{Res}_B \, \xi, 1_B \rangle = 0$. Hence, by linearity, we have

$$(\mathrm{Defres}_{S_n} \chi)(1) = \langle \mathrm{Res}_B \, \chi, 1_B \rangle \tag{2}$$

for any character χ of S_{mn}. It now follows from Theorem 1.1 and Frobenius reciprocity that

$$a_{\lambda/\mu,(1^n)} = \left\langle \chi^{\lambda/\mu}, \mathrm{Ind}_{S_m \times \cdots \times S_m}^{S_{mn}} 1_{S_m} \times \cdots \times 1_{S_m} \right\rangle$$

for any skew-partition λ/μ of mn. It is clear from Definition 1.4 that $a_{\lambda/\mu,(1^n)}$ is the number of semi-standard tableaux of shape λ/μ and type (m^n). Therefore, by setting $\mu = \varnothing$ in the previous equation, we obtain a special case of Young's rule (see [4, 2.8.5] or [8, Proposition 7.18.7]). Proposition 6.2 in §6 below gives a related result on the degrees of the deflations of the irreducible characters of S_{mn} to S_n.

We end this introduction by mentioning that an important motivation for Theorem 1.1 is Problem 9 on plethysms of Schur functions in Stanley's survey article [9]. For m, $n \in \mathbf{N}$, let $\phi^{(m^n)}$ be the permutation character of S_{mn} acting on all unordered set partitions of $\{1, 2, \ldots, mn\}$ into n sets each of size m. Equivalently, $\phi^{(m^n)} = \mathrm{Ind}_{S_m \wr S_n}^{S_{mn}} 1$. Stated in the language of character theory, Stanley's Problem 9 asks for a combinatorial description of the multiplicities $\langle \phi^{(m^n)}, \chi^\lambda \rangle$ that makes it clear that they are non-negative. Our Theorem 1.1 gives a new formula for these multiplicities, which while not obviously non-negative, may be a step towards a solution.

Corollary 1.3. *Let m, $n \in \mathbf{N}$. If λ is a partition of mn then*

$$\left\langle \phi^{(m^n)}, \chi^\lambda \right\rangle = \sum_\gamma \frac{a_{\lambda,\gamma}}{z_\gamma}$$

where the sum is over all partitions γ of n and z_γ is the size of the centralizer in S_n of an element of cycle type γ.

Proof. Using Frobenius reciprocity, then the inflation-deflation reciprocity relation

$$\langle \mathrm{Def}_{S_n} \psi, \chi^\nu \rangle = \left\langle \psi, \mathrm{Inf}_{S_n}^{S_m \wr S_n} \chi^\nu \right\rangle,$$

where ψ is a character of $S_m \wr S_n$ and ν is a partition of n, we have

$$\begin{aligned} \left\langle \mathrm{Ind}_{S_m \wr S_n}^{S_{mn}} 1, \chi^\lambda \right\rangle_{S_{mn}} &= \left\langle 1_{S_m \wr S_n}, \mathrm{Res}_{S_m \wr S_n} \chi^\lambda \right\rangle_{S_m \wr S_n} \\ &= \left\langle 1_{S_n}, \mathrm{Defres}_{S_n} \chi^\lambda \right\rangle_{S_n}. \end{aligned}$$

The corollary now follows from Theorem 1.1. □

A long-standing related conjecture is Foulkes' Conjecture, which asserts that if $m \leq n$ then

$$\left\langle \phi^{(m^n)}, \chi^\lambda \right\rangle \geq \left\langle \phi^{(n^m)}, \chi^\lambda \right\rangle$$

for all partitions λ of mn. Equivalent formulations of Foulkes' Conjecture exist in the language of general linear groups, symmetric polynomials, and geometric invariant theory. Despite having been attacked from all these directions (and more), it has only been proved when $m \leq 4$ (see [5]), and asymptotically when n is very large compared to m (see [1, page 352]).

For further background, and some recent results on the constituents of $\phi^{(m^n)}$, we refer the reader to [7]. Corollary 1.3 gives an algorithm for testing Foulkes' Conjecture for a single character χ^λ of S_{mn} that seems likely to be significantly faster than more direct methods, such as those requiring the character values of $\phi^{(m^n)}$ and $\phi^{(n^m)}$ to be calculated on all partitions of mn.

2. Outline

The remainder of this paper proceeds as follows. Throughout, we shall adopt the convention that if α is a partition of $r \in \mathbf{N}$, then $g_\alpha \in S_r$ is an element of cycle type α, and z_α is the size of the centralizer of g_α in S_r. (The choice of g_α within the conjugacy class is irrelevant.)

In §3 we prove Proposition 3.2, which implies that if χ is a character of S_{mn} and $g \in S_n$, then $(\mathrm{Defres}_{S_n} \chi)(g)$ is the average value of χ on the coset of the base group $S_m \times \cdots \times S_m$ in $S_m \wr S_n$ corresponding to g. Equation (2) above is a special case of this result. In the case when $g \in S_n$ is an n-cycle, we obtain Proposition 3.5(ii), which implies that if λ/μ is a skew-partition of mn, then

$$\mathrm{Defres}_{S_n} \chi^{\lambda/\mu}(g) = \sum_{\alpha} \frac{\chi^{\lambda/\mu}(g_{n\alpha})}{z_\alpha} \tag{3}$$

where the sum is over all partitions α of m, and $n\alpha$ denotes the partition obtained from α by multiplying each of its parts by n.

In §4 we prove Theorem 2.1 below, which gives a formula for the character values $\chi^{\lambda/\mu}(g_{n\alpha})$ appearing on the right-hand side of (3). The statement of this theorem uses the n-quotient of λ/μ, as defined in §4.1 below; in this section we also define what it means for a skew-partition to be n-decomposable, and define an n-sign, $\varepsilon_n(\lambda/\mu) \in \{1, -1\}$, for each such skew-partition.

Theorem 2.1. *Let m, $n \in \mathbf{N}$ and let λ/μ be a skew-partition of mn. Let α be a partition of m. If λ/μ is not n-decomposable then $\chi^{\lambda/\mu}(g_{n\alpha}) = 0$. If λ/μ is n-decomposable, with n-quotient $(\lambda^{(0)}/\mu^{(0)}, \ldots, \lambda^{(n-1)}/\mu^{(n-1)})$, then*

$$\chi^{\lambda/\mu}(g_{n\alpha}) = \varepsilon_n(\lambda/\mu)\, \mathrm{Ind}_{S_{\ell_0} \times \cdots \times S_{\ell_{n-1}}}^{S_m} \big(\chi^{\lambda^{(0)}/\mu^{(0)}} \times \cdots \times \chi^{\lambda^{(n-1)}/\mu^{(n-1)}}\big)(g_\alpha)$$

where $|\lambda^{(i)}/\mu^{(i)}| = \ell_i$.

In the special case when $\mu = \varnothing$, it is easily seen from the definitions in §4.1 that the partition λ/μ is n-decomposable if and only if λ has empty n-core, and that the n-quotient and n-sign of λ/μ (as a skew-partition) are equal to the usual n-quotient and sign of λ, as defined in [4, 2.7.29] and [4, page 81], respectively. Specializing further to the case $\alpha = (1^m)$, we obtain a well-known formula (see Corollary 2.7.33 in [4]) for the value of χ^λ on the product of m disjoint n-cycles, namely

$$\chi^\lambda(g_{(n,\ldots,n)}) = \varepsilon_n(\lambda/\varnothing) \frac{m!}{|\lambda^{(0)}|! \ldots |\lambda^{(n-1)}|!} \chi^{\lambda^{(0)}}(1) \ldots \chi^{\lambda^{(n-1)}}(1)$$

where λ is a partition with empty n-core and n-quotient $(\lambda^{(0)}, \ldots, \lambda^{(n-1)})$; if the n-core of λ is non-empty then the value is zero. Theorem 2.1 is a significant generalization of this result.

In §5 we combine the results of §3 and §4 to prove Proposition 5.4 which shows that Theorem 1.1 holds when $g \in S_n$ is an n-cycle. We then use this proposition to prove Theorem 1.1 by induction on the number of cycles in g; our use of skew-partitions allows us to mirror a standard proof of the Murnaghan–Nakayama rule, and so deal with g one cycle at a time.

