CHARACTER DEFLATIONS AND A GENERALIZATION OF THE MURNAGHAN–NAKAYAMA RULE

ANTON EVSEEV, ROWENA PAGET AND MARK WILDON

ABSTRACT. Given natural numbers m and n, we define a deflation map from the characters of the symmetric group S_{mn} to the characters of S_n. This map is obtained by first restricting a character of S_{mn} to the wreath product $S_m \wr S_n$, and then taking the sum of the irreducible constituents of the restricted character on which the base group $S_m \times \cdots \times S_m$ acts trivially. We prove a combinatorial formula which gives the values of the images of the irreducible characters of S_{mn} under this map. We also prove an analogous result for more general deflation maps in which the base group is not required to act trivially. These results generalize the Murnaghan–Nakayama rule and special cases of the Littlewood–Richardson rule. As a corollary we obtain a new combinatorial formula for the character multiplicities that are the subject of the long-standing Foulkes' Conjecture. Using this formula we verify Foulkes' Conjecture in some new cases.

1. INTRODUCTION

Tableaux combinatorics is a pivotal theme in the representation theory of the symmetric groups. Fundamental results in this area include the Murnaghan–Nakayama rule for the values taken by irreducible characters of symmetric groups and the Littlewood–Richardson rule (as well as its special case, Young's rule), which determines the restrictions of irreducible characters to Young subgroups of symmetric groups.

The two main results of this paper are Theorems 1.5 and 6.3, which give a combinatorial description of the restrictions of characters of symmetric groups to their maximal imprimitive subgroups. Theorem 1.5 is a simultaneous generalization of the Murnaghan–Nakayama rule and Young's rule. Theorem 6.3 gives a further generalization, in which Young's rule is replaced by a family of special cases of the Littlewood–Richardson rule.

As a corollary of Theorem 1.5, we obtain in Proposition 5.1 a new recursive formula for the character multiplicities that are the subject of Foulkes'

Conjecture, a long-standing problem which spans representation theory, invariant theory and algebraic combinatorics. We use this formula to verify Foulkes' Conjecture in some new cases, extending the results in [14]. Figures 3 and 4 in §5 show some of the data computed using this formula.

1.1. **Character deflations.** We now introduce the ideas needed to state Theorem 1.5. By a construction originally due to Frobenius, the irreducible characters of the symmetric group S_r are canonically labelled by the partitions of r. As is usual, we write χ^λ for the irreducible character labelled by the partition λ, and $\chi^{\lambda/\mu}$ for the character labelled by the skew-partition λ/μ. We refer the reader to [8, Chapter 2] or [19, §7.18] for a construction of these characters and to [19, page 309] for background on skew-partitions.

For each $r \in \mathbb{N}$, it is well known (see, for example, [4, Exercise 5.2.8]) that the maximal imprimitive subgroups of S_r are precisely the imprimitive wreath products $S_m \wr S_n \le S_r$ for $m, n \in \mathbb{N}$ such that $mn = r$. Let ϑ be a character of S_m, and let V be a representation of S_m affording ϑ. Then $V^{\otimes n}$ is a representation of the base group $S_m \times \cdots \times S_m$. The complement S_n of this base group acts on $V^{\otimes n}$ by permuting the factors:

$$g(v_1 \otimes \cdots \otimes v_n) = v_{g^{-1}(1)} \otimes \cdots \otimes v_{g^{-1}(n)}$$

for $g \in S_n$ and $v_1, \ldots, v_n \in V$. These two actions combine to give a representation of $S_m \wr S_n$ on $V^{\otimes n}$ (see [8, 4.3.6]). We shall denote by $\widetilde{\vartheta^{\times n}}$ the character of $S_m \wr S_n$ afforded by this representation. We also need the characters of $S_m \wr S_n$ whose kernel contains $S_m \times \cdots \times S_m$. These characters are precisely the inflations of the characters of S_n to $S_m \wr S_n$ along the canonical surjection $S_m \wr S_n \twoheadrightarrow S_n$. If ν is a partition of n, we denote by $\mathrm{Inf}_{S_n}^{S_m \wr S_n} \chi^\nu$ the irreducible character of $S_m \wr S_n$ constructed in this way. It is easily seen that the characters $\widetilde{\vartheta^{\times n}} \, \mathrm{Inf}_{S_n}^{S_m \wr S_n} \chi^\nu$ obtained by multiplying characters of these two types are irreducible. (By [8, Theorem 4.3.33], any irreducible character of $S_m \wr S_n$ is induced from a suitable product of characters of this form. We note that this result will not be used in this paper.)

Given a finite group G, we let $\mathcal{C}(G)$ denote the abelian group of virtual characters of G.

Definition 1.1. Let $m, n \in \mathbb{N}$ and let ϑ be an irreducible character of S_m. Let ξ be an irreducible character of $S_m \wr S_n$. We define

$$\mathrm{Def}_{S_n}^\vartheta \xi = \begin{cases} \chi^\nu & \text{if } \xi = \widetilde{\vartheta^{\times n}} \, \mathrm{Inf}_{S_n}^{S_m \wr S_n} \chi^\nu \text{ where } \nu \text{ is a partition of } n \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathrm{Def}_{S_n}^\vartheta : \mathcal{C}(S_m \wr S_n) \to \mathcal{C}(S_n)$ be the group homomorphism defined by linear extension of this definition. Given $\psi \in \mathcal{C}(S_m \wr S_n)$, we say that $\mathrm{Def}_{S_n}^\vartheta \psi$

is the *deflation of ψ with respect to ϑ*. Let $\mathrm{Defres}_{S_n}^{\vartheta} : \mathcal{C}(S_{mn}) \to \mathcal{C}(S_n)$ be the group homomorphism defined by

$$\mathrm{Defres}_{S_n}^{\vartheta} \chi = \mathrm{Def}_{S_n}^{\vartheta} \mathrm{Res}_{S_m \wr S_n}^{S_{mn}} \chi$$

for $\chi \in \mathcal{C}(S_{mn})$.

In the case when ϑ is the trivial character of S_m, we shall omit ϑ and simply write Def_{S_n} and Defres_{S_n}. If V is a complex representation of $S_m \wr S_n$ with character χ, then $\mathrm{Def}_{S_n} \chi$ is the character of the maximal subrepresentation of V on which the base group $S_m \times \cdots \times S_m$ acts trivially.

Theorem 1.5 gives a combinatorial rule for the values of $\mathrm{Defres}_{S_n} \chi^{\lambda/\mu}$ where λ/μ is a skew-partition of mn. In order to state this rule, we review and extend the definition of a border-strip tableau (see [19, §7.17]).

Recall that a skew-partition σ/τ is said to be a *border strip* (or rim hook) if the Young diagram of σ/τ is connected and contains no 2×2 square. The *length* of the border strip σ/τ is $|\sigma/\tau|$ and its *height* is defined to be one less than its number of non-empty rows. If λ/μ is a skew-partition, then we define a *border-strip tableau* of shape λ/μ to be an assignment of the elements of a set $J \subseteq \mathbb{N}$ to the boxes of the Young diagram of λ/μ so that the rows and columns are non-decreasing, and for each $j \in J$, the boxes labelled j form a border strip; if $J = \{1, \ldots, k\}$, and for each $j \in J$ the border strip formed by the boxes labelled j has length α_j, then we say that the tableau has *type* $(\alpha_1, \ldots, \alpha_k)$. We need the following three further definitions, which are illustrated in Figure 1 and Example 1.6 below.

Definition 1.2. Let T be a border-strip tableau. The *sign* of T is defined by $\mathrm{sgn}(T) = (-1)^h$, where h is the sum of the heights of the border strips forming T.
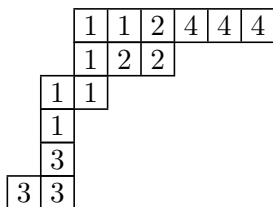

FIGURE 1. A border-strip tableau of shape $(8, 5, 3, 2, 2, 2)/(2, 2, 1, 1, 1)$ and type $(6, 3, 3, 3)$. The heights of the border strips labelled $1, 2, 3, 4$ are $3, 1, 1, 0$ respectively, and the sign of this border-strip tableau is thus -1.

Definition 1.3. Let λ/τ be a border strip in a partition λ. If the lowest-numbered row of λ met by λ/τ is row k then we define the *row number* of λ/τ to be k, and write $N(\lambda/\tau) = k$.

Note that if T is a border-strip tableau of shape λ/μ and type $(\alpha_1, \ldots, \alpha_k)$ then there are partitions

$$\mu = \lambda^0 \subset \lambda^1 \subset \cdots \subset \lambda^{k-1} \subset \lambda^k = \lambda$$

such that for each $j \in \{1, \ldots, k\}$, the border strip in T labelled j is λ^j/λ^{j-1}.

Definition 1.4. Let $m, n \in \mathbb{N}$ and let λ/μ be a skew-partition of mn. Given a composition $\gamma = (\gamma_1, \ldots, \gamma_d)$ of n, let $\gamma^{\star m} = (\gamma_1, \ldots, \gamma_1, \ldots, \gamma_d, \ldots, \gamma_d)$ denote the composition of mn obtained from γ by repeating each part m times. An *m-border-strip tableau* of *shape* λ/μ and *type* γ is a border-strip tableau of shape λ/μ and type $\gamma^{\star m}$ such that for each $j \in \{1, 2, \ldots, d\}$, the row numbers of the border strips

$$\lambda^{(j-1)m+1}/\lambda^{(j-1)m}, \ldots, \lambda^{jm}/\lambda^{jm-1}$$

corresponding to the m parts in $\gamma^{\star m}$ of length γ_j satisfy

$$(1) \qquad N(\lambda^{(j-1)m+1}/\lambda^{jm}) \geq \cdots \geq N(\lambda^{jm}/\lambda^{jm-1}).$$

Let

$$a_{\lambda/\mu,\gamma} = \sum_T \operatorname{sgn}(T)$$

where the sum is over all m-border-strip tableaux T of shape λ/μ and type γ.

Theorem 1.5. *Let* $m, n \in \mathbb{N}$ *and let* λ/μ *be a skew-partition of* mn. *If* γ *is a composition of* n *and* $g \in S_n$ *has cycle type* γ *then*

$$(\operatorname{Defres}_{S_n} \chi^{\lambda/\mu})(g) = a_{\lambda/\mu,\gamma}.$$

Example 1.6. Let $\lambda = (6, 5, 3, 2)$ and let $\mu = (3, 1)$. The three different 2-border-strip tableaux of shape λ/μ and type $\gamma = (1, 2, 3)$ are shown below.

As required by Definition 1.4, for each $j \in \{1, 2, 3\}$, the row number of the border strip labelled $2j - 1$ in each tableau is at least the row number of the border strip labelled $2j$. Thus the first border strip corresponding to each part of γ is added no higher up in each partition diagram than the second. The sums of the heights of the border strips forming these tableaux are 4, 4 and 3 and so their signs are $+1$, $+1$ and -1, respectively.

By Definition 1.4 we have $a_{\lambda/\mu,\gamma} = 1$. Hence Theorem 1.5 implies that $(\mathrm{Defres}_{S_6} \chi^{(6,5,3,2)/(3,1)})(g) = 1$ if $g \in S_6$ has cycle type $(1,2,3)$.

Deflation is closely related to plethysm of Schur functions (see [12, §I.8]). In fact, using the standard correspondence between characters of symmetric groups and symmetric functions, one can show that the special case $\gamma = (n)$ of Theorem 1.5 is equivalent to a result proved in [3, Section 9]. Also, in the special case $\mu = \varnothing$, the combinatorial description of Theorem 1.5 can be shown (using our Lemma 4.2 below) to be equivalent to the one given for plethysm by Macdonald [12, §I.8, Example 8]. These connections are discussed in more detail in §7.

We prove Theorem 1.5 in §§2–4. The only prerequisites, apart from some basic character theory, are the Murnaghan–Nakayama rule and the combinatorics of the abacus. In addition to being self-contained, our proof is highly combinatorial in the sense that the key steps, given in §3, can all be stated in terms of explicit bijections between certain classes of tableaux.

1.2. **Some special cases.** It is clear than if $m = 1$ then $\mathrm{Defres}_{S_n} \chi = \chi$ for any character χ of S_n, and so the special case $m = 1$ of Theorem 1.5 asserts that $\chi^{\lambda/\mu}(g) = a_{\lambda/\mu,\gamma}$ for any skew-partition λ/μ of n and any element $g \in S_n$ of cycle type γ. Equivalently,

$$\chi^{\lambda/\mu}(g) = \sum_T \mathrm{sgn}(T)$$

where the sum is over all border-strip tableaux of shape λ/μ and type γ. This is the Murnaghan–Nakayama rule, as stated in [19, Equation (7.75)]. It should be noted that we require the Murnaghan–Nakayama rule in §3.3 below, and so our work does not provide a new proof of this result. In practice the Murnaghan–Nakayama rule is most frequently used as a recursive formula for the values of characters or skew characters. Equation (10) at the end of §4 formulates Theorem 1.5 in this way.

As Stanley observes in [19, page 348], it is far from obvious that the character values given by the Murnaghan–Nakayama rule applied to a skew-partition λ/μ and a composition γ are independent of the order of the parts of γ. This remark applies even more strongly to Theorem 1.5. For example, the reader may check that if $\lambda/\mu = (6,5,3,2)/(3,1)$, as in Example 1.6, and $\gamma' = (2,1,3)$, then there is a unique 2-border-strip tableau of shape $(6,5,3,2)/(3,1)$ and type γ'. Thus $a_{\lambda/\mu,\gamma'} = 1$, but the sums defining $a_{\lambda/\mu,\gamma}$ and $a_{\lambda/\mu,\gamma'}$ are different.

Another special case of Theorem 1.5 worth noting occurs when g is the identity element of S_n. If ξ is an irreducible character of $S_m \wr S_n$ then either

the base group $B = S_m \times \cdots \times S_m$ is contained in the kernel of ξ and $\langle \mathrm{Res}_B\, \xi, 1_B \rangle = \xi(1)$, or $\langle \mathrm{Res}_B\, \xi, 1_B \rangle = 0$. Hence, by linearity, we have

$$(2) \qquad\qquad (\mathrm{Defres}_{S_n} \chi)(1) = \langle \mathrm{Res}_B\, \chi, 1_B \rangle$$

for any character χ of S_{mn}. It now follows from Theorem 1.5 and Frobenius reciprocity that

$$a_{\lambda/\mu,(1^n)} = \left\langle \chi^{\lambda/\mu}, \mathrm{Ind}_{S_m \times \cdots \times S_m}^{S_{mn}} 1_{S_m} \times \cdots \times 1_{S_m} \right\rangle$$

for any skew-partition λ/μ of mn. It is clear from Definition 1.4 that $a_{\lambda/\mu,(1^n)}$ is the number of semi-standard tableaux of shape λ/μ and type (m^n). Therefore, by setting $\mu = \varnothing$ in the previous equation, we obtain a special case of Young's rule (see [8, 2.8.5] or [19, Proposition 7.18.7]).

1.3. **Outline of the paper.** The remainder of the paper proceeds as follows. Throughout, we shall adopt the convention that if α is a partition of $r \in \mathbb{N}$, then $g_\alpha \in S_r$ is an element of cycle type α, and z_α is the size of the centralizer of g_α in S_r. (The choice of g_α within the conjugacy class is irrelevant.) If $\alpha = (\alpha_1, \ldots, \alpha_k)$, we write $n\alpha = (n\alpha_1, \ldots, n\alpha_k)$.

In §2 we prove Proposition 2.2, which implies that if χ is a character of S_{mn} and $g \in S_n$, then $(\mathrm{Defres}_{S_n} \chi)(g)$ is the average value of χ on the coset of the base group $S_m \times \cdots \times S_m$ in $S_m \wr S_n$ corresponding to g. Equation (2) above is a special case of this result. In the case when $g \in S_n$ is an n-cycle, we obtain Proposition 2.6(ii), which implies that if λ/μ is a skew-partition of mn, then

$$(3) \qquad\qquad \mathrm{Defres}_{S_n} \chi^{\lambda/\mu}(g) = \sum_\alpha \frac{\chi^{\lambda/\mu}(g_{n\alpha})}{z_\alpha}$$

where the sum is over all partitions α of m, and $n\alpha$ denotes the partition obtained from α by multiplying each of its parts by n.

In §3 we state a theorem of Farahat (see [5, Section 4]), which gives a formula for the character values $\chi^{\lambda/\mu}(g_{n\alpha})$ appearing on the right-hand side of (3). We then give a character-theoretic proof of this theorem.

In §4 we combine the results of §3 and §4 to show that Theorem 1.5 holds when $g \in S_n$ is an n-cycle (Proposition 4.3) and then to deduce it in general.

In §5 we apply our results on deflations to Foulkes' Conjecture on permutation characters of the symmetric group. In particular, we prove a new recursive formula for the character multiplicities that appear in this conjecture. Using this formula we check Foulkes' Conjecture in some new cases, extending the results in [14].

In §6, we consider the more general deflation maps $\mathrm{Def}_{S_n}^\vartheta$. When $\vartheta = \chi^{(a,1^b)}$ is labelled by a hook partition, Theorem 6.3 gives a combinatorial

description of the value of $\mathrm{Defres}^{\vartheta}_{S_n} \chi^{\lambda/\mu}$ on an n-cycle. This result general-izes the case $\gamma = (n)$ of Theorem 1.5 and may be viewed as a simultaneous generalization of the Murnaghan–Nakayama rule and a special case of the Littlewood–Richardson rule. We also give an illustrative example showing how our methods can be used to compute values of deflated characters in the non-hook case.

Finally, in §7, we discuss the aforementioned connections between Theorem 1.5 and results in [3, 12] stated in terms of symmetric functions.

2. Deflation by averaging

Let m, $n \in \mathbb{N}$. We shall think of $S_m \wr S_n$ as the group of permutations of

$$\{1, \ldots, m\} \times \{1, \ldots, n\}$$

that leaves invariant the set of blocks of the form $\Delta_j = \{(1, j), \ldots, (m, j)\}$, $1 \le j \le n$. Given $h_1, \ldots, h_n \in S_m$ and $g \in S_n$, we write $(h_1, \ldots, h_n; g)$ for the permutation which sends (i, j) to $(h_{gj} i, gj)$. This left action is equivalent to the action defined in [8, 4.1.18]. Let $B = S_m \times \cdots \times S_m$ denote the base group in the wreath product. As shorthand, if $k = (h_1, \ldots, h_n) \in B$ then we shall write $(k\,; g)$ for $(h_1, \ldots, h_n; g)$.

Lemma 2.1. *Let m, $n \in \mathbb{N}$, let ϑ be an irreducible character of S_m, and let ξ be an irreducible character of $S_m \wr S_n$. Let $g \in S_n$. If $\xi = \widetilde{\vartheta^{\times n}} \mathrm{Inf}^{S_m \wr S_n}_{S_n} \chi^\nu$ for some partition ν of n then*

$$\frac{1}{|B|} \sum_{k \in B} \xi(k\,;g) \overline{\widetilde{\vartheta^{\times n}}}(k\,;g) = \chi^\nu(g),$$

and if $\xi \in \mathrm{Irr}(S_m \wr S_n)$ is not of this form then the left-hand side is zero.

Proof. Suppose that the left-hand side is non-zero. The character $\widetilde{\vartheta^{\times n}}$ of $S_m \wr S_n$ restricts to the irreducible character $\vartheta \times \cdots \times \vartheta$ of B. Hence we may draw upon [7, Lemma 8.14]: suppose $N \triangleleft G$, χ is an irreducible character of G whose restriction to N is also irreducible and $\xi \in \mathrm{Irr}(G)$. Under these conditions [7, Lemma 8.14(b)] states that if $\sum_{x \in Ng} \xi(x)\overline{\chi(x)} \ne 0$, for some $g \in G$, then $\mathrm{Res}^G_N \chi$ is a constituent of $\mathrm{Res}^G_N \xi$, and part (c) of the same lemma establishes that $\frac{1}{|N|} \sum_{x \in Ng} |\chi(x)|^2 = 1$. We apply the former with $G = S_m \wr S_n$, $N = B$ and $\chi = \widetilde{\vartheta^{\times n}}$ to deduce that $\left\langle \mathrm{Res}_B \xi, \mathrm{Res}_B \widetilde{\vartheta^{\times n}} \right\rangle \ne 0$. It follows by Frobenius Reciprocity that ξ is a constituent of

$$\mathrm{Ind}^{S_m \wr S_n}_B (\vartheta \times \cdots \times \vartheta) = \sum_\nu \chi^\nu(1) \widetilde{\vartheta^{\times n}} \mathrm{Inf}^{S_m \wr S_n}_{S_n} \chi^\nu,$$

where the sum is over all partitions ν of n. Since ξ is irreducible we must have $\xi = \widetilde{\vartheta^{\times n}} \mathrm{Inf}^{S_m \wr S_n}_{S_n} \chi^\nu$ for some ν. Therefore the left-hand side in the

lemma is

$$\frac{\chi^\nu(g)}{|B|} \sum_{k \in B} \big(\widetilde{\vartheta^{\times n}}(k\,;g)\big)^2,$$

which is equal to $\chi^\nu(g)$ by [7, Lemma 8.14(c)] as stated above. $\square$

By Definition 1.1, we have $\mathrm{Defres}^\vartheta_{S_n}(\widetilde{\vartheta^{\times n}}\,\mathrm{Inf}^{S_m \wr S_n}_{S_n}\chi^\nu)(g) = \chi^\nu(g)$ for all $g \in S_n$. The next proposition therefore follows immediately from Lemma 2.1.

Proposition 2.2. *Let $m,\,n \in \mathbb{N}$, let ϑ be an irreducible character of S_m, and let ψ be a character of $S_m \wr S_n$. If $g \in S_n$ then*

$$(\mathrm{Def}^\vartheta_{S_n}\psi)(g) = \frac{1}{|B|}\sum_{k \in B}\psi(k\,;g)\widetilde{\vartheta^{\times n}}(k\,;g). \qquad \square$$

Corollary 2.3. *Let $m,\,n \in \mathbb{N}$, let ϑ be an irreducible character of S_m, and let ψ be a character of $S_m \wr S_n$. If $g \in S_n$ is an n-cycle then*

$$(\mathrm{Def}^\vartheta_{S_n}\psi)(g) = \frac{1}{m!}\sum_{h \in S_m}\psi(h,1,\ldots,1;g)\vartheta(h).$$

Proof. Suppose that g is the n-cycle $(x_1\,x_2\,\ldots\,x_n)$. By [8, 4.2.8], the permutations $(h_1,\ldots,h_n;g)$ and $(h'_1,\ldots,h'_n;g) \in S_m \wr S_n$ are conjugate in $S_m \wr S_n$ if and only if the elements $h_{x_n}h_{x_{n-1}}\ldots h_{x_1}$ and $h'_{x_n}h'_{x_{n-1}}\ldots h'_{x_1}$ are conjugate in S_m. In particular, each conjugacy class of $S_m \wr S_n$ which meets $\{(k\,;g) : k \in B\}$ has a representative of the form $(h,1,\ldots,1;g)$. Moreover, the number of elements $(h_1,h_2,\ldots,h_n;g)$ conjugate to $(h,1,\ldots,1;g)$ is $m!^{n-1}|h^{S_m}|$, since $h_2,\ldots,h_n$ may be chosen arbitrarily, and then h_1 must be chosen so that $h_{x_n}h_{x_{n-1}}\cdots h_{x_1} \in h^{S_m}$. It follows that

$$\sum_{k \in B}\psi(k\,;g)\widetilde{\vartheta^{\times n}}(k\,;g) = m!^{n-1}\sum_{h \in S_m}\psi(h,1,\ldots,1;g)\widetilde{\vartheta^{\times n}}(h,1,\ldots,1;g).$$

The explicit formula given in [8, Lemma 4.3.9] to compute the character value $\widetilde{\vartheta^{\times n}}(h_1,h_2,\ldots,h_n;g)$ is particularly straightforward in the case when g is an n-cycle: if $g = (x_1\,x_2\,\ldots\,x_n)$ then $\widetilde{\vartheta^{\times n}}(h_1,h_2,\ldots,h_n;g) = \vartheta(h_{x_n}h_{x_{n-1}}\cdots h_{x_1})$. It then follows by applying this to the expression we previously obtained that

$$\sum_{k \in B}\psi(k\,;g)\widetilde{\vartheta^{\times n}}(k\,;g) = m!^{n-1}\sum_{h \in S_m}\psi(h,1,\ldots,1;g)\vartheta(h).$$

Now apply Proposition 2.2 to the left-hand side. $\square$

The following definition and lemma allow for a more convenient statement of Corollary 2.3.

Definition 2.4. Let m, $n \in \mathbb{N}$, let $g \in S_n$ be an n-cycle, and let ψ be a character of $S_m \wr S_n$. We define $\omega(\psi)$ to be the class function on S_m such that

$$\omega(\psi)(h) = \psi(h, 1, \ldots, 1; g)$$

for all $h \in S_m$.