The results in §3, together with several results in §5, are proved for the more general deflation maps $\mathrm{Def}^{\vartheta}_{S_n}$ defined in Definition 1.1. While we are not able to prove a combinatorial rule of the strength of Theorem 1.1 in this general setting, we are still able to prove some results: these, together with an illustrative example, are given in §6.

3. Deflation by averaging

Let m, $n \in \mathbf{N}$. We shall think of $S_m \wr S_n$ as the group of permutations of

$$\{1, \ldots, m\} \times \{1, \ldots, n\}$$

that leaves invariant the set of blocks of the form $\Delta_j = \{(1, j), \ldots, (m, j)\}$, $1 \leq j \leq n$. Given $h_1, \ldots, h_n \in S_m$ and $g \in S_n$, we write $(h_1, \ldots, h_n; g)$ for the permutation which sends (i, j) to $(h_{gj}i, gj)$. This left action is equivalent to the action defined in [4, 4.1.18]. Let $B = S_m \times \cdots \times S_m$ denote the base group in the wreath product. As shorthand, if $k = (h_1, \ldots, h_n) \in B$ then we shall write $(k\,; g)$ rather than $(h_1, \ldots, h_n; g)$.

Lemma 3.1. *Let m, $n \in \mathbf{N}$, let ϑ be an irreducible character of S_m, and let ξ be an irreducible character of $S_m \wr S_n$. If $\xi = \widetilde{\vartheta^{\times n}} \operatorname{Inf}_{S_n}^{S_m \wr S_n} \chi^\nu$ for some partition ν of n then*

$$\frac{1}{|B|} \sum_{k \in B} \xi(k\,; g) \widetilde{\vartheta^{\times n}}(k\,; g) = \chi^\nu(g),$$

and if $\xi \in \operatorname{Irr}(S_m \wr S_n)$ is not of this form then the left-hand side is zero.

Proof. Suppose that the left-hand side is non-zero. The character $\widetilde{\vartheta^{\times n}}$ of $S_m \wr S_n$ restricts to the irreducible character $\vartheta \times \cdots \times \vartheta$ of B. Hence, by [3, Lemma 8.14(b)], applied with $G = S_m \wr S_n$ and $N = B$, we have $\left\langle \operatorname{Res}_B \xi, \operatorname{Res}_B \widetilde{\vartheta^{\times n}} \right\rangle \neq 0$. It follows by Frobenius Reciprocity that ξ is a constituent of

$$\operatorname{Ind}_B^{S_m \wr S_n} (\vartheta \times \cdots \times \vartheta) = \sum_\nu \chi^\nu(1) \widetilde{\vartheta^{\times n}} \operatorname{Inf}_{S_n}^{S_m \wr S_n} \chi^\nu,$$

where the sum is over all partitions ν of n. Since ξ is irreducible we must have $\xi = \widetilde{\vartheta^{\times n}} \operatorname{Inf}_{S_n}^{S_m \wr S_n} \chi^\nu$ for some ν. Therefore the left-hand side in the lemma is

$$\frac{\chi^\nu(g)}{|B|} \sum_{k \in B} \bigl(\widetilde{\vartheta^{\times n}}(k\,; g)\bigr)^2$$

which is equal to $\chi^\nu(g)$ by [3, Lemma 8.14(c)]. □

By Definition 1.1, we have $\mathrm{Defres}^{\vartheta}_{S_n}(\widetilde{\vartheta^{\times n}}\,\mathrm{Inf}^{S_m \wr S_n}_{S_n}\chi^\nu)(g) = \chi^\nu(g)$ for all $g \in S_n$. The next proposition therefore follows immediately from Lemma 3.1.

Proposition 3.2. *Let m, $n \in \mathbf{N}$, let ϑ be an irreducible character of S_m, and let ψ be a character of $S_m \wr S_n$. If $g \in S_n$ then*

$$(\mathrm{Def}^{\vartheta}_{S_n}\psi)(g) = \frac{1}{|B|}\sum_{k \in B}\psi(k\,;g)\widetilde{\vartheta^{\times n}}(k\,;g). \qquad \square$$

Corollary 3.3. *Let m, $n \in \mathbf{N}$, let ϑ be an irreducible character of S_m, and let ψ be a character of $S_m \wr S_n$. If $g \in S_n$ is an n-cycle then*

$$(\mathrm{Def}^{\vartheta}_{S_n}\psi)(g) = \frac{1}{m!}\sum_{h \in S_m}\psi(h,1,\ldots,1;g)\vartheta(h).$$

Proof. Suppose that g is the n-cycle $(x_1\,x_2\,\ldots\,x_n)$. By [4, 4.2.8], the permutations $(h_1,\ldots,h_n;g)$ and $(h'_1,\ldots,h'_n;g) \in S_m \wr S_n$ are conjugate in $S_m \wr S_n$ if and only if the elements $h_{x_n}h_{x_{n-1}}\ldots h_{x_1}$ and $h'_{x_n}h'_{x_{n-1}}\ldots h'_{x_1}$ are conjugate in S_m. In particular, each conjugacy class of $S_m \wr S_n$ which meets $\{(k\,;g) : k \in B\}$ has a representative of the form $(h,1,\ldots,1;g)$. Moreover, the number of elements $(h_1,h_2,\ldots,h_n;g)$ conjugate to $(h,1,\ldots,1;g)$ is $m!^{n-1}|h^{S_m}|$, since $h_2,\ldots,h_n$ may be chosen arbitrarily, and then h_1 must be chosen so that $h_{x_n}h_{x_{n-1}}\cdots h_{x_1} \in h^{S_m}$. It follows that

$$\begin{aligned}\sum_{k \in B}\psi(k\,;g)\widetilde{\vartheta^{\times n}}(k\,;g) &= m!^{n-1}\sum_{h \in S_m}\psi(h,1,\ldots,1;g)\widetilde{\vartheta^{\times n}}(h,1,\ldots,1;g)\\ &= m!^{n-1}\sum_{h \in S_m}\psi(h,1,\ldots,1;g)\vartheta(h)\end{aligned}$$

where the second equality uses Lemma 4.3.9 in [4]. Now apply Proposition 3.2 to the left-hand side. □

The following definition and lemma allow for a more convenient statement of Corollary 3.3.

Definition 3.1. Let m, $n \in \mathbf{N}$, let $g \in S_n$ be an n-cycle, and let ψ be a character of $S_m \wr S_n$. We define $\omega(\psi)$ to be the class function on S_m such that

$$\omega(\psi)(h) = \psi(h,1,\ldots,1;g)$$

for all $h \in S_m$.

Recall that if α is a partition of $m \in \mathbf{N}$, then we denote by $n\alpha$ the partition of mn obtained by multiplying each of the parts of α by n.

Lemma 3.4. *Let m, $n \in \mathbf{N}$. If $g \in S_n$ is an n-cycle and $h \in S_m$ has cycle type α then $(h,1,\ldots,1;g) \in S_m \wr S_n$ has cycle type $n\alpha$.*

Proof. It suffices to show that if $\mathcal{O}$ is an orbit of h on $\{1, 2, \ldots, m\}$ then $\mathcal{O} \times \{1, \ldots, n\}$ is an orbit of $(h, 1, \ldots, 1; g)$ in its action on $\{1, \ldots, m\} \times \{1, \ldots, n\}$. We leave this to the reader as an easy exercise. □

The next proposition follows easily from Lemma 3.4 and Corollary 3.3. The character value $\chi(g_{n\alpha})$ in part (i) is the subject of Theorem 2.1; combining this theorem with part (ii) gives Equation (8) in §5 below. Note also that part (ii) of the proposition implies Equation (3) in §2.