Lemma 2.5. *Let m, $n \in \mathbb{N}$. If $g \in S_n$ is an n-cycle and $h \in S_m$ has cycle type α then $(h, 1, \ldots, 1; g) \in S_m \wr S_n$ has cycle type $n\alpha$.*

Proof. It suffices to show that if $\mathcal{O}$ is an orbit of h on $\{1, 2, \ldots, m\}$ then $\mathcal{O} \times \{1, \ldots, n\}$ is an orbit of $(h, 1, \ldots, 1; g)$ in its action on $\{1, \ldots, m\} \times \{1, \ldots, n\}$. We leave this to the reader as an easy exercise. $\square$

The next proposition follows easily from Lemma 2.5 and Corollary 2.3.

Proposition 2.6. *Let m, $n \in \mathbb{N}$, and let χ be a character of S_{mn}.*

(i) *If α is a partition of m then*

$$\omega(\mathrm{Res}_{S_m \wr S_n} \chi)(g_\alpha) = \chi(g_{n\alpha}).$$

(ii) *If ϑ is an irreducible character of S_m and $g \in S_n$ is an n-cycle then*

$$(\mathrm{Defres}_{S_n}^{\vartheta} \chi)(g) = \langle \omega(\mathrm{Res}_{S_m \wr S_n} \chi), \vartheta \rangle$$
$$= \sum_\alpha \frac{\chi(g_{n\alpha})}{z_\alpha} \vartheta(g_\alpha)$$

where the sum is over all partitions α of m. $\square$

The character value $\chi(g_{n\alpha})$ in part (i) is the subject of Theorem 3.3; we shall see that combining this theorem with part (ii) gives Equation (8) in §4 below. Note also that part (ii) of the proposition implies Equation (3) in §2.

3. Skew characters

In this section we state and prove a result on the values of skew characters on elements of the form $g_{n\alpha}$ (Theorem 3.3). First, we give the necessary combinatorial definitions. In several arguments we shall refer to James' abacus notation for partitions, as described in [8, page 78].

3.1. **Quotients of skew-partitions.** We shall define n-quotients and n-signs for the following class of skew-partitions.

Definition 3.1. Let m, $n \in \mathbb{N}$. We say that a skew-partition λ/μ of mn is *n-decomposable* if there exists a border-strip tableau of shape λ/μ and type (n^m).

Definition 3.2. Let m, $n \in \mathbb{N}$ and let λ/μ be an n-decomposable skew-partition of mn. Let $\Gamma(\lambda)$ be an abacus display for λ on an n-runner abacus using tn beads for some $t \in \mathbb{N}$. Let $\Gamma(\mu)$ be the abacus display for μ obtained by performing an appropriate sequence of m upward bead moves on $\Gamma(\lambda)$. (This is possible since λ/μ is n-decomposable.) Let $(\lambda^{(0)}, \ldots, \lambda^{(n-1)})$ and $(\mu^{(0)}, \ldots, \mu^{(n-1)})$ be the n-quotients of λ and μ corresponding to $\Gamma(\lambda)$ and $\Gamma(\mu)$, respectively. The *n-quotient* of λ/μ is defined to be

$$(\lambda^{(0)}/\mu^{(0)}, \ldots, \lambda^{(n-1)}/\mu^{(n-1)}).$$

We define the *n-sign* of λ/μ, denoted $\varepsilon_n(\lambda/\mu)$ to be the sign of any border-strip tableau of shape λ/μ and type (n^m).

To avoid cumbersome restatements, we adopt the convention that $\lambda^{(i)}/\mu^{(i)}$ always has the meaning of Definition 3.2 above. It is clear from the abacus that $\mu^{(i)}$ is a subpartition of $\lambda^{(i)}$ for each $i \in \{0, \ldots, n-1\}$, and so the n-quotient is well-defined. It follows from Proposition 3.13 in [15], or our Proposition 3.6 below, that the n-sign of a skew-partition is well defined. See §3.4 below for an example of these definitions and all the results in this section.

We remark that it appears to be impossible to define the n-core of an arbitrary skew-partition. The example $\lambda/\mu = (2,2)/(1)$ and $n = 2$ illustrates the obstacles that arise. Representing λ on a 2-runner abacus as

$$\begin{array}{cc} \circ & \circ \\ \bullet & \bullet \end{array}$$

we see that either bead may be moved up, giving two different skew-partitions from which no border strip of length 2 can be removed, namely $(2)/(1)$ and $(1,1)/(1)$. The 2-quotients corresponding to these bead moves, namely $((1), \varnothing)$ and $(\varnothing, (1))$, are also different.

Theorem 3.3 (Farahat). *Let m, $n \in \mathbb{N}$ and let λ/μ be a skew-partition of mn. Let α be a partition of m. If λ/μ is not n-decomposable then $\chi^{\lambda/\mu}(g_{n\alpha}) = 0$. If λ/μ is n-decomposable and $(\lambda^{(0)}/\mu^{(0)}, \ldots, \lambda^{(n-1)}/\mu^{(n-1)})$ is its n-quotient, then*

$$\chi^{\lambda/\mu}(g_{n\alpha}) = \varepsilon_n(\lambda/\mu) \operatorname{Ind}_{S_{\ell_0} \times \cdots \times S_{\ell_{n-1}}}^{S_m} \left(\chi^{\lambda^{(0)}/\mu^{(0)}} \times \cdots \times \chi^{\lambda^{(n-1)}/\mu^{(n-1)}} \right)(g_\alpha)$$

where $|\lambda^{(i)}/\mu^{(i)}| = \ell_i$.

This result, stated in the alternative language of star diagrams, was first proved in [5, Section 4]. The special case where μ is the n-core of λ also follows from the correction by Thrall and Robinson [21] to Section 7 of Robinson [18] or, alternatively, from Littlewood's result in [11, Section 2].

Farahat's proof depends on an algebraic argument using Schur functions. A character-theoretic proof is given by Kerber, Sänger, and Wagner: see [10, Equation 3.6]. In the remainder of this section, we give a shorter character-theoretic proof of Theorem 3.3, expressing each side of the theorem as a sum, and then constructing a bijection between the summands. An example to illustrate this bijection is given in §3.4. (Our bijection is similar to that defined using *Brettspiele* in [10].)

3.2. **A model for induction from a Young subgroup.** The following general result on the values of a character induced from a Young subgroup will be used in the proof of Theorem 3.3. (The notation is chosen to be consistent with this later use.)

Lemma 3.4. *Let $(\ell_0, \ldots, \ell_{n-1})$ be a composition of $m \in \mathbb{N}$. For each $i \in \{0, \ldots, n-1\}$, let ϑ_i be a character of S_{ℓ_i}. If $g \in S_m$ then*

$$\mathrm{Ind}_{S_{\ell_0} \times \cdots \times S_{\ell_{n-1}}}^{S_m} (\vartheta_0 \times \cdots \times \vartheta_{n-1})(g) = \sum_{\mathbf{t}} \vartheta_0(g_{\alpha_0(\mathbf{t})}) \ldots \vartheta_{n-1}(g_{\alpha_{n-1}(\mathbf{t})})$$

where the sum is over all $(\ell_0, \ldots, \ell_{n-1})$-tabloids $\mathbf{t}$ such that $g\mathbf{t} = \mathbf{t}$, and $\alpha_i(\mathbf{t})$ is the cycle type of the permutation induced by g on the entries of row $i + 1$ of $\mathbf{t}$.

Proof. Let $\mathbf{t}_1, \ldots, \mathbf{t}_N$ be the $(\ell_0, \ldots, \ell_{n-1})$-tabloids. Let $\mathbf{s}$ be an $(\ell_0, \ldots, \ell_{n-1})$-tabloid fixed by the Young subgroup $S_{\ell_0} \times \cdots \times S_{\ell_{n-1}}$. For each j such that $1 \leq j \leq N$, choose $x_j \in S_m$ such that $\mathbf{t}_j = x_j \mathbf{s}$. Let $\vartheta = \vartheta_0 \times \cdots \times \vartheta_{n-1}$. For each $g \in S_m$ we have

$$\left(\mathrm{Ind}_{S_{\ell_0} \times \cdots \times S_{\ell_{n-1}}}^{S_m} \vartheta\right)(g) = \sum_j \vartheta(x_j^{-1} g x_j)$$

where the sum is over all j such that

$$x_j^{-1} g x_j \in S_{\ell_0} \times \cdots \times S_{\ell_{n-1}},$$

or, equivalently, over all j such that $g\mathbf{t}_j = \mathbf{t}_j$. If $\Delta_1, \ldots, \Delta_q$ are the orbits of g on row $i + 1$ of $\mathbf{t}_j$, then $x_j^{-1}\Delta_1, \ldots, x_j^{-1}\Delta_q$ are the orbits of $x_j^{-1} g x_j$ on row $i + 1$ of $\mathbf{s}$. Hence $x_j^{-1} g x_j$ acts with cycle type $\alpha_i(\mathbf{t}_j)$ on row $i + 1$ of $\mathbf{s}$ and so

$$\vartheta(x_j^{-1} g x_j) = \vartheta_0(g_{\alpha_0(\mathbf{t_j})}) \ldots \vartheta_{n-1}(g_{\alpha_{n-1}(\mathbf{t_j})}).$$

The lemma follows. $\square$

3.3. **Proof of Theorem 3.3.** Let $m, n \in \mathbb{N}$ and let λ/μ be a skew-partition of mn. Let α be a partition of m. If there is a border-strip tableau of shape λ/μ and type $n\alpha$ then it is clear from the abacus that λ/μ is n-decomposable. Hence if λ/μ is not n-decomposable then, by the Murnaghan–Nakayama rule, $\chi^{\lambda/\mu}(g_{n\alpha}) = 0$.

We may therefore assume that λ/μ is n-decomposable. Let $\ell_i = |\lambda^{(i)}/\mu^{(i)}|$ for each $i \in \{0, \ldots, n-1\}$ and let $H = S_{\ell_0} \times \cdots \times S_{\ell_{n-1}}$. To show that

$$(4) \qquad \chi^{\lambda/\mu}(g_{n\alpha}) = \varepsilon_n(\lambda/\mu) \operatorname{Ind}_H^{S_m} \left(\chi^{\lambda^{(0)}/\mu^{(0)}} \times \cdots \times \chi^{\lambda^{(n-1)}/\mu^{(n-1)}} \right)(g_\alpha),$$

we shall use the following generalization of border-strip tableaux.

Definition 3.5. Let $m, n \in \mathbb{N}$. Let λ/μ be an n-decomposable skew-partition and let $\alpha = (\alpha_1, \ldots, \alpha_k)$ be a composition of m. A n-quotient border-strip tableau of shape λ/μ and type α is an n-tuple $(T_0, \ldots, T_{n-1})$ of border-strip tableaux such that

(a) for each $i \in \{0, \ldots, n-1\}$, the shape of T_i is $\lambda^{(i)}/\mu^{(i)}$, and

(b) for each $j \in \{1, \ldots, k\}$, the boxes in the T_i labelled j lie in a single tableau, where they form a border strip of length α_j.

By the Murnaghan–Nakayama rule we have

$$\chi^{\lambda/\mu}(g_{n\alpha}) = \sum \operatorname{sgn}(T)$$

where the sum is over all border-strip tableaux of shape λ/μ and type $n\alpha$. The bijection in the following proposition implies that

$$(5) \qquad \chi^{\lambda/\mu}(g_{n\alpha}) = \varepsilon_n(\lambda/\mu) \sum \operatorname{sgn}(T_0) \ldots \operatorname{sgn}(T_{n-1})$$

where the sum is over all n-quotient border-strip tableaux $(T_0, \ldots, T_{n-1})$ of shape λ/μ and type α. An illustrative example of the bijection is given in Figure 2 in §3.4 below.

Proposition 3.6. *Let λ/μ be an n-decomposable skew-partition of mn and let $\alpha = (\alpha_1, \ldots, \alpha_k)$ be a partition of m. There is a canonical bijection between border-strip tableaux of shape λ/μ and type $n\alpha$ and n-quotient border-strip tableaux of shape λ/μ and type α. Under this bijection, if T is mapped to $(T_0, \ldots, T_{n-1})$, then*

$$\operatorname{sgn}(T) = \varepsilon_n(\lambda/\mu) \operatorname{sgn}(T_0) \ldots \operatorname{sgn}(T_{n-1}).$$

Proof. Let T be a border-strip tableau of shape λ/μ and type $n\alpha$. The abacus gives a canonical bijection between border strips in λ/μ of length $n\ell$ and border strips of length ℓ in the skew-partitions $\lambda^{(i)}/\mu^{(i)}$ for $i \in \{0, \ldots, n-1\}$. If the border strip of length $n\alpha_k$ in T corresponds to a

border strip of length α_k in $\lambda^{(i_k)}/\mu^{(i_k)}$, then we label the corresponding boxes in the Young diagram of $\lambda^{(i_k)}/\mu^{(i_k)}$ by k. Removing these border strips from the tableaux concerned and iterating the process with the border strip of length $n\alpha_{k-1}$, and so on, we obtain a canonical bijection between border-strip tableaux of shape λ/μ and type $n\alpha$ and n-quotient border-strip tableaux of shape λ/μ and type α.

It only remains to prove the assertion about signs. Since $\varepsilon_n(\lambda/\mu)$ is the common sign of any λ/μ-tableau of shape λ/μ and type (n^m), it suffices to show that if T is a λ/μ-tableau of type $n\alpha$ and U is a λ/μ-tableau of type $n\beta$ then

$$\mathrm{sgn}(T)\mathrm{sgn}(U) = \mathrm{sgn}(T_0)\dots\mathrm{sgn}(T_{n-1})\mathrm{sgn}(U_0)\dots\mathrm{sgn}(U_{n-1}).$$

Starting from $\Gamma(\lambda)$ with the beads numbered in order of their positions, perform the sequence of bead moves corresponding to T, then perform the inverse of the sequence of bead moves corresponding to U. Let σ be the resulting permutation of the beads. Each time a border strip of height ℓ is removed or added, the permutation required to restore the order of numbers is an $\ell+1$-cycle. Therefore $\mathrm{sgn}\,\sigma = \mathrm{sgn}(T)\mathrm{sgn}(U)$. On the other hand, σ permutes the beads on each runner amongst themselves, and a similar argument now shows that $\mathrm{sgn}(\sigma) = \mathrm{sgn}(T_0)\dots\mathrm{sgn}(T_{n-1})\mathrm{sgn}(U_0)\dots\mathrm{sgn}(U_{n-1})$. $\qquad\square$

Comparing Equations (4) and (5) we see that to complete the proof of Theorem 3.3, it suffices to show that

$$(6) \quad \sum \mathrm{sgn}(T_0)\dots\mathrm{sgn}(T_{n-1}) = \mathrm{Ind}_H^{S_m}\left(\chi^{\lambda^{(0)}/\mu^{(0)}} \times \cdots \times \chi^{\lambda^{(n-1)}/\mu^{(n-1)}}\right)(g_\alpha)$$

where the sum is over all n-quotient border-strip tableaux $(T_0,\dots,T_{n-1})$ of shape λ/μ and type α. In fact Equation (6) follows from Lemma 3.4 and the Murnaghan–Nakayama rule, by some manipulations that are essentially formal. By Lemma 3.4 we have

$$(7) \quad \begin{aligned} &\mathrm{Ind}_H^{S_m}\left(\chi^{\lambda^{(0)}/\mu^{(0)}} \times \cdots \times \chi^{\lambda^{(n-1)}/\mu^{(n-1)}}\right)(g_\alpha) \\ &\qquad = \sum_{\mathbf{t}} \chi^{\lambda^{(0)}/\mu^{(0)}}(g_{\alpha_0(\mathbf{t})})\cdots\chi^{\lambda^{(n-1)}/\mu^{(n-1)}}(g_{\alpha_{n-1}(\mathbf{t})}) \end{aligned}$$

where the sum is over all $(\ell_0,\dots,\ell_{n-1})$-tabloids $\mathbf{t}$ such that $g_\alpha\mathbf{t} = \mathbf{t}$ and $\alpha_i(\mathbf{t})$ is the cycle type of the permutation of row $i+1$ of $\mathbf{t}$ induced by g_α.

For such a tabloid $\mathbf{t}$, let

$$f(\mathbf{t}) = \sum \mathrm{sgn}(T_0)\dots\mathrm{sgn}(T_{n-1})$$

where the sum is over all n-quotient border-strip tableaux $(T_0,\dots,T_{n-1})$ of shape λ/μ and type α, such that T_i has a border strip labelled j (of length α_j) if and only if the elements of the orbit of g_α corresponding to the

part α_j lie in row $i + 1$ of $\mathbf{t}$. The Murnaghan–Nakayama rule implies that if $g_\alpha \mathbf{t} = \mathbf{t}$ then

$$\chi^{\lambda^{(0)}/\mu^{(0)}}(g_{\alpha_0(\mathbf{t})}) \cdots \chi^{\lambda^{(n-1)}/\mu^{(n-1)}}(g_{\alpha_{n-1}(\mathbf{t})}) = f(\mathbf{t}),$$

and so, by Equation (7),

$$\mathrm{Ind}_H^{S_m}\big(\chi^{\lambda^{(0)}/\mu^{(0)}} \times \cdots \times \chi^{\lambda^{(n-1)}/\mu^{(n-1)}}\big)(g_\alpha) = \sum f(\mathbf{t})$$

where the sum is over all $(\ell_0, \ldots, \ell_{n-1})$-tabloids $\mathbf{t}$ such that $g_\alpha \mathbf{t} = \mathbf{t}$. Every n-quotient border-strip tableau of shape λ/μ and type α corresponds to some tabloid $\mathbf{t}$ such that $g_\alpha \mathbf{t} = \mathbf{t}$. Thus

$$\sum \mathrm{sgn}(T_0) \ldots \mathrm{sgn}(T_{n-1}) = \sum f(\mathbf{t})$$

where the left-hand sum is over all n-quotient border-strip tableaux of shape λ/μ and type α, and the right-hand sum is over all $(\ell_0, \ldots, \ell_{n-1})$-tabloids $\mathbf{t}$ such that $g_\alpha \mathbf{t} = \mathbf{t}$. Equation (6) now follows on comparing the two preceding equations.

3.4. **Example.** We give an example of the correspondences used in the proof of Theorem 3.3. Take $m = 5$ and $n = 3$, and let

$$\lambda/\mu = (8, 5, 3, 2, 2, 2)/(2, 2, 1, 1, 1).$$

Any border-strip tableau of shape λ/μ and type (3^5) has either two or four 3-border-strips of height 1, with the rest of height 0, so $\varepsilon_3(\lambda/\mu) = 1$. Let $\alpha = (2, 1, 1, 1)$. By the Murnaghan–Nakayama rule, $\chi^{\lambda/\mu}(g_{3\alpha})$ is the sum of the signs of the four border-strip tableaux of type λ/μ and type $3\alpha = (6, 3, 3, 3)$ shown in Figure 2. Their signs are $+1$, -1, -1, -1, respectively, so $\chi^{\lambda/\mu}(g_{3\alpha}) = -2$. These tableaux are in bijection with the four 3-quotient border-strip tableaux of shape λ/μ and type $\alpha = (2, 1, 1, 1)$ shown in Figure 2; since $\mathrm{sgn}_3(\lambda/\mu) = 1$, the bijection is sign preserving.

The 3-quotient of λ/μ is $\big((1, 1, 1), (3, 1)/(1, 1), \varnothing\big)$, so the characters of S_3 and S_2 we must consider are the sign character and the trivial character, respectively. Taking $g_{(2,1,1,1)} = (12) \in S_5$, and following the end of the proof of Theorem 3.3, we see that there are four $(3, 2)$-tabloids fixed by (12), namely

$$\underline{\begin{array}{ccc} 3 & 4 & 5 \\ \hline 1 & 2 \end{array}}, \quad \underline{\begin{array}{ccc} 1 & 2 & 3 \\ \hline 4 & 5 \end{array}}, \quad \underline{\begin{array}{ccc} 1 & 2 & 4 \\ \hline 3 & 5 \end{array}}, \quad \underline{\begin{array}{ccc} 1 & 2 & 5 \\ \hline 3 & 4 \end{array}},$$

in the order corresponding to the tableaux shown in Figure 2. The corresponding values of the function f used in the proof of Theorem 3.3 are $+1$, -1, -1, -1 respectively. It should be noted that in general there will be several tableaux corresponding to each product of character values

$$\chi^{\lambda^{(0)}/\mu^{(0)}}(g_{\alpha_0(\mathbf{t})}) \cdots \chi^{\lambda^{(n-1)}/\mu^{(n-1)}}(g_{\alpha_{n-1}(\mathbf{t})});$$
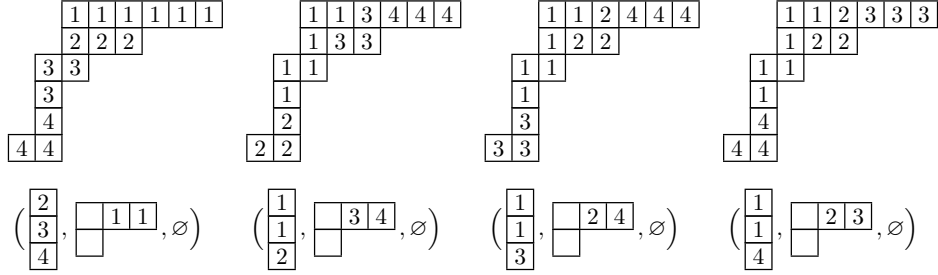
FIGURE 2. The bijection in Proposition 3.6 between border-strip tableaux of shape $(8,5,3,2,2,2)/(2,2,1,1,1)$ and type $(6,3,3,3)$ and 3-quotient border-strip tableaux of the same shape and type $(2,1,1,1)$. The shapes of the border-strip tableaux forming each 3-quotient border-strip tableau are given by the 3-quotient of $(8,5,3,2,2,2)/(2,2,1,1,1)$, namely $\big((1,1,1),(3,1)/(1,1),\varnothing\big)$. To make clear the skew-shape, the tableaux of shape $(3,1)/(1,1)$ are drawn as $(3,1)$-tableaux with two empty boxes.

it is a special feature of this example that each $f(\mathbf{t})$ has a single summand, and so the bijection extends all the way to tabloids.

4. PROOF OF THEOREM 1.5

We shall prove Theorem 1.5 by induction on the number of parts of γ. Most of the work occurs in proving the base case when γ has a single part. In the first step we combine the results of §3 and §4. For later use in §6, we state the following proposition for a general deflation map.

Proposition 4.1. *Let $m, n \in \mathbb{N}$, let ϑ be an irreducible character of S_m, and let λ/μ be a skew-partition of mn. Let $g \in S_n$ be an n-cycle. If λ/μ is not n-decomposable, then $\mathrm{Defres}_{S_n}^{\vartheta}(g) = 0$. If λ/μ is n-decomposable, then*

$$(\mathrm{Defres}_{S_n}^{\vartheta}\, \chi^{\lambda/\mu})(g) = \varepsilon_n(\lambda/\mu) \left\langle \mathrm{Ind}_H^{S_m}\big(\chi^{\lambda^{(0)}/\mu^{(0)}} \times \cdots \times \chi^{\lambda^{(n-1)}/\mu^{(n-1)}}\big), \vartheta \right\rangle,$$

where $H = S_{|\lambda^{(0)}/\mu^{(0)}|} \times \cdots \times S_{|\lambda^{(n-1)}/\mu^{(n-1)}|}$.

Proof. If λ/μ is not n-decomposable then, by Theorem 3.3, $\chi^{\lambda/\mu}(g_{n\alpha}) = 0$ for all partitions α of m. Hence, by Proposition 2.6(ii), $(\mathrm{Defres}_{S_n}^{\vartheta}\, \chi^{\lambda/\mu})(g) = 0$. If λ/μ is n-decomposable then, using Proposition 2.6(i), we may restate Theorem 3.3 as

$$(8) \quad \omega(\mathrm{Res}_{S_m \wr S_n}\, \chi^{\lambda/\mu}) = \varepsilon_n(\lambda/\mu) \,\mathrm{Ind}_H^{S_m}\big(\chi^{\lambda^{(0)}/\mu^{(0)}} \times \cdots \times \chi^{\lambda^{(n-1)}/\mu^{(n-1)}}\big).$$

The result now follows from Proposition 2.6(ii).   □

It is clear from the condition on row numbers in Equation (1) in Definition 1.4 that if λ/μ is a skew-partition of mn then there is at most one

m-border-strip tableau of shape λ/μ and type (n). The following lemma gives a more precise condition. Recall that a skew-partition σ/τ is said to be a *horizontal strip* if the Young diagram of σ/τ has no two boxes in the same column.

Lemma 4.2. *Let m, $n \in \mathbb{N}$ and let λ/μ be a skew-partition of mn. If λ/μ is n-decomposable, and each $\lambda^{(i)}/\mu^{(i)}$ is a horizontal strip, then there is a unique m-border-strip tableau of shape λ/μ and type (n); this tableau has sign $\varepsilon_n(\lambda/\mu)$. Otherwise there are no such tableaux.*

Proof. Suppose that T is an m-border-strip tableau of type (n) and shape λ/μ. Then by Definition 3.1, λ/μ is n-decomposable. Let $\Gamma(\lambda)$ be an n-runner abacus display for λ using tn beads for some $t \in \mathbb{N}$. Let $\Gamma(\mu)$ be the abacus display for μ obtained from $\Gamma(\lambda)$ by an appropriate sequence of m single bead moves, so that in each move a bead is slid upwards into a gap immediately above it. We label the positions on the abacus from top to bottom so that the positions in row r of an abacus display are numbered $(r-1)n, \ldots, rn-1$ (as usual). Observe that the row number of a border strip of length n in T corresponding to a bead in position p of $\Gamma(\lambda)$ is the number of beads in positions $p+1$, $p+2$, $\ldots$ of $\Gamma(\lambda)$. Therefore if $p < p'$ and the beads in positions p and p' of $\Gamma(\lambda)$ both correspond to border strips in T, then in the sequence of bead moves corresponding to T, the bead in position p' is moved upwards before the bead in position p.