Proposition 3.5. *Let m, $n \in \mathbf{N}$, and let χ be a character of S_{mn}.*

(i) *If α is a partition of m then*

$$\omega(\mathrm{Res}_{S_m \wr S_n} \chi)(g_\alpha) = \chi(g_{n\alpha}).$$

(ii) *If ϑ is an irreducible character of S_m and $g \in S_n$ is an n-cycle then*

$$\begin{aligned}(\mathrm{Defres}^{\vartheta}_{S_n} \chi)(g) &= \langle \omega(\mathrm{Res}_{S_m \wr S_n} \chi), \vartheta \rangle \\ &= \sum_{\alpha} \frac{\chi(g_{n\alpha})}{z_\alpha} \vartheta(g_\alpha)\end{aligned}$$

where the sum is over all partitions α of m. □

4. Skew characters

In this section we prove Theorem 2.1. The combinatorial definitions used in this theorem are given in §4.1 below. In §4.2 we give a model that allows the character values of a character induced from a Young subgroup to be easily computed; we use this model in the proof of Theorem 2.1 in §4.3. An example to illustrate the main bijection used in the proof is given in §4.4. In several arguments we shall refer to James' abacus notation for partitions, as described in [4, page 78].

4.1. **Quotients of skew-partitions.** We shall define n-quotients and n-signs for the following class of skew-partitions.

Definition 4.1. Let $m, n \in \mathbf{N}$. A skew-partition λ/μ of mn is *n-decomposable* if there exists a border-strip tableau of shape λ/μ and type (n^m).

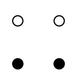
Definition 4.2. Let m, $n \in \mathbf{N}$ and let λ/μ be an n-decomposable skew-partition of mn. Let $\Gamma(\lambda)$ be an abacus display for λ on an n-runner abacus using tn beads for some $t \in \mathbf{N}$. Let $\Gamma(\mu)$ be the abacus display for μ obtained by performing an appropriate sequence of m upward bead moves on $\Gamma(\lambda)$. (This is possible since λ/μ is n-decomposable.) Let $(\lambda^{(0)}, \ldots, \lambda^{(n-1)})$ and $(\mu^{(0)}, \ldots, \mu^{(n-1)})$ be the n-quotients of λ and μ corresponding to $\Gamma(\lambda)$ and $\Gamma(\mu)$, respectively. The *n-quotient* of λ/μ is defined to be

$$(\lambda^{(0)}/\mu^{(0)}, \ldots, \lambda^{(n-1)}/\mu^{(n-1)}).$$

The *natural numbering* assigns numbers to the beads in $\Gamma(\lambda)$ and $\Gamma(\mu)$ in order of their positions. We define the *level* of a bead to be the number of beads above it on its runner. Let $\rho \in S_{tn}$ be the permutation such that $\rho(b) = c$ if the bead numbered b in $\Gamma(\lambda)$ has the same level and lies on the same runner as the bead numbered c in $\Gamma(\mu)$. We call ρ the *relabelling permutation* associated to $\Gamma(\lambda)$ and $\Gamma(\mu)$. We define the *n-sign* of λ/μ by $\varepsilon_n(\lambda/\mu) = \operatorname{sgn} \rho$.

To avoid cumbersome restatements, we adopt the convention that $\lambda^{(i)}/\mu^{(i)}$ always has the meaning in Definition 4.2 above. It is clear from the abacus that $\mu^{(i)}$ is a subpartition of $\lambda^{(i)}$ for each $i \in \{0, \dots, n-1\}$, and so the n-quotient is well-defined. In practice, it is often most convenient to calculate $\varepsilon_n(\lambda/\mu)$ by using Proposition 3.13 in [6], which implies that if T is any border-strip tableau of shape λ/μ and type (n^m), then $\operatorname{sgn}(T) = \varepsilon_n(\lambda/\mu)$. (This result also follows from our Proposition 4.2.) For an example, see Figure 2 below.

We remark that it appears to be impossible to define the n-core of an arbitrary skew-partition. The example $\lambda/\mu = (2,2)/(1)$ and $n = 2$ illustrates the obstacles that arise. Representing λ on a 2-runner abacus as



we see that either bead may be moved up, giving two different skew-partitions from which no border strip of length 2 can be removed, namely $(2)/(1)$ and $(1,1)/(1)$. The 2-quotients corresponding to these bead moves, namely $((1), \varnothing)$ and $(\varnothing, (1))$, are also different.

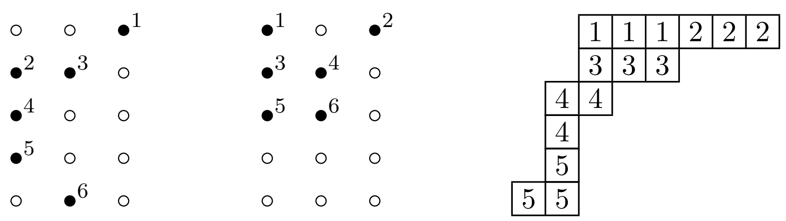


FIGURE 2. Abacus displays for $\lambda = (8,5,3,2,2,2)$ and $\mu = (2,2,1,1,1)$ showing the numbers assigned to beads in the natural numbering defined in Definition 4.2. The relabelling permutation is $(12)(34)$, so $\operatorname{sgn}_3(\lambda/\mu) = 1$. This agrees with the sign of the 3-border-strip tableau of shape λ/μ and type (3^5) also shown.

4.2. **A model for induction from a Young subgroup.** The following general result on the values of a character induced from a Young subgroup will be used in the proof of Theorem 2.1. (The notation is chosen to be consistent with this later use.)

Lemma 4.1. *Let $(\ell_0, \ldots, \ell_{n-1})$ be a composition of $m \in \mathbf{N}$. For each $i \in \{0, \ldots, n-1\}$, let ϑ_i be a character of S_{ℓ_i}. If $g \in S_m$ then*

$$\operatorname{Ind}_{S_{\ell_0} \times \cdots \times S_{\ell_{n-1}}}^{S_m} (\vartheta_0 \times \cdots \times \vartheta_{n-1})(g) = \sum_{\mathbf{t}} \vartheta_0(g_{\alpha_0(\mathbf{t})}) \ldots \vartheta_{n-1}(g_{\alpha_{n-1}(\mathbf{t})})$$

where the sum is over all $(\ell_0, \ldots, \ell_{n-1})$-tabloids $\mathbf{t}$ such that $g\mathbf{t} = \mathbf{t}$, and $\alpha_i(\mathbf{t})$ is the cycle type of the permutation induced by g on the entries of row $i+1$ of $\mathbf{t}$.

Proof. Let $\mathbf{t}_1, \ldots, \mathbf{t}_N$ be the $(\ell_0, \ldots, \ell_{n-1})$-tabloids. Let $\mathbf{s}$ be a $(\ell_0, \ldots, \ell_{n-1})$-tabloid fixed by the Young subgroup $S_{\ell_0} \times \cdots \times S_{\ell_{n-1}}$. For each j such that $1 \leq j \leq N$, choose $x_j \in S_m$ such that $\mathbf{t}_j = x_j \mathbf{s}$. Let $\vartheta = \vartheta_0 \times \cdots \times \vartheta_{n-1}$. For each $g \in S_m$ we have

$$\bigl(\operatorname{Ind}_{S_{\ell_0} \times \cdots \times S_{\ell_{n-1}}}^{S_m} \vartheta\bigr)(g) = \sum_j \vartheta(x_j^{-1} g x_j)$$

where the sum is over all j such that

$$x_j^{-1} g x_j \in S_{\ell_0} \times \cdots \times S_{\ell_{n-1}},$$

or, equivalently, over all j such that $g\mathbf{t}_j = \mathbf{t}_j$. If $\Delta_1, \ldots, \Delta_q$ are the orbits of g on row $i+1$ of $\mathbf{t}_j$, then $x_j^{-1}\Delta_1, \ldots, x_j^{-1}\Delta_q$ are the orbits of $x_j^{-1} g x_j$ on row $i+1$ of $\mathbf{s}$. Hence $x_j^{-1} g x_j$ acts with cycle type $\alpha_i(\mathbf{t}_j)$ on row $i+1$ of $\mathbf{s}$ and so

$$\vartheta(x_j^{-1} g x_j) = \vartheta_0(g_{\alpha_0(\mathbf{t_j})}) \ldots \vartheta_{n-1}(g_{\alpha_{n-1}(\mathbf{t_j})}).$$

The lemma follows. □

4.3. **Proof of Theorem 2.1.** Let $m, n \in \mathbf{N}$ and let λ/μ be a skew-partition of mn. Let α be a partition of m. If there is a border-strip tableau of shape λ/μ and type $n\alpha$ then it is clear from the abacus that λ/μ is n-decomposable. Hence if λ/μ is not n-decomposable then, by the Murnaghan–Nakayama rule, $\chi^{\lambda/\mu}(g_{n\alpha}) = 0$.