Let $i \in \{0, \ldots, n-1\}$. Suppose that $\Gamma(\lambda)$ has beads in positions $nq + i$ and $nq' + i$ where $q < q'$, and that $\Gamma(\mu)$ has no beads in positions $n(q+1)+i$, $\ldots$, $nq' + i$. The bead in position $nq + i$ of $\Gamma(\lambda)$ prevents the bead initially in position $nq' + i$ from reaching its final position in $\Gamma(\mu)$. Therefore the bead in position $nq + i$ must be moved before the bead in position $nq' + i$ reaches its final position. This contradicts the previous paragraph. Hence there exist $x_j \in \mathbb{N}_0$ and $y_j \in \mathbb{N}_0$ such that the beads on runner i of $\Gamma(\lambda)$ are in positions $\{i + nx_j : 1 \le j \le s\}$, the beads on runner i of $\Gamma(\mu)$ are in positions $\{i + ny_j : 1 \le j \le s\}$ and

$$(9) \qquad\qquad y_1 \le x_1 < y_2 \le x_2 < \cdots < y_s \le x_s.$$

It easily follows that $\lambda^{(i)}/\mu^{(i)}$ is a horizontal strip. Then, by Proposition 3.6, $\operatorname{sgn}(T) = \varepsilon_n(\lambda/\mu)$.

Conversely, suppose that λ/μ is n-decomposable and each $\lambda^{(i)}/\mu^{(i)}$ is a horizontal strip. Then the inequality (9) on the bead positions in each runner holds. We now describe a sequence of m single upward bead moves that transforms $\Gamma(\lambda)$ into $\Gamma(\mu)$, and thus corresponds to a border-strip tableau T of shape λ/μ and type $(n)^{\star m}$. At each step, locate the bead with maximal

position p such that there is no bead in position p of $\Gamma(\mu)$. Slide that bead up into position $p - n$. This is possible by inequality (9). The row numbers of the border strips of length n corresponding to this sequence of moves are increasing, so T is a m-border-strip tableau of shape λ/μ and type (n). Uniqueness is clear since there is always at most one m-border strip tableau of shape λ/μ and type (n). $\square$

We can now complete the proof of the base case.

Proposition 4.3. *Let m, $n \in \mathbb{N}$ and let $g \in S_n$ be an n-cycle. If λ/μ is a skew-partition of mn then*

$$(\mathrm{Defres}_{S_n} \chi^{\lambda/\mu})(g) = a_{\lambda/\mu,(n)}.$$

Proof. If λ/μ is not n-decomposable then $a_{\lambda/\mu,(n)} = 0$ by Lemma 4.2. Proposition 4.1 implies that the result holds in this case.

Now suppose that λ/μ is n-decomposable. Let $\ell_i = |\lambda^{(i)}/\mu^{(i)}|$ for each $i \in \{0, \ldots, n-1\}$. By Proposition 4.1 and Frobenius reciprocity, we have

$$(\mathrm{Defres}_{S_n} \chi^{\lambda/\mu})(g) = \varepsilon_n(\lambda/\mu) \left\langle \chi^{\lambda^{(0)}/\mu^{(0)}}, 1_{S_{\ell_0}} \right\rangle \cdots \left\langle \chi^{\lambda^{(n-1)}/\mu^{(n-1)}}, 1_{S_{\ell_{n-1}}} \right\rangle.$$

Theorem 2.3.13(ii) of [8] states that if σ/τ is a skew-partition of ℓ then

$$\left\langle \chi^{\sigma/\tau}, 1_{S_\ell} \right\rangle = \begin{cases} 1 & \text{if } \sigma_1 \geq \tau_1 \geq \sigma_2 \geq \tau_2 \geq \cdots \\ 0 & \text{otherwise,} \end{cases}$$

and it is easily seen that this first condition precisely picks out the cases where σ/τ is a horizontal strip. Therefore $(\mathrm{Defres}_{S_n} \chi^{\lambda/\mu})(g) = \varepsilon_n(\lambda/\mu)$ if each $\lambda^{(i)}/\mu^{(i)}$ is a horizontal strip, and otherwise $(\mathrm{Defres}_{S_n} \chi^{\lambda/\mu})(g) = 0$. The proposition now follows from Lemma 4.2. $\square$

For the inductive step we need the following lemma and proposition. The former is well-known and can be deduced from [8, 2.3.12]. We write $\mu \subseteq \lambda$ if μ is a subpartition of λ (i.e. the Young diagram of μ is contained in that of λ).

Lemma 4.4. *Let λ/μ be a skew-partition of r. If $1 \leq c < r$ then*

$$\mathrm{Res}_{S_c \times S_{r-c}} \chi^{\lambda/\mu} = \sum_\tau \chi^{\tau/\mu} \times \chi^{\lambda/\tau}$$

where the sum is over all partitions τ such that $\mu \subseteq \tau \subseteq \lambda$ and $|\tau/\mu| = c$.

For later use we state and prove the following proposition for a general deflation map.

Proposition 4.5. *Let m, $n \in \mathbb{N}$ and let λ/μ be a skew-partition of mn. Let ϑ be an irreducible character of S_m. Let $g \in S_n$. If $g = kh$ where $k \in S_\ell$ and $h \in S_{n-\ell}$, then*

$$(\mathrm{Defres}_{S_n}^{\vartheta} \chi^{\lambda/\mu})(g) = \sum_{\tau} (\mathrm{Defres}_{S_\ell}^{\vartheta} \chi^{\tau/\mu})(k)(\mathrm{Defres}_{S_{n-\ell}}^{\vartheta} \chi^{\lambda/\tau})(h)$$

where the sum is over all partitions τ such that $\mu \subseteq \tau \subseteq \lambda$ and $|\tau/\mu| = m\ell$.

Proof. Let B be the base group of the wreath product $S_m \wr S_n \le S_{mn}$. Choose a subgroup $S_{m\ell} \times S_{m(n-\ell)} \le S_{mn}$ containing B. If ψ is a character of $S_m \wr S_n$ then it is easily checked that

$$\mathrm{Res}_{S_\ell \times S_{n-\ell}}(\mathrm{Def}_{S_n}^{\vartheta} \psi) = (\mathrm{Def}_{S_\ell}^{\vartheta} \times \mathrm{Def}_{S_{n-\ell}}^{\vartheta})(\mathrm{Res}_{S_m \wr S_\ell \times S_m \wr S_{n-\ell}}^{S_m \wr S_n} \psi).$$

Hence

$$\mathrm{Res}_{S_\ell \times S_{n-\ell}}(\mathrm{Defres}_{S_n}^{\vartheta} \chi) = (\mathrm{Defres}_{S_\ell}^{\vartheta} \times \mathrm{Defres}_{S_{n-\ell}}^{\vartheta})(\mathrm{Res}_{S_{m\ell} \times S_{m(n-\ell)}}^{S_{mn}} \chi)$$

for any character χ of S_{mn}. The proposition now follows from the expression for $\mathrm{Res}_{S_{m\ell} \times S_{m(n-\ell)}} \chi^{\lambda/\mu}$ given in Lemma 4.4. $\square$

We are now ready to prove Theorem 1.5. Let m, $n \in \mathbb{N}$ and let λ/μ be a skew-partition of mn. Let $\gamma = (\gamma_1, \dots, \gamma_d)$ be a composition of n. Let $g \in S_n$ have cycle type γ and let $h \in S_{n-\gamma_1}$ have cycle type $(\gamma_2, \dots, \gamma_d)$. Note that, by Lemma 4.2, if τ/μ is a skew-partition of $m\gamma_1$ then there is at most one m-border-strip tableau of shape τ/μ and type (γ_1). We shall denote this tableau by $T_{\tau/\mu}$ when it exists. By Definition 1.4, $a_{\tau/\mu,(\gamma_1)} = \mathrm{sgn}(T_{\tau/\mu})$ (or is zero if no such tableau exists). It follows that Proposition 4.3 may be restated as $(\mathrm{Defres}_{S_{\gamma_1}} \chi^{\tau/\mu})(k) = \mathrm{sgn}(T_{\tau/\mu})$, where $k \in S_{\gamma_1}$ is a γ_1-cycle and $(\mathrm{Defres}_{S_{\gamma_1}} \chi^{\tau/\mu})(k) = 0$ if no tableau $T_{\tau/\mu}$ exists. It therefore follows from Proposition 4.5 that

$$(10) \qquad (\mathrm{Defres}_{S_n} \chi^{\lambda/\mu})(g) = \sum_{\tau} \mathrm{sgn}(T_{\tau/\mu})(\mathrm{Defres}_{S_{n-\gamma_1}} \chi^{\lambda/\tau})(h)$$

where the sum is over all partitions τ such that $\mu \subseteq \tau \subseteq \lambda$, $|\tau/\mu| = m\gamma_1$ and there is an m-border-strip tableau of shape τ/μ. By induction on the number of parts of γ we have

$$(\mathrm{Defres}_{S_n} \chi^{\lambda/\mu})(g) = \sum_{\tau} \mathrm{sgn}(T_{\tau/\mu}) a_{\lambda/\tau,(\gamma_2,\dots,\gamma_d)}$$

with the same conditions on the sum. It is clear that if T is an m-border-strip tableau of shape λ/μ and type γ then the border strips in T corresponding to the m parts of length γ_1 in $\gamma^{\star m}$ form an m-border-strip tableau of shape τ/μ for some τ. Therefore the right-hand side of the previous equation is $a_{\lambda/\mu,\gamma}$. This completes the proof of Theorem 1.5.

5. An application to Foulkes' Conjecture

For m, $n \in \mathbb{N}$, let $\phi^{(m^n)}$ be the permutation character of S_{mn} acting on all unordered set partitions of $\{1, 2, \ldots, mn\}$ into n sets each of size m. Equivalently, $\phi^{(m^n)} = \mathrm{Ind}_{S_m \wr S_n}^{S_{mn}} 1$. Foulkes' Conjecture asserts that if $m \leq n$ then

$$\left\langle \phi^{(m^n)}, \chi^\lambda \right\rangle \geq \left\langle \phi^{(n^m)}, \chi^\lambda \right\rangle$$

for all partitions λ of mn. Equivalent formulations of Foulkes' Conjecture exist in the language of general linear groups, symmetric polynomials, and geometric invariant theory. Despite having been attacked from all these directions (and more), Foulkes' Conjecture has only been proved when $m \leq 4$ (see [2] and [13]), asymptotically when n is very large compared to m (see [1, page 352]) and, in a computational result of Müller and Neunhöffer [14], when $m + n \leq 17$. For further background we refer the reader to [20, Problem 9]. For some recent results on the constituents of $\phi^{(m^n)}$ see [16] and [6].

In this section we use character deflations to prove a new recursive formula for the character multiplicities in Foulkes' Conjecture. Firstly, using Frobenius reciprocity, then the inflation-deflation reciprocity relation

$$(11) \qquad \langle \mathrm{Def}_{S_n} \psi, \chi \rangle = \left\langle \psi, \mathrm{Inf}_{S_n}^{S_m \wr S_n} \chi \right\rangle,$$

where ψ is a character of $S_m \wr S_n$ and χ is a character of S_n, we observe that

$$\left\langle \phi^{(m^n)}, \chi^\lambda \right\rangle = \left\langle \mathrm{Defres}_{S_n} \chi^\lambda, 1_{S_n} \right\rangle.$$

Proposition 5.1. *Let m, $n \in \mathbb{N}$. If λ is a partition of mn then*

$$\left\langle \phi^{(m^n)}, \chi^\lambda \right\rangle = \frac{1}{n} \sum_{\ell=1}^{n} \sum_{\mu} \varepsilon_\ell(\lambda/\mu) \left\langle \phi^{(m^{n-\ell})}, \chi^\mu \right\rangle$$

where the second sum is over all partitions μ of ℓm such that there exists an m-border strip tableau of shape λ/μ and type (ℓ).