We may therefore assume that λ/μ is n-decomposable. Let $\ell_i = |\lambda^{(i)}/\mu^{(i)}|$ for each $i \in \{0, \ldots, n-1\}$ and let $H = S_{\ell_0} \times \cdots \times S_{\ell_{n-1}}$. To show that

$$\chi^{\lambda/\mu}(g_{n\alpha}) = \varepsilon_n(\lambda/\mu) \operatorname{Ind}_H^{S_m} \bigl(\chi^{\lambda^{(0)}/\mu^{(0)}} \times \cdots \times \chi^{\lambda^{(n-1)}/\mu^{(n-1)}}\bigr)(g_\alpha), \tag{4}$$

we shall use the following generalization of border-strip tableaux.

Definition 4.3. Let m, $n \in \mathbf{N}$. Let λ/μ be an n-decomposable skew-partition and let $\alpha = (\alpha_1, \ldots, \alpha_k)$ be a composition of m. A *n-quotient border-strip tableau* of *shape* λ/μ and *type* α is an n-tuple $(T_0, \ldots, T_{n-1})$ of border-strip tableaux such that

(a) for each $i \in \{0, \ldots, n-1\}$, the shape of T_i is $\lambda^{(i)}/\mu^{(i)}$, *and*

(b) for each $j \in \{1, \ldots, k\}$, the boxes in the T_i labelled j lie in a single tableau, where they form a border strip of length α_j.

By the Murnaghan–Nakayama rule we have

$$\chi^{\lambda/\mu}(g_{n\alpha}) = \sum \operatorname{sgn}(T)$$

where the sum is over all border-strip tableaux of shape λ/μ and type $n\alpha$. The bijection in the following proposition implies that

$$\chi^{\lambda/\mu}(g_{n\alpha}) = \varepsilon_n(\lambda/\mu) \sum \operatorname{sgn}(T_0) \ldots \operatorname{sgn}(T_{n-1}) \tag{5}$$

where the sum is over all n-quotient border-strip tableaux $(T_0, \ldots, T_{n-1})$ of shape λ/μ and type α. An illustrative example of the bijection is given in Figure 3 in §4.4 below.

Proposition 4.2. *Let λ/μ be an n-decomposable skew-partition of mn and let $\alpha = (\alpha_1, \ldots, \alpha_k)$ be a partition of m. There is a canonical bijection between border-strip tableaux of shape λ/μ and type $n\alpha$ and n-quotient border-strip tableaux of shape λ/μ and type α. Under this bijection, if T is mapped to $(T_0, \ldots, T_{n-1})$, then*

$$\operatorname{sgn}(T) = \varepsilon_n(\lambda/\mu) \operatorname{sgn}(T_0) \ldots \operatorname{sgn}(T_{n-1}).$$

Proof. Let T be a border-strip tableau of shape λ/μ and type α. The abacus gives a canonical bijection between border strips in λ/μ of length $n\ell$ and border strips of length ℓ in the skew-partitions $\lambda^{(i)}/\mu^{(i)}$ for $i \in \{0, \ldots, n-1\}$. If the border strip of length $n\alpha_k$ in T corresponds to a border strip of length α_k in $\lambda^{(i_k)}/\mu^{(i_k)}$, then we label the corresponding boxes in the Young diagram of $\lambda^{(i_k)}/\mu^{(i_k)}$ by k. Removing these border strips from the tableaux concerned and iterating the process with the border strip of length $n\alpha_{k-1}$, and so on, we obtain a canonical bijection between border-strip tableaux of shape λ/μ and type $n\alpha$ and n-quotient border-strip tableaux of shape λ/μ and type α.

It only remains to prove the assertion about signs. Let $\Gamma(\lambda)$ be an abacus display for λ on an n-runner abacus using tn beads for some $t \in \mathbf{N}$, and let $\Gamma(\mu)$ be the abacus display for μ obtained from $\Gamma(\lambda)$. Let $\rho \in S_{tn}$ be the relabelling permutation defined in Definition 4.2 in §4.1.

Let T be a border-strip tableau of shape λ/μ and type $n\alpha$. For each $j \in \{1,\ldots,k\}$, let λ^j/λ^{j-1} be the shape of the border strip in T corresponding to the part $n\alpha_j$ of $n\alpha$. On the abacus, T corresponds to a sequence of upward bead moves that turn $\Gamma(\lambda)$ into $\Gamma(\mu)$ *via* abacus displays $\Gamma(\lambda^{k-1}), \ldots, \Gamma(\lambda^1)$ for the partitions $\lambda^{k-1}, \ldots, \lambda^1$. In the move from $\Gamma(\lambda^j)$ to $\Gamma(\lambda^{j-1})$ specified by T, a single bead is lifted (possibly over some intermediate beads) into a gap α_j positions higher on its runner, and no other bead is moved. Define a permutation $\sigma_j \in S_{tn}$ so that $\sigma_j(b) = c$ if the bead numbered b in the natural numbering of $\Gamma(\lambda^j)$ becomes the bead numbered c in the natural numbering of $\Gamma(\lambda^{j-1})$, after the move from $\Gamma(\lambda^j)$ to $\Gamma(\lambda^{j-1})$ described above. It is easy to see that if the border strip λ^j/λ^{j-1} has height h then σ_j is a cycle of length $h+1$, and so $\operatorname{sgn}\sigma_j = (-1)^h$. (This is proved in Lemma 3.12 in [6].) Hence, if $\pi = \sigma_1\ldots\sigma_k$, then $\operatorname{sgn}\pi = \operatorname{sgn}(T)$. Moreover, $\pi(b) = c$ if and only if the bead numbered b in $\Gamma(\lambda)$ is the same bead (possibly after a sequence of upward moves on its runner) as the bead numbered c in $\Gamma(\mu)$.

Suppose that T is mapped by our bijection to $(T_0,\ldots,T_{n-1})$ where T_i is a border-strip tableau of shape $\lambda^{(i)}/\mu^{(i)}$. For each $i \in \{0,\ldots,n-1\}$, let $X_i \subseteq \{1,\ldots,tn\}$ be the numbers assigned to the beads on runner i of $\Gamma(\lambda)$. It is clear that $\rho^{-1}\pi$ restricts to a permutation of X_i; we denote this permutation by π_i. If $X_i = \{b_0,\ldots,b_{s-1}\}$ where $b_0 < b_1 < \ldots < b_{s-1}$, then $\pi_i(b_r) = b_{r'}$ if and only if, after the sequence of bead moves specified by T, the bead of level r in $\Gamma(\lambda)$ on runner i has level r' in $\Gamma(\mu)$. Hence the permutation π_i of X_i records the bead moves corresponding to T that take place on runner i, that is, the bead moves corresponding to the border-strip tableau T_i of shape $\lambda^{(i)}/\mu^{(i)}$. As before, we have $\operatorname{sgn}(T_i) = \operatorname{sgn}\pi_i$. Since

$$\operatorname{sgn}\rho^{-1}\pi = \prod_{i=0}^{n-1}\operatorname{sgn}\pi_i$$

and $\operatorname{sgn}\rho = \varepsilon_n(\lambda/\mu)$, we have

$$\operatorname{sgn}(T) = \varepsilon_n(\lambda/\mu)\prod_{i=0}^{n-1}\operatorname{sgn}(T_i).$$

This completes the proof of the proposition. □

Comparing Equations (4) and (5) we see that to complete the proof of Theorem 2.1, it suffices to show that

$$(6)\quad \sum \operatorname{sgn}(T_0)\ldots\operatorname{sgn}(T_{n-1}) = \operatorname{Ind}_H^{S_m}\big(\chi^{\lambda^{(0)}/\mu^{(0)}} \times \cdots \times \chi^{\lambda^{(n-1)}/\mu^{(n-1)}}\big)(g_\alpha)$$

where the sum is over all n-quotient border-strip tableaux $(T_0,\ldots,T_{n-1})$ of shape λ/μ and type α. We shall see that Equation (6) follows from