Proof. We have seen that

$$\left\langle \phi^{(m^n)}, \chi^\lambda \right\rangle = \left\langle \mathrm{Defres}_{S_n} \chi^\lambda, 1_{S_n} \right\rangle = \frac{1}{n!} \sum_{g \in S_n} (\mathrm{Defres}_{S_n} \chi^\lambda)(g).$$

We may write each $g \in S_n$ as a product of an ℓ-cycle containing the letter 1 and some $h \in S_{n-\ell}$ acting on the remaining letters. The number of possible such ℓ-cycles is $(n-1)!/(n-\ell)!$, hence

$$\left\langle \phi^{(m^n)}, \chi^\lambda \right\rangle = \frac{1}{n!} \sum_{\ell=1}^{n} \frac{(n-1)!}{(n-\ell)!} \sum_{h \in S_{n-\ell}} (\mathrm{Defres}_{S_n} \chi^\lambda)(xh)$$

where x is the ℓ-cycle $(1\,2\,\ldots\,\ell)$. We now apply Proposition 4.5 to see that

$$\left\langle \phi^{(m^n)}, \chi^\lambda \right\rangle = \frac{1}{n} \sum_{\ell=1}^{n} \frac{1}{(n-\ell)!} \sum_{h \in S_{n-\ell}} \sum_{\mu} (\mathrm{Defres}_{S_\ell} \chi^{\lambda/\mu})(x)(\mathrm{Defres}_{S_{n-\ell}} \chi^\mu)(h),$$

where the sum is over partitions $\mu \subseteq \lambda$ with $|\lambda/\mu| = m\ell$. Since x is an ℓ-cycle, Proposition 4.3 shows that $(\mathrm{Defres}_{S_\ell} \chi^{\lambda/\mu})(x)$ is $\varepsilon_\ell(\lambda/\mu)$ if there exists an m-border-strip tableau of shape λ/μ and type (ℓ) and is zero otherwise. Thus

$$
\begin{aligned}
\left\langle \phi^{(m^n)}, \chi^\lambda \right\rangle &= \frac{1}{n} \sum_{\ell=1}^{n} \sum_{\mu} \varepsilon_\ell(\lambda/\mu) \frac{1}{(n-\ell)!} \sum_{h \in S_{n-\ell}} (\mathrm{Defres}_{S_{n-\ell}} \chi^\mu)(h) \\
&= \frac{1}{n} \sum_{\ell=1}^{n} \sum_{\mu} \varepsilon_\ell(\lambda/\mu) \left\langle \mathrm{Defres}_{S_{n-\ell}} \chi^\mu, 1_{S_{n-\ell}} \right\rangle \\
&= \frac{1}{n} \sum_{\ell=1}^{n} \sum_{\mu} \varepsilon_\ell(\lambda/\mu) \left\langle \phi^{(m^{n-\ell})}, \chi^\mu \right\rangle
\end{aligned}
$$

where, in each case, the second sum is over all partitions $\mu \subseteq \lambda$ for which there exists an m-border-strip tableau of shape λ/μ and type (ℓ). $\square$

Proposition 5.1 gives an algorithm for testing Foulkes' Conjecture for a single character χ^λ of S_{mn} that is far faster than more direct methods, such as those requiring the character values of $\phi^{(m^n)}$ and $\phi^{(n^m)}$ to be calculated on all partitions of mn. Timings suggest that it can be significantly faster than the algorithm used by SYMMETRICA [9], although some of this gain comes at the expense of increased use of memory. For example, to calculate all the multiplicities $\langle \phi^{(6^{11})}, \chi^\lambda \rangle$ for λ a partition of 66 takes 34 minutes using the Haskell [17] implementation of Proposition 5.1 available from the third author's website[1], compared to 350 minutes for SYMMETRICA using the function `COMPLETE_COMPLETE_PLET`, both running on the same machine.

The graphs in Figures 3 and 4 show a number of intriguing features of the character multiplicities appearing in Foulkes' Conjecture. In particular, it seems plausible that if Foulkes' Conjecture is false, then a counterexample occurs when the relevant partition is either very large or very small in the lexicographic order on partitions of mn with at most n parts.

Using SYMMETRICA, Foulkes' Conjecture has been checked for all m and n with $m + n \leq 17$ in [14]. Using Proposition 5.1 and the software already mentioned, we have extended this range. The relevant data is available from the third author's website.

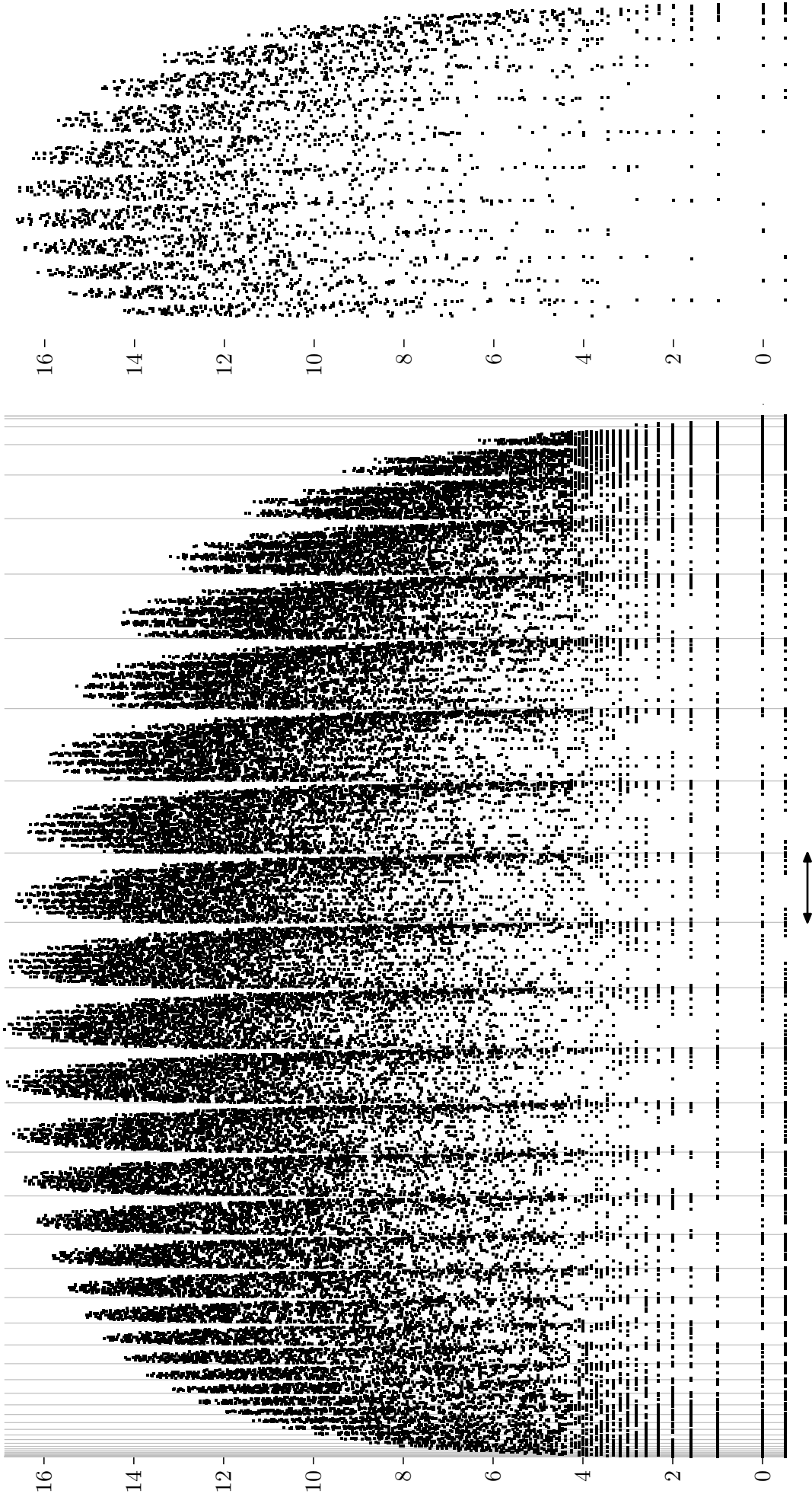
[1]See `www.ma.rhul.ac.uk/~uvah099/`.

FIGURE 3. The left-hand graph shows $\log_2\left\langle\phi^{(7^8)},\chi^\lambda\right\rangle$ for all partitions of 56 with at most 8 parts. Partitions are ordered lexicographically, with the smallest partition (7^8) at the far right. If the multiplicity is zero then the point is placed below the x axis. Vertical lines separate partitions with equal largest parts. The right-hand graph shows an enlarged view of the multiplicities for partitions with first part 19; the range of these partitions is indicated by the arrow in the left-hand graph.
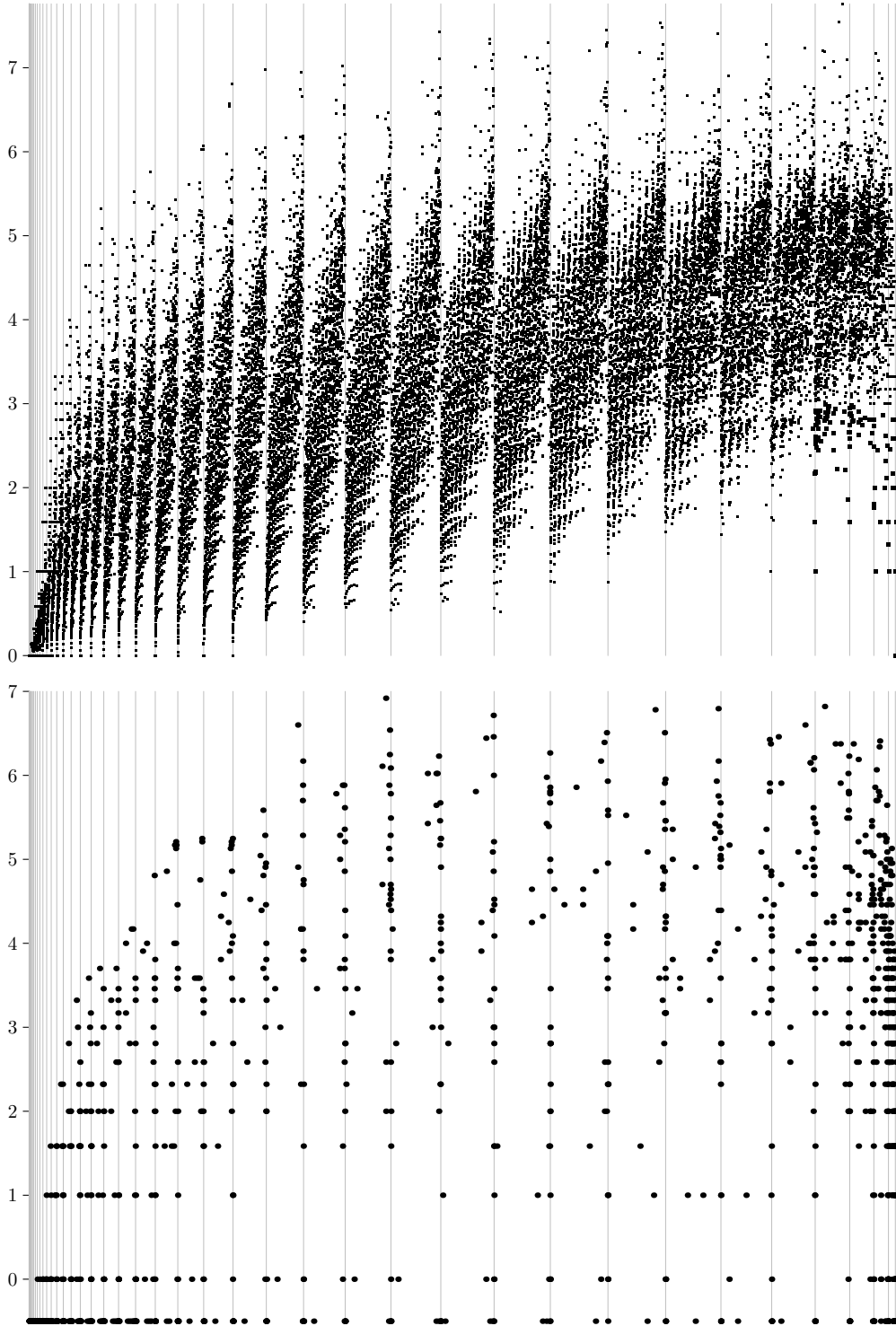
FIGURE 4. The top graph shows $\log_2\langle\phi^{(7^8)},\chi^\lambda\rangle - \log_2\langle\phi^{(8^7)},\chi^\lambda\rangle$ for the partitions of 56 with at most 7 parts for which the smaller multiplicity is non-zero. To increase their visibility a small number of points have been enlarged. Partitions are ordered lexicographically, with the smallest partition (8^7) at the far right. The lower graph shows $\log_2\langle\phi^{(7^8)},\chi^\lambda\rangle$ for those partitions for which $\langle\phi^{(8^7)},\chi^\lambda\rangle = 0$; if $\langle\phi^{(7^8)},\chi^\lambda\rangle = 0$ then the point is drawn below the axis.

Corollary 5.2. *If $m \leq n$ and $m + n \leq 19$ then*

$$\left\langle \phi^{(m^n)}, \chi^\lambda \right\rangle \geq \left\langle \phi^{(n^m)}, \chi^\lambda \right\rangle$$

for all partitions λ of mn. □

6. Generalized deflations

In this section we discuss deflation with respect to an arbitrary irreducible character ϑ of S_m. In the case where ϑ is labelled by a hook partition, $\vartheta = \chi^{(a,1^b)}$ where $a + b = m$, we give a combinatorial description, generalizing Theorem 1.5. We also prove some other general results, and show how these, together with the results of §4, may be used to calculate the values of an irreducible character of S_{mn} deflated with respect to an arbitrary character of S_m.

Firstly we deduce from Theorem 1.5 an analogous result for deflations with respect to the sign character. As is usual, if λ is a partition then we denote by λ' the conjugate partition to λ.

Proposition 6.1. *Let m, $n \in \mathbb{N}$ and let λ/μ be a skew-partition of mn. If γ is a composition of n and $g \in S_n$ has cycle type γ then*

$$(\mathrm{Defres}_{S_n}^{\mathrm{sgn}_{S_m}})(g) = \begin{cases} a_{\lambda'/\mu',\gamma} & \textit{if } m \textit{ is even} \\ \mathrm{sgn}_{S_n}(g)\, a_{\lambda'/\mu',\gamma} & \textit{if } m \textit{ is odd.} \end{cases}$$

Proof. It is easily seen that

$$\mathrm{Res}_{S_m \wr S_n}\, \mathrm{sgn}_{S_{mn}} = \begin{cases} \widetilde{\mathrm{sgn}_{S_m}^{\times n}} & \text{if } m \text{ is even} \\ \widetilde{\mathrm{sgn}_{S_m}^{\times n}}\, \mathrm{sgn}_{S_n} & \text{if } m \text{ is odd.} \end{cases}$$

Hence if χ is any character of S_{mn} then

$$\mathrm{Defres}_{S_n}^{\mathrm{sgn}_{S_m}} \chi = \eta\, \mathrm{Defres}_{S_n}(\chi\, \mathrm{sgn}_{S_{mn}})$$

where $\eta = 1_{S_n}$ if m is even and $\eta = \mathrm{sgn}_{S_n}$ if m is odd. Theorem 7.15.6 of [19] (which is written in the language of skew Schur functions) can be restated in character theoretic terms as follows: if λ/μ is a skew-partition of mn then $\chi^{\lambda/\mu}\, \mathrm{sgn}_{S_{mn}} = \chi^{\lambda'/\mu'}$. Therefore, by Theorem 1.5, we have

$$\mathrm{Defres}_{S_n}^{\mathrm{sgn}_{S_m}}(\chi^{\lambda'/\mu'}) = \eta(g) a_{\lambda'/\mu',\gamma},$$

as required. □

It would also have been possible to prove Proposition 6.1 directly from Proposition 4.1, by reasoning along the same lines as the proof of Theorem 1.5 in §5.

We now turn our attention to the case where $\vartheta = \chi^{(a,1^b)}$ for a hook partition $(a, 1^b)$. For convenience, we use the language of skew shapes (see, for example, [22]). A *skew shape* is a finite subset κ of $\mathbb{N} \times \mathbb{N}$ that is convex with respect to the partial order $\leq_p$ defined by $(i, j) \leq_p (i', j')$ if and only if $i \leq i'$ and $j \leq j'$. We identify the skew-partition λ/μ in which λ has t parts with the skew shape $\{(i, j) : 1 \leq j \leq t, \mu_i + 1 \leq j \leq \lambda_i\}$. We define $\chi^\kappa = \chi^{\lambda/\mu}$ and $\varepsilon_n(\kappa) = \varepsilon_n(\lambda/\mu)$.

Suppose κ is a non-empty skew shape. Then we define the *initial box* of κ to be (i_κ, j_κ) where $(i_\kappa, j_\kappa) \in \kappa$ and, for any $i \leq i_\kappa$ and $j \geq j_\kappa$, if $(i, j) \in \kappa$ then $i = i_\kappa$ and $j = j_\kappa$. Similarly, we define the *terminal box* of κ to be (k_κ, ℓ_κ) where $(k_\kappa, \ell_\kappa) \in \kappa$ and, for any $k \geq k_\kappa$ and $\ell \leq \ell_\kappa$, if $(k, \ell) \in \kappa$ then $k = k_\kappa$ and $\ell = \ell_\kappa$. The initial and terminal boxes exist due to the convexity of κ. For all $(i, j) \in \kappa$, $i_\kappa \leq i \leq k_\kappa$ and $\ell_\kappa \leq j \leq j_\kappa$.

We use the term *border strip* in this context to mean a connected skew shape which contains no 2×2 square. We say that D is a *border-strip n-diagram* if D is a finite set of disjoint border strips, each of length n, such that $\bigcup D$ is a skew shape. We say that D is a *horizontal border-strip n-diagram* if whenever (i_ρ, j_ρ) is the initial box of some border strip $\rho \in D$, we have $(i, j_\rho) \notin \bigcup D$ for all $i < i_\rho$. Similarly, D is a *vertical border-strip n-diagram* if whenever (k_ρ, ℓ_ρ) is the terminal box of some border strip $\rho \in D$, we have $(k_\rho, \ell) \notin \bigcup D$ for all $\ell < \ell_\rho$.

We define a relation $\mathcal{R}$ on the set of border strips of length n by $(\rho_1, \rho_2) \in \mathcal{R}$ if ρ_1 and ρ_2 are disjoint border strips of length n and there exist $z \in \rho_1$ and $w \in \rho_2$ such that $z <_p w$. Observe that when $n \geq 2$ the relation $\mathcal{R}$ is not transitive. We can now state the following combinatorial definition, which is illustrated in Figure 5 overleaf.

Definition 6.2. Let κ be a skew shape of size mn, let a, b be positive integers such that $a + b = m$, and D and E be two border-strip n-diagrams such that $\kappa = \bigcup(D \cup E)$. We say that (D, E) is an $(a, 1^b)$-*like border-strip n-diagram of shape κ* if the following conditions are satisfied:

(1) $|D| = a$ and $|E| = b + 1$;
(2) $D \cap E = \{\sigma\}$ where σ is a border strip of length n which contains the initial box of κ;
(3) D is a horizontal border-strip n-diagram, and E is a vertical border-strip n-diagram;
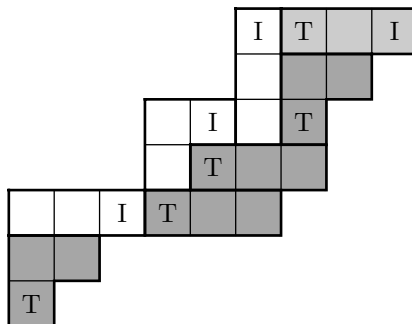(4) there do not exist $\rho_D \in D$ and $\rho_E \in E$ such that $(\rho_E, \rho_D) \in \mathcal{R}$.

FIGURE 5. A $(4, 1^4)$-like border-strip 3-diagram of shape $(9, 8, 7, 7, 6, 2, 1)/(5, 5, 3, 3)$. Border strips in the horizontal border-strip 3-diagram are shown in light grey and white, with their initial boxes labelled I; border strips in the vertical border-strip 3-diagram are shown in light grey and dark grey with their terminal boxes labelled T.

We denote the set of $(a, 1^b)$-like border-strip n-diagrams of shape κ by $\mathcal{B}_{a,b}^\kappa$.

We note that, by a simple counting argument, whenever $(D, E) \in \mathcal{B}_{a,b}^\kappa$, the elements of D are pairwise disjoint, as well as those of E, and we have $(\bigcup D) \cap (\bigcup E) = \sigma$. Definition 6.2 is inspired by the Littlewood–Richardson Rule (see [8, 2.8.13] or [19, A1.3.3]) in the case of a hook partition $(a, 1^b)$. In this case the rule states that the multiplicity of $\chi^{(a, 1^b)}$ in χ^κ is the number of ways to represent κ as a union of two skew shapes α and β satisfying the following: (1) the size of α is a and the size of β is $b + 1$; (2) $\alpha \cap \beta = \{x\}$ where x is the initial box of κ; (3) α is a horizontal strip and β is a vertical strip; (4) there do not exist $y \in \alpha$ and $z \in \beta$ such that $z <_p y$. Thus

$$(\mathrm{Def}_{S_1}^{\chi^{(a, 1^b)}} \chi^\kappa)(1_{S_1}) = |\mathcal{B}_{a,b}^\kappa|.$$

This is the case $n = 1$ of the following theorem.

Theorem 6.3. *Let κ be a skew shape of size mn. Let a, b be positive integers such that $a + b = m$. Let $g \in S_n$ be an n-cycle. Then*

$$(\mathrm{Defres}_{S_n}^{\chi^{(a, 1^b)}} \chi^\kappa)(g) = \varepsilon_n(\kappa)|\mathcal{B}_{a,b}^\kappa|.$$

We observe that an (m)-like border-strip n-diagram of shape κ may be viewed as an m-border-strip tableau of shape κ and type (n) by labelling the border strips in the unique way so that the condition on row numbers in Definition 1.4 holds. Thus Theorem 6.3 is a generalization of Theorem 1.5 in the case of an n-cycle.

Example 6.4. We compute $(\mathrm{Defres}_{S_3}^{\chi^{(3, 1)}} \chi^{(4, 4, 4)})(g)$ where g is a 3-cycle. There are two $(3, 1)$-like border-strip 3-diagrams of shape $(4, 4, 4)$, namely

$(\{\rho_1, \rho_2, \rho_3\}, \{\rho_3, \rho_4\})$ and $(\{\rho_5, \rho_6, \rho_7\}, \{\rho_7, \rho_8\})$, where ρ_i denotes the border strip consisting of the boxes labelled by i in the tableaux below.

$$
\begin{array}{|c|c|c|c|}
\hline
1 & 2 & 2 & 3 \\
\hline
1 & 2 & 3 & 3 \\
\hline
1 & 4 & 4 & 4 \\
\hline
\end{array}
\ ,\quad
\begin{array}{|c|c|c|c|}
\hline
5 & 6 & 7 & 7 \\
\hline
5 & 6 & 7 & 8 \\
\hline
5 & 6 & 8 & 8 \\
\hline
\end{array}
\ .
$$

Since $\varepsilon_3(4,4,4) = 1$, Theorem 6.3 implies that $(\mathrm{Defres}_{S_3}^{\chi^{(3,1)}} \chi^{(4,4,4)})(g) = 2$.

The proof of Theorem 6.3 requires the following definition, in which we assume that $a + b = m$.

Definition 6.5. Let κ be a skew shape of size mn. Define $\mathcal{D}_{a,b}^{\kappa}$ to be the set of pairs (D, E) where D and E are border-strip n-diagrams such that $\kappa = \bigcup(D \cup E)$ and the following conditions are satisfied:

(1') $|D| = a$ and $|E| = b$;

(3) D is a horizontal border-strip n-diagram, and E is a vertical border-strip n-diagram;

(4) there do not exist $\rho_D \in D$ and $\rho_E \in E$ such that $(\rho_E, \rho_D) \in \mathcal{R}$.

Note that since $|\kappa| = mn$, the border-strips in D and E are necessarily disjoint. We need the following two lemmas on $\mathcal{D}_{a,b}^{\kappa}$; their proofs are given after the proof of Theorem 6.3.

Lemma 6.6. *Let κ be a skew shape of size mn, and let a, b be positive integers such that $a + b = m$. Then*

$$\langle \omega(\mathrm{Res}_{S_m \wr S_n} \chi^{\kappa}), \ \mathrm{Ind}_{S_a \times S_b}^{S_m}(\chi^{(a)} \times \chi^{(1^b)}) \rangle = \varepsilon_n(\kappa)|\mathcal{D}_{a,b}^{\kappa}|.$$

Lemma 6.7. *Let κ be a skew shape of size mn. Let $0 \le a < m$ and $b = m - a$. Then*

$$|\mathcal{D}_{a,b}^{\kappa}| = |\mathcal{B}_{a,b}^{\kappa}| + |\mathcal{B}_{a+1,b-1}^{\kappa}|.$$

Proof of Theorem 6.3. The proof is by induction on b. Recall that g denotes an n-cycle. In the case $b = 0$ we have

$$(\mathrm{Defres}_{S_n}^{\chi^{(m)}} \chi^{\kappa})(g) = a_{\kappa,(n)} = \varepsilon_n(\kappa)|\mathcal{D}_{m,0}^{\kappa}| = \varepsilon_n(\kappa)|\mathcal{B}_{m,0}^{\kappa}|,$$

where the first equality follows from Theorem 1.5 and the second by Lemma 4.2.