Lemma 4.1 and the Murnaghan–Nakayama rule, by some manipulations that are essentially formal. By Lemma 4.1 we have

$$\begin{aligned} &\operatorname{Ind}_H^{S_m}\left(\chi^{\lambda^{(0)}/\mu^{(0)}} \times \cdots \times \chi^{\lambda^{(n-1)}/\mu^{(n-1)}}\right)(g_\alpha) \\ &\qquad = \sum_{\mathbf{t}} \chi^{\lambda^{(0)}/\mu^{(0)}}(g_{\alpha_0(\mathbf{t})}) \ldots \chi^{\lambda^{(n-1)}/\mu^{(n-1)}}(g_{\alpha_{n-1}(\mathbf{t})}) \end{aligned} \tag{7}$$

where the sum is over all $(\ell_0, \ldots, \ell_{n-1})$-tabloids $\mathbf{t}$ such that $g_\alpha \mathbf{t} = \mathbf{t}$ and $\alpha_i(\mathbf{t})$ is the cycle type of the permutation of row $i+1$ of $\mathbf{t}$ induced by g_α.

For such a tabloid $\mathbf{t}$, let

$$f(\mathbf{t}) = \sum \operatorname{sgn}(T_0) \ldots \operatorname{sgn}(T_{n-1})$$

where the sum is over all n-quotient border-strip tableaux $(T_0, \ldots, T_{n-1})$ of shape λ/μ and type α, such that T_i has a border strip labelled j (of length α_j) if and only if the elements of the orbit of g_α corresponding to the part α_j lie in row $i+1$ of $\mathbf{t}$. It follows easily from the Murnaghan–Nakayama rule that if $g_\alpha \mathbf{t} = \mathbf{t}$ then

$$\chi^{\lambda^{(0)}/\mu^{(0)}}(g_{\alpha_0(\mathbf{t})}) \ldots \chi^{\lambda^{(n-1)}/\mu^{(n-1)}}(g_{\alpha_{n-1}(\mathbf{t})}) = f(\mathbf{t}),$$

and so, by Equation (7),

$$\operatorname{Ind}_H^{S_m}\left(\chi^{\lambda^{(0)}/\mu^{(0)}} \times \cdots \times \chi^{\lambda^{(n-1)}/\mu^{(n-1)}}\right)(g_\alpha) = \sum f(\mathbf{t})$$

where the sum is over all $(\ell_0, \ldots, \ell_{n-1})$ tabloids $\mathbf{t}$ such that $g_\alpha \mathbf{t} = \mathbf{t}$. Every n-quotient border-strip tableau of shape λ/μ and type α corresponds to some tabloid $\mathbf{t}$ such that $g_\alpha \mathbf{t} = \mathbf{t}$. Thus

$$\sum \operatorname{sgn}(T_0) \ldots \operatorname{sgn}(T_{n-1}) = \sum f(\mathbf{t})$$

where the left-hand sum is over all n-quotient border-strip tableaux of shape λ/μ and type α, and the right-hand sum is over all $(\ell_0, \ldots, \ell_{n-1})$-tabloids $\mathbf{t}$ such that $g_\alpha \mathbf{t} = \mathbf{t}$. Equation (6) now follows on comparing the two preceding equations.

4.4. **Example.** We give an example of the correspondences used in the proof of Theorem 2.1. Let $\lambda/\mu = (8, 5, 3, 2, 2, 2)/(2, 2, 1, 1, 1)$ as in Figure 2, and let $\alpha = (2, 1, 1, 1)$. By the Murnaghan–Nakayama rule, $\chi^{\lambda/\mu}(g_{3\alpha})$ is the sum of the signs of the four border-strip tableaux of type λ/μ and type $3\alpha = (6, 3, 3, 3)$ shown in Figure 3 overleaf. Their signs are $+1$, -1, -1, -1, respectively, so $\chi^{\lambda/\mu}(g_{3\alpha}) = -2$. These tableaux are in bijection with the four 3-quotient border-strip tableaux of shape λ/μ and type $\alpha = (2, 1, 1, 1)$ shown in Figure 3; since $\operatorname{sgn}_3(\lambda/\mu) = 1$, the bijection is sign preserving.

The 3-quotient of λ/μ is $\big((1, 1, 1), (3, 1)/(1, 1), \varnothing\big)$, so the characters of S_3 and S_2 we must consider are the sign character and the trivial character, respectively. Taking $g_{(2,1,1,1)} = (12) \in S_5$, and following the end of the

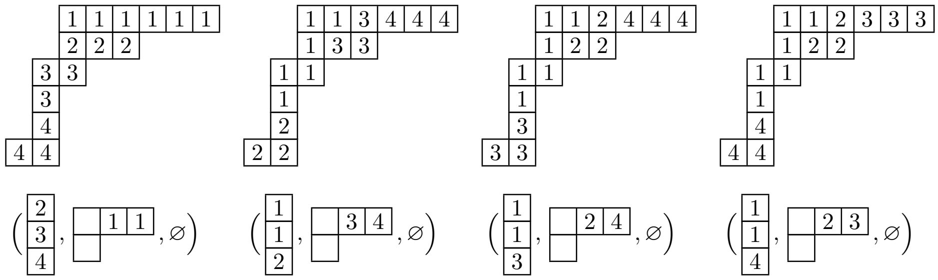


FIGURE 3. The bijection in Proposition 4.2 between border-strip tableaux of shape $(8,5,3,2,2,2)/(2,2,1,1,1)$ and type $(6,3,3,3)$ and 3-quotient border-strip tableaux of the same shape and type $(2,1,1,1)$. The shapes of the border-strip tableaux forming each 3-quotient border-strip tableau are given by the 3-quotient of $(8,5,3,2,2,2)/(2,2,1,1,1)$, namely $\big((1,1,1),(3,1)/(1,1),\varnothing\big)$. To make clear the skew-shape, the tableaux of shape $(3,1)/(1,1)$ are drawn as $(3,1)$-tableaux with two empty boxes.

proof of Theorem 2.1, we see that there are four $(3,2)$-tabloids fixed by (12), namely

$$\begin{array}{c}\hline 3\ 4\ 5 \\ \hline 1\ 2 \\ \hline\end{array}, \quad \begin{array}{c}\hline 1\ 2\ 3 \\ \hline 4\ 5 \\ \hline\end{array}, \quad \begin{array}{c}\hline 1\ 2\ 4 \\ \hline 3\ 5 \\ \hline\end{array}, \quad \begin{array}{c}\hline 1\ 2\ 5 \\ \hline 3\ 4 \\ \hline\end{array},$$

in the order corresponding to the tableaux shown in Figure 3. The corresponding values of the function f used in the proof of Theorem 2.1 are $+1$, -1, -1, -1 respectively. It should be noted that in general there will be several tableaux corresponding to each product of character values

$$\chi^{\lambda^{(0)}/\mu^{(0)}}(g_{\alpha_0(\mathbf{t})})\ldots\chi^{\lambda^{(n-1)}/\mu^{(n-1)}}(g_{\alpha_{n-1}(\mathbf{t})});$$

it is a special feature of this example that each $f(\mathbf{t})$ has a single summand, and so the bijection extends all the way to tabloids.

5. PROOF OF THEOREM 1.1

We shall prove Theorem 1.1 by induction on the number of parts of γ. Most of the work occurs in proving the base case when γ has a single part. In the first step we combine the results of §3 and §4. For later use in §6, we state the following proposition for a general deflation map.

Proposition 5.1. *Let m, $n \in \mathbf{N}$, let ϑ be an irreducible character of S_m, and let λ/μ be a skew-partition of mn. Let $g \in S_n$ be an n-cycle. If λ/μ is not n-decomposable, then* $\mathrm{Defres}^{\vartheta}_{S_n}(g) = 0$. *If λ/μ is n-decomposable, then*

$$(\mathrm{Defres}^{\vartheta}_{S_n}\chi^{\lambda/\mu})(g) = \varepsilon_n(\lambda/\mu)\left\langle \mathrm{Ind}_H^{S_m}\big(\chi^{\lambda^{(0)}/\mu^{(0)}}\times\cdots\times\chi^{\lambda^{(n-1)}/\mu^{(n-1)}}\big),\vartheta\right\rangle.$$

where $H = S_{|\lambda^{(0)}/\mu^{(0)}|}\times\cdots\times S_{|\lambda^{(n-1)}/\mu^{(n-1)}|}$.

Proof. If λ/μ is not n-decomposable then, by Theorem 2.1, $\chi^{\lambda/\mu}(g_{n\alpha}) = 0$ for all partitions α of m. Hence, by Proposition 3.5(ii), $(\mathrm{Defres}^{\vartheta}_{S_n} \chi^{\lambda/\mu})(g) = 0$. If λ/μ is n-decomposable then, using Proposition 3.5(i), we may restate Theorem 2.1 as

$$\omega(\mathrm{Res}_{S_m \wr S_n} \chi^{\lambda/\mu}) = \varepsilon_n(\lambda/\mu)\,\mathrm{Ind}_H^{S_m}\big(\chi^{\lambda^{(0)}/\mu^{(0)}} \times \cdots \times \chi^{\lambda^{(n-1)}/\mu^{(n-1)}}\big). \tag{8}$$

The result now follows from Proposition 3.5(ii). □

In the case where $\mu = \varnothing$, Proposition 5.1 combined with the Littlewood–Richardson rule (see [4, 2.8.13] or [8, A1.3.3]) yields the following corollary.

Corollary 5.2. *Let m, $n \in \mathbf{N}$, let λ be a partition of mn. Let κ be a partition of m and let $\vartheta = \chi^{\kappa}$. Let $g \in S_n$ be an n-cycle. If $(\mathrm{Defres}^{\vartheta}_{S_n} \chi^{\lambda})(g) \neq 0$ then λ has empty n-core and moreover, $\lambda^{(i)} \subseteq \kappa$ for each $i \in \{0, \ldots, n-1\}$, where $(\lambda^{(0)}, \ldots, \lambda^{(n-1)})$ is the n-quotient of λ. In this case*

$$(\mathrm{Defres}^{\vartheta}_{S_n} \chi^{\lambda})(g) = \varepsilon_n(\lambda/\varnothing) c^{\kappa}_{\lambda^{(0)} \ldots \lambda^{(n-1)}}$$

where $c^{\kappa}_{\lambda^{(0)} \ldots \lambda^{(n-1)}}$ denotes a generalized Littlewood–Richardson coefficient.

We note that there is no immediate analogue of Corollary 5.2 for skew-partitions because the Littlewood–Richardson rule only applies directly to characters labelled by partitions. However, it is possible, using [8, Equation (7.64)] to rewrite a character $\chi^{\lambda/\mu}$ as an integral linear combination of characters χ^{ν}, where the coefficients are obtained by the Littlewood–Richardson rule; then Corollary 5.2 may be applied.

It is clear from the condition on row numbers in Equation (1) in Definition 1.4 that if λ/μ is a skew-partition of mn then there is at most one m-border-strip tableau of shape λ/μ and type (n). The following lemma gives a more precise condition. Recall that a skew-partition σ/τ is said to be a *horizontal strip* if the Young diagram of σ/τ has no two boxes in the same column.

Lemma 5.3. *Let m, $n \in \mathbf{N}$ and let λ/μ be a skew-partition of mn. If λ/μ is n-decomposable, and each $\lambda^{(i)}/\mu^{(i)}$ is a horizontal strip, then there is a unique m-border-strip tableau of shape λ/μ and type (n); this tableau has sign $\varepsilon_n(\lambda/\mu)$. Otherwise there are no such tableaux.*

Proof. Suppose that T is an m-border-strip tableau of shape λ/μ and type (n). Then by Definition 4.1, λ/μ is n-decomposable. Let $\Gamma(\lambda)$ be an n-runner abacus display for λ using tn beads for some $t \in \mathbf{N}$. Let $\Gamma(\mu)$ be the abacus display for μ obtained from $\Gamma(\lambda)$ by an appropriate sequence of m single bead moves, so that in each move a bead is slid upwards into a gap immediately above it. We label the positions on the abacus from top to bottom so that the positions in row r of an abacus display are numbered $(r-1)n, \ldots, rn-1$ (as usual). Observe that the row number of a border strip of length n in T corresponding to a bead in position p of $\Gamma(\lambda)$ is the number of beads in positions $p+1$, $p+2$, $\ldots$ of $\Gamma(\lambda)$. Therefore if $p < p'$ and the beads in positions p and p' of $\Gamma(\lambda)$ both correspond to border strips in T, then in the sequence of bead moves corresponding to T, the bead in position p' is moved upwards before the bead in position p.

Let $i \in \{0, \ldots, n-1\}$. Suppose that $\Gamma(\lambda)$ has beads in positions $nq+i$ and $nq'+i$ where $q < q'$, and that $\Gamma(\mu)$ has no beads in positions $n(q+1)+i$, $\ldots$, $nq'+i$. The bead in position $nq+i$ of $\Gamma(\lambda)$ prevents the bead initially in position $nq'+i$ from reaching its final position in $\Gamma(\mu)$. Therefore the bead in position $nq+i$ must be moved before the bead in position $nq'+i$ reaches its final position. This contradicts the previous paragraph. Hence there exist $x_j \in \mathbf{N}_0$ and $y_j \in \mathbf{N}_0$ such that the beads on runner i of $\Gamma(\lambda)$ are in positions $\{i + nx_j : 1 \leq j \leq s\}$, the beads on runner i of $\Gamma(\mu)$ are in positions $\{i + ny_j : 1 \leq j \leq s\}$ and

$$y_1 \leq x_1 < y_2 \leq x_2 < \cdots < y_s \leq x_s. \tag{9}$$

It easily follows that $\lambda^{(i)}/\mu^{(i)}$ is a horizontal strip. Then, by Proposition 4.2, $\mathrm{sgn}(T) = \varepsilon_n(\lambda/\mu)$.

Conversely, suppose that λ/μ is n-decomposable and each $\lambda^{(i)}/\mu^{(i)}$ is a horizontal strip. Then the inequality (9) on the bead positions in each runner holds. We now describe a sequence of m single upward bead moves that transforms $\Gamma(\lambda)$ into $\Gamma(\mu)$, and thus corresponds to a border-strip tableau T of shape λ/μ and type $(n)^{\star m}$. At each step, locate the bead with maximal position p such that there is no bead in position p of $\Gamma(\mu)$. Slide that bead up into position $p-n$. This is possible by inequality (9). The row numbers of the border strips of length n corresponding to this sequence of moves are increasing, so T is a m-border-strip tableau of shape λ/μ and type (n). Uniqueness is clear since there is always at most one m-border strip tableau of shape λ/μ and type (n). □

We can now complete the proof of the base case.

Proposition 5.4. *Let m, $n \in \mathbf{N}$ and let $g \in S_n$ be an n-cycle. If λ/μ is a skew-partition of mn then*

$$(\mathrm{Defres}_{S_n} \chi^{\lambda/\mu})(g) = a_{\lambda/\mu,(n)}.$$

Proof. If λ/μ is not n-decomposable then $a_{\lambda/\mu,(n)} = 0$ by Lemma 5.3. Proposition 5.1 implies that the result holds in this case.

Now suppose that λ/μ is n-decomposable. Let $\ell_i = |\lambda^{(i)}/\mu^{(i)}|$ for each $i \in \{0, \ldots, n-1\}$. By Proposition 5.1 and Frobenius reciprocity, we have

$$(\mathrm{Defres}_{S_n} \chi^{\lambda/\mu})(g) = \varepsilon_n(\lambda/\mu) \left\langle \chi^{\lambda^{(0)}/\mu^{(0)}}, 1_{S_{\ell_0}} \right\rangle \cdots \left\langle \chi^{\lambda^{(n-1)}/\mu^{(n-1)}}, 1_{S_{\ell_{n-1}}} \right\rangle.$$

It follows from [4, Theorem 2.3.13(ii)] that if σ/τ is a skew-partition of ℓ then

$$\left\langle \chi^{\sigma/\tau}, 1_{S_\ell} \right\rangle = \begin{cases} 1 & \text{if } \sigma/\tau \text{ is a horizontal strip} \\ 0 & \text{otherwise.} \end{cases}$$

(This result is also equivalent to Pieri's rule, as stated in [8, 7.15.9].) Therefore $(\mathrm{Defres}_{S_n} \chi^{\lambda/\mu})(g) = \varepsilon_n(\lambda/\mu)$ if each $\lambda^{(i)}/\mu^{(i)}$ is a horizontal strip, and otherwise $(\mathrm{Defres}_{S_n} \chi^{\lambda/\mu})(g) = 0$. The proposition now follows from Lemma 5.3. □

For the inductive step we need the following lemma and proposition. For later use we state and prove Proposition 5.6 for a general deflation map.

Lemma 5.5. *Let λ/μ be a skew-partition of r. If $1 \leq c < r$ then*

$$\mathrm{Res}_{S_c \times S_{r-c}} \chi^{\lambda/\mu} = \sum_{\tau} \chi^{\tau/\mu} \times \chi^{\lambda/\tau}$$

where the sum is over all partitions τ such that $\mu \subseteq \tau \subseteq \lambda$ and $|\tau/\mu| = c$.

Proof. Let $|\lambda| = s$. It follows from [4, 2.3.12] that

$$\mathrm{Res}_{S_{s-r} \times S_r} \chi^{\lambda} = \sum_{\mu} \chi^{\mu} \times \chi^{\lambda/\mu}$$

where the sum is over all partitions μ of $s-r$ such that $\mu \subseteq \lambda$. So

$$\mathrm{Res}_{S_{s-r} \times S_{r-c} \times S_c} \chi^{\lambda} = \sum_{\mu} \chi^{\mu} \times \mathrm{Res}_{S_{r-c} \times S_c} \chi^{\lambda/\mu}$$

with the summation as above. But by the same result we have

$$\begin{aligned} \mathrm{Res}_{S_{s-r} \times S_{r-c} \times S_c} \chi^{\lambda} &= \sum_{\tau} \mathrm{Res}_{S_{s-r} \times S_{r-c}} \chi^{\tau} \times \chi^{\lambda/\tau} \\ &= \sum_{\tau} \Big(\sum_{\mu} \chi^{\mu} \times \chi^{\tau/\mu} \Big) \times \chi^{\lambda/\tau} \\ &= \sum_{\mu} \chi^{\mu} \times \Big(\sum_{\tau} \chi^{\tau/\mu} \times \chi^{\lambda/\tau} \Big) \end{aligned}$$

where the sums are over all partitions τ of $s-c$ and μ of $s-r$ such that $\mu \subseteq \tau \subseteq \lambda$. The lemma follows on comparing the summands for χ^{μ}. □

Proposition 5.6. *Let m, $n \in \mathbf{N}$ and let λ/μ be a skew-partition of mn. Let ϑ be an irreducible character of S_m. Let $g \in S_n$. If $g = kh$ where $k \in S_\ell$ and $h \in S_{n-\ell}$, then*

$$(\mathrm{Defres}^{\vartheta}_{S_n} \chi^{\lambda/\mu})(g) = \sum_{\tau} (\mathrm{Defres}^{\vartheta}_{S_\ell} \chi^{\tau/\mu})(k)(\mathrm{Defres}^{\vartheta}_{S_{n-\ell}} \chi^{\lambda/\tau})(h)$$

where the sum is over all partitions τ such that $\mu \subseteq \tau \subseteq \lambda$ and $|\tau/\mu| = m\ell$.

Proof. Let B be the base group of the wreath product $S_m \wr S_n \leq S_{mn}$. Choose a subgroup $S_{m\ell} \times S_{m(n-\ell)} \leq S_{mn}$ containing B. If ψ is a character of $S_m \wr S_n$ then it is easily checked that

$$\mathrm{Res}_{S_\ell \times S_{n-\ell}}(\mathrm{Def}^{\vartheta}_{S_n} \psi) = (\mathrm{Def}^{\vartheta}_{S_\ell} \times \mathrm{Def}^{\vartheta}_{S_{n-\ell}})(\mathrm{Res}^{S_m \wr S_n}_{S_m \wr S_\ell \times S_m \wr S_{n-\ell}} \psi).$$

Hence

$$\mathrm{Res}_{S_\ell \times S_{n-\ell}}(\mathrm{Defres}^{\vartheta}_{S_n} \chi) = (\mathrm{Defres}^{\vartheta}_{S_\ell} \times \mathrm{Defres}^{\vartheta}_{S_{n-\ell}})(\mathrm{Res}^{S_{mn}}_{S_{m\ell} \times S_{m(n-\ell)}} \chi)$$

for any character χ of S_{mn}. The proposition now follows from the expression for $\mathrm{Res}_{S_{m\ell} \times S_{m(n-\ell)}} \chi^{\lambda/\mu}$ given in Lemma 5.5. □

We are now ready to prove Theorem 1.1. Let m, $n \in \mathbf{N}$ and let λ/μ be a skew-partition of mn. Let $\gamma = (\gamma_1, \ldots, \gamma_d)$ be a composition of n. Let $g \in S_n$ have cycle type γ and let $h \in S_{n-\gamma_1}$ have cycle type $(\gamma_2, \ldots, \gamma_d)$. Note that, by Lemma 5.3, if τ/μ is a skew-partition of $m\gamma_1$ then there is at most one m-border-strip tableau of shape τ/μ and type (γ_1). We shall denote this tableau by $T_{\tau/\mu}$ when it exists. By Definition 1.4, $a_{\tau/\mu,(\gamma_1)} = \mathrm{sgn}(T_{\tau/\mu})$ (or is zero if no such tableau exists). Therefore Proposition 5.4 may be restated as $(\mathrm{Defres}_{S_{\gamma_1}} \chi^{\tau/\mu})(k) = \mathrm{sgn}(T_{\tau/\mu})$, where $k \in S_{\gamma_1}$ is a γ_1-cycle and $(\mathrm{Defres}_{S_{\gamma_1}} \chi^{\tau/\mu})(k) = 0$ if no tableau $T_{\tau/\mu}$ exists. It therefore follows from Proposition 5.6 that

$$(\mathrm{Defres}_{S_n} \chi^{\lambda/\mu})(g) = \sum_{\tau} \mathrm{sgn}(T_{\tau/\mu})(\mathrm{Defres}_{S_{n-\gamma_1}} \chi^{\lambda/\tau})(h) \tag{10}$$

where the sum is over all partitions τ such that $\mu \subseteq \tau \subseteq \lambda$, $|\tau/\mu| = m\gamma_1$ and there is an m-border-strip tableau of shape τ/μ. By induction on the number of parts of γ we have

$$(\mathrm{Defres}_{S_n} \chi^{\lambda/\mu})(g) = \sum_{\tau} \mathrm{sgn}(T_{\tau/\mu}) a_{\lambda/\tau,(\gamma_2,\ldots,\gamma_d)}$$

with the same conditions on the sum. It is clear that if T is an m-border-strip tableau of shape λ/μ and type γ then the border strips in T corresponding to the m parts of length γ_1 in $\gamma^{\star m}$ form an m-border-strip tableau of shape τ/μ for some τ. Therefore the right-hand side of the previous equation is $a_{\lambda/\mu,\gamma}$. This completes the proof of Theorem 1.1.

6. Generalized deflations

Using Theorem 1.1 it is possible to prove an analogous result for deflations with respect to the signed character. As is usual, if λ is a partition then we denote by λ' the conjugate partition to λ.

Proposition 6.1. *Let m, $n \in \mathbf{N}$ and let λ/μ be a skew-partition of mn. If γ is a composition of n and $g \in S_n$ has cycle type γ then*

$$(\mathrm{Defres}_{S_n}^{\mathrm{sgn}_{S_m}})(g) = \begin{cases} a_{\lambda'/\mu',\gamma} & \text{if } m \text{ is even} \\ \mathrm{sgn}_{S_n}(g)\, a_{\lambda'/\mu',\gamma} & \text{if } m \text{ is odd.} \end{cases}$$

Proof. It is easily seen that

$$\mathrm{Res}_{S_m \wr S_n} \mathrm{sgn}_{S_{mn}} = \begin{cases} \widetilde{\mathrm{sgn}_{S_m}^{\times n}} & \text{if } m \text{ is even} \\ \widetilde{\mathrm{sgn}_{S_m}^{\times n}}\, \mathrm{sgn}_{S_n} & \text{if } m \text{ is odd.} \end{cases}$$

Hence if χ is any character of S_{mn} then

$$\mathrm{Defres}_{S_n}^{\mathrm{sgn}_{S_m}} \chi = \eta\, \mathrm{Defres}_{S_n}(\chi\, \mathrm{sgn}_{S_{mn}})$$

where $\eta = 1_{S_n}$ if m is even and $\eta = \mathrm{sgn}_{S_n}$ if m is odd. It follows from Theorems 7.15.6 and 7.17.5 in [8] that if λ/μ is a skew-partition of mn then $\chi^{\lambda/\mu}\, \mathrm{sgn}_{S_{mn}} = \chi^{\lambda'/\mu'}$. Therefore, by Theorem 1.1, we have

$$\mathrm{Defres}_{S_n}^{\mathrm{sgn}_{S_m}}(\chi^{\lambda'/\mu'}) = \eta(g) a_{\lambda'/\mu',\gamma},$$

as required. □

It would also have been possible to prove Proposition 6.1 directly from Proposition 5.1, by reasoning along the same lines as the proof of Theorem 1.1 in §5.

In Corollary 5.2, Proposition 5.1 was used together with the Littlewood–Richardson rule to describe the value on an n-cycle of an arbitrary deflation to S_n of an irreducible character. Proposition 6.2 below gives the degrees of the deflations of the irreducible characters of S_{mn} to S_n. It may be proved in the same way as Equation (2) in §1. Note that the right-hand side equals the generalized Littlewood–Richardson coefficient $c^{\lambda}_{\kappa \ldots \kappa}$.

Proposition 6.2. *Let m, $n \in \mathbf{N}$. Let κ be a partition of m and let $\vartheta = \chi^\kappa$. Then*

$$\mathrm{Defres}_{S_n}^{\vartheta}(\chi^\lambda)(1_{S_n}) = \left\langle \mathrm{Ind}_{S_m \times \cdots \times S_m}^{S_{mn}} \chi^\kappa \times \cdots \times \chi^\kappa, \chi^\lambda \right\rangle$$

for any partition λ of mn. □

We end by showing how Propositions 5.1, 5.6, and 6.2 and Corollary 5.2 may be used to calculate the values of an irreducible character of S_{mn} deflated with respect to an arbitrary character of S_m.

Example 6.3. Let $\vartheta = \chi^{(2,1)}$. We shall find $(\mathrm{Defres}^{\vartheta}_{S_4}\,\chi^{(6,4,2)})(g)$ in the cases where $g \in S_4$ is a transposition or a double transposition.

Firstly take g to be a transposition. By Proposition 5.6 we have

$$(\mathrm{Defres}^{\vartheta}_{S_4}\chi^{(6,4,2)})(g) = \sum_{\tau} (\mathrm{Defres}^{\vartheta}_{S_2}\,\chi^{\tau})(1_{S_2})(\mathrm{Defres}^{\vartheta}_{S_2}\,\chi^{(6,4,2)/\tau})(k)$$

where $k = (12) \in S_2$ and the sum is over all partitions τ of 6. Using Proposition 6.2 on the first term we obtain

$$(\mathrm{Defres}^{\vartheta}_{S_4}\chi^{(6,4,2)})(g) = \sum_{\tau} c^{\tau}_{(2,1)(2,1)}(\mathrm{Defres}^{\vartheta}_{S_2}\,\chi^{(6,4,2)/\tau})(k)$$

with the same conditions on the sum. By Proposition 5.1, we need only consider those partitions τ such that $(6,4,2)/\tau$ is 2-decomposable. The 2-quotient of $(6,4,2)$ is $\big((2),(3,1)\big)$, so the τ we must consider have 2-quotient $\big(\varnothing,(3)\big)$, $\big((1),(2)\big)$, $\big(\varnothing,(2,1)\big)$, $\big((2),(1)\big)$, $\big((1),(1,1)\big)$; hence, the partitions τ are (6), $(4,2)$, $(4,1,1)$, $(3,3)$, $(2,2,2)$. Calculation shows that $c^{(6)}_{(2,1)(2,1)} = 0$, and $c^{\tau}_{(2,1)(2,1)} = 1$ in the other four cases. So by Proposition 5.1 we have

$$(\mathrm{Defres}^{\vartheta}_{S_4}\,\chi^{(6,4,2)})(g) = \sum_{\tau \in P} \varepsilon_2\big((6,4,2)/\tau\big)\left\langle \mathrm{Ind}^{S_3}_{H}\,\chi^{(2)/\tau^{(0)}} \times \chi^{(3,1)/\tau^{(1)}}, \chi^{(2,1)}\right\rangle$$

where $P = \{(4,2),(4,1,1),(3,3),(2,2,2)\}$ and $H = S_{|(2)/\tau^{(0)}|} \times S_{|(3,1)/\tau^{(1)}|}$. The contributions to the sum from the elements of P, in the order given above, are $+2$, -1, -1 and $+1$ respectively. For example, the 2-quotient of $(6,4,2)/(4,2)$ is $\big((2)/(1),(3,1)/(2)\big)$ and $\varepsilon_2\big((6,4,2)/(4,2)\big) = 1$, so the contribution from $(4,2)$ is

$$\left\langle \mathrm{Ind}^{S_3}_{S_1\times S_2}\,\chi^{(2)/(1)} \times \chi^{(3,1)/(2)}, \chi^{(2,1)}\right\rangle = \left\langle \mathrm{Ind}^{S_3}_{1}\,1, \chi^{(2,1)}\right\rangle = \chi^{(2,1)}(1_{S_3}) = 2.$$

Therefore $(\mathrm{Defres}^{\vartheta}_{S_4}\,\chi^{(6,4,2)})(g) = 1$.

Similar arguments can be used in the case where g is a double transposition. By Proposition 5.6 we have

$$(\mathrm{Defres}^{\vartheta}_{S_4}\chi^{(6,4,2)})(g) = \sum_{\tau} (\mathrm{Defres}^{\vartheta}_{S_2}\,\chi^{\tau})(k)(\mathrm{Defres}^{\vartheta}_{S_2}\,\chi^{(6,4,2)/\tau})(h)$$

where k and h are transpositions, and the sum is over all partitions τ of 6. By Proposition 5.1, we need only consider those partitions τ such that $(6,4,2)/\tau$ is 2-decomposable, namely (6), $(4,2)$, $(4,1,1)$, $(3,3)$, $(2,2,2)$. By Corollary 5.2, we may rule out (6), since its 2-quotient is $\big(\varnothing,(3)\big)$, and we see that $(\mathrm{Defres}^{\vartheta}_{S_4}\,\chi^{(6,4,2)})(g)$ equals

$$\sum_{\tau\in P} \varepsilon_2\big(\tau/\varnothing\big)c^{(2,1)}_{\tau^{(0)}\tau^{(1)}}\varepsilon_2\big((6,4,2)/\tau\big)\left\langle \mathrm{Ind}^{S_3}_{H}\,\chi^{(2)/\tau^{(0)}} \times \chi^{(3,1)/\tau^{(1)}}, \chi^{(2,1)}\right\rangle$$

where $P = \{(4,2),(4,1,1),(3,3),(2,2,2)\}$ and $H = S_{|(2)/\tau^{(0)}|} \times S_{|(3,1)/\tau^{(1)}|}$. Noting that $\varepsilon_2\big(\tau/\varnothing\big)\varepsilon_2\big((6,4,2)/\tau\big) = \varepsilon_2\big((6,4,2)/\varnothing\big) = +1$, the contributions to the sum from the elements of P, in the order given above, are $+2$, $+1$, $+1$ and $+1$ respectively. Therefore $(\mathrm{Defres}^{\vartheta}_{S_4}\chi^{(6,4,2)})(g) = 5$.

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