For $b > 0$, we combine Proposition 2.6 with Lemmas 6.6 and 6.7, Young's Rule and the inductive hypothesis to get

$$(\mathrm{Defres}_{S_n}^{\chi^{(a,1^b)}} \chi^\kappa)(g) = \langle \omega(\mathrm{Res}_{S_m \wr S_n} \chi^\kappa), \chi^{(a,1^b)} \rangle$$

$$= \langle \omega(\mathrm{Res}_{S_m \wr S_n} \chi^\kappa), \mathrm{Ind}_{S_a \times S_b}^{S_m}(\chi^{(a)} \times \chi^{(1^b)}) - \chi^{(a+1,1^{b-1})} \rangle$$

$$= \varepsilon_n(\kappa)|\mathcal{D}_{a,b}^\kappa| - (\mathrm{Defres}_{S_n}^{\chi^{(a+1,1^{b-1})}} \chi^\vartheta)(g)$$

$$= \varepsilon_n(\kappa)|\mathcal{D}_{a,b}^\kappa| - \varepsilon_n(\kappa)|\mathcal{B}_{a+1,b-1}^\kappa|$$

$$= \varepsilon_n(\kappa)|\mathcal{B}_{a,b}^\kappa|,$$

as required. $\qquad\square$

It remains to demonstrate the truth of the two lemmas.

Proof of Lemma 6.6. Let g denote an n-cycle. In the case $b = 0$,

$$\langle \omega(\mathrm{Res}_{S_m \wr S_n} \chi^\kappa), \ \chi^{(m)} \rangle = (\mathrm{Defres}_{S_n} \chi^\kappa)(g) = a_{\kappa,(n)} = \varepsilon_n(\kappa)|\mathcal{D}_{m,0}^\kappa|$$

by Proposition 2.6, Theorem 1.5 and Lemma 4.2. Similarly, the case $a = 0$ follows using Proposition 6.1:

$$\langle \omega(\mathrm{Res}_{S_m \wr S_n} \chi^\kappa), \ \chi^{(1^m)} \rangle = (\mathrm{Defres}_{S_n}^{\mathrm{sgn}\, S_m} \chi^\kappa)(g) = \varepsilon_n(\kappa)|\mathcal{D}_{0,m}^\kappa|.$$

In the general case, let $\kappa = \lambda/\mu$. By Frobenius reciprocity and Lemma 4.4,

$$\langle \omega(\mathrm{Res}_{S_m \wr S_n} \chi^\kappa), \ \mathrm{Ind}_{S_a \times S_b}^{S_m}(\chi^{(a)} \times \chi^{(1^b)}) \rangle =$$
$$\langle \sum_\tau \omega(\mathrm{Res}_{S_a \wr S_n} \chi^{\tau/\mu}) \times \omega(\mathrm{Res}_{S_b \wr S_n} \chi^{\lambda/\tau}), \chi^{(a)} \times \chi^{(1^b)} \rangle,$$

where the sum is over all partitions τ such that $\mu \subseteq \tau \subseteq \lambda$ and $|\tau/\mu| = an$. Using the two extreme cases, it follows that

$$\langle \omega(\mathrm{Res}_{S_m \wr S_n} \chi^\kappa), \ \mathrm{Ind}_{S_a \times S_b}^{S_m}(\chi^{(a)} \times \chi^{(1^b)}) \rangle = \sum_\tau \varepsilon_n(\tau/\mu)|\mathcal{D}_{a,0}^{\tau/\mu}|\varepsilon_n(\lambda/\tau)|\mathcal{D}_{0,b}^{\lambda/\tau}|$$

$$= \varepsilon_n(\lambda/\mu) \sum_\tau |\mathcal{D}_{a,0}^{\tau/\mu}| |\mathcal{D}_{0,b}^{\lambda/\tau}|$$

$$= \varepsilon_n(\lambda/\mu)|\mathcal{D}_{a,b}^{\lambda/\mu}|$$

as required. $\qquad\square$

The proof of Lemma 6.7 relies upon the following simple result.

Lemma 6.8. *Let ρ_1 and ρ_2 be disjoint border strips, each of length n, such that $(\rho_1, \rho_2) \in \mathcal{R}$. For $t \in \{1, 2\}$, let (i_t, j_t) and (k_t, ℓ_t) be the initial and terminal boxes respectively of ρ_t. Then*

(1) *There exists $(r, s) \in \rho_2$ such that either $r \geq i_1$ and $s > j_1$, or $r > k_1$ and $s \geq \ell_1$.*

(2) *There exists $(t, q) \in \rho_1$ such that either $t < i_2$ and $q \leq j_2$, or $t \leq k_2$ and $q < \ell_2$.*

Proof. We prove the first statement only; the second is entirely analogous. Suppose that $(\rho_1, \rho_2) \in \mathcal{R}$ but there exists no $(r, s) \in \rho_2$ satisfying the stated conditions. Since $(\rho_1, \rho_2) \in \mathcal{R}$, there exist $(a, b) \in \rho_1$ and $(e, f) \in \rho_2$ such that $(a, b) <_p (e, f)$. In particular, $i_1 \leq e$ and $\ell_1 \leq f$, and our assumption implies that $f \leq j_1$ and $e \leq k_1$. Thus (e, f) belongs to the rectangle $[i_1, k_1] \times [\ell_1, j_1]$. The border strip ρ_1 divides its complement in $[i_1, k_1] \times [\ell_1, j_1]$ into two connected components, with (e, f) lying to the south east of ρ_1 (as $(a, b) \in \rho_1$ satisfies $(a, b) <_p (e, f)$).

Let $\tau = \{(c, j_1 + 1) : i_1 \leq c \leq k_1 + 1\} \cup \{(k_1 + 1, d) : \ell_1 \leq d \leq j_1 + 1\}$, and observe that our assumption ensures that $\tau \cap \rho_2 = \varnothing$. The set $\rho_1 \cup \tau$ is the boundary of a certain region Δ. Since $(e, f) \in \Delta$ and ρ_2 does not intersect the boundary, the whole border strip ρ_2 must be contained in Δ and, in particular, $\rho_2 \subset [i_1, k_1] \times [\ell_1, j_1]$. However, as ρ_1 is a border strip of the same length with initial and terminal boxes (i_1, k_1) and (ℓ_1, j_1) respectively, ρ_2 must contain the corners (i_1, k_1) and (ℓ_1, j_1) and hence intersect ρ_1, contrary to our hypotheses. □

This lemma can be used to verify that if D is a horizontal border-strip n-diagram with initial box $(i_\sigma, j_\sigma) \in \sigma \in D$ then $(\sigma, \rho) \notin \mathcal{R}$ for all $\rho \in D$. Indeed, if $(\sigma, \rho) \in \mathcal{R}$ then by Lemma 6.8(i) there exists $(r, s) \in \rho$ such that either $r \geq i_\sigma$ and $s > j_\sigma$, or $r > k_\sigma$ and $s \geq \ell_\sigma$. The 'either' case is impossible because (i_ρ, j_ρ) is the initial box of D. Hence (i_ρ, j_ρ) must lie in $[i_\sigma, r] \times [s, j_\sigma]$, to the south of σ, and so a box of σ lies above (i_ρ, j_ρ), contradicting the horizontality of D. Similarly, if E is a vertical border-strip n-diagram with initial box $(i_\sigma, j_\sigma) \in \sigma \in E$ then $(\rho, \sigma) \notin \mathcal{R}$ for all $\rho \in E$: indeed if $(\rho, \sigma) \in \mathcal{R}$ then by Lemma 6.8(ii) there exists $(t, q) \in \rho$ such that either $t < i_\sigma$ and $q \leq j_\sigma$, or $t \leq k_\sigma$ and $q < \ell_\sigma$. The 'either' case is again ruled out because (i_σ, j_σ) is the initial box of E. Hence

$$(t, q) <_p (k_\sigma, q) \leq_p (k_\sigma, \ell_\sigma),$$

and so (k_σ, q) is a box of E because $\bigcup E$ is convex. But this box lies to the left of the terminal box of σ, a contradiction. We shall use these observations in the proof of Lemma 6.7.

Proof of Lemma 6.7. We construct a bijection

$$f \colon \mathcal{B}_{a,b}^\kappa \sqcup \mathcal{B}_{a+1,b-1}^\kappa \to \mathcal{D}_{a,b}^\kappa.$$

Given $(D, E) \in \mathcal{B}^\kappa_{a,b} \sqcup \mathcal{B}^\kappa_{a+1,b-1}$, let σ denote the unique element of $D \cap E$. For $(D, E) \in \mathcal{B}^\kappa_{a,b}$, we set $f(D, E) = (D, E \setminus \{\sigma\})$, and for $(D, E) \in \mathcal{B}^\kappa_{a+1,b-1}$ we set $f(D, E) = (D \setminus \{\sigma\}, E)$.

Firstly, we verify that $f(D, E) \in \mathcal{D}^\kappa_{a,b}$. Suppose that $(D, E) \in \mathcal{B}^\kappa_{a,b}$ (as the second case is exactly analogous). The conditions (1′), (3), (4) on $(D, E \setminus \{\sigma\})$ follow immediately provided that $\bigcup(E \setminus \{\sigma\})$ is a skew shape. Take $x <_p y <_p z$ with $x, z \in \bigcup(E \setminus \{\sigma\})$. We know y lies in the skew shape $\bigcup E$, so suppose that $y \in \sigma$. Then if $z \in \rho_E \in E \setminus \{\sigma\}$ we have $(\sigma, \rho_E) \in \mathcal{R}$, contrary to condition (4) of Definition 6.2.

To see that f is the bijection we require, we define its inverse map

$$h \colon \mathcal{D}^\kappa_{a,b} \to \mathcal{B}^\kappa_{a,b} \sqcup \mathcal{B}^\kappa_{a+1,b-1}.$$

For $(D, E) \in \mathcal{D}^\kappa_{a,b}$, let σ denote the border strip of $D \cup E$ containing the initial box of κ. Then if $\sigma \in D$ we set $h(D, E) = (D, E \cup \{\sigma\})$ and if $\sigma \in E$ we set $h(D, E) = (D \cup \{\sigma\}, E)$.

We verify that $h(D, E) \in \mathcal{B}^\kappa_{a,b} \sqcup \mathcal{B}^\kappa_{a+1,b-1}$. Suppose $\sigma \in D$. To see that $\bigcup(E \cup \{\sigma\})$ is a skew shape, take $x <_p y <_p z$ with $x, z \in \bigcup(E \cup \{\sigma\})$. If x, z lie in the skew shape $\bigcup E$ then so does y, and similarly if $x, z \in \sigma$ then $y \in \sigma$. Two cases remain. Firstly, if $x \in \rho \in E$ and $z \in \sigma \in D$ then $(\rho, \sigma) \in \mathcal{R}$, contrary to the definition of $\mathcal{D}^\kappa_{a,b}$. Secondly, suppose that $x \in \sigma \in D$ and $z \in \bigcup E$, and, for a contradiction, that $y \in \rho \in D \setminus \{\sigma\}$. Then $(\sigma, \rho) \in \mathcal{R}$, which is impossible by the first observation following Lemma 6.8. (The proof that $D \cup \{\sigma\}$ is a skew shape in the case $\sigma \in E$ is entirely analogous.)

Next we verify that if $\sigma \in D$ then $E \cup \{\sigma\}$ is a vertical border-strip n-diagram. Let (i_σ, j_σ) be the initial box of σ and let $(k_\sigma, \ell_\sigma) \in \sigma$ be the terminal box of σ. Since E is a vertical border-strip n-diagram, it suffices to check firstly that there is no box $(k_\sigma, \ell) \in \rho_E \in E$ for $\ell < \ell_\sigma$, and secondly that, if $\rho \in E$ has terminal box (k_ρ, ℓ_ρ), then $(k_\rho, j) \notin \sigma$ for any $j < \ell_\rho$. The first statement is a consequence of the definition of $\mathcal{D}^\kappa_{a,b}$ since the existence of such a box implies $(\rho_E, \sigma) \in \mathcal{R}$. If the second statement fails with $(k_\rho, j) \in \sigma$, then $(\sigma, \rho) \in \mathcal{R}$ and Lemma 6.8(i) implies that there is a box $(r, s) \in \rho$ with either $r \geq i_\sigma$ and $s > j_\sigma$, or $r > k_\sigma$ and $s \geq \ell_\sigma$. The 'either' case is ruled out because (i_σ, j_σ) is the initial box of σ. In the 'or' case, since $(k_\rho, j) \in \sigma$, we have $k_\sigma \geq k_\rho$, and therefore $r > k_\sigma \geq k_\rho$, contradicting that $(r, s) \in \rho$. (The proof that if $\sigma \in E$ then $D \cup \{\sigma\}$ is a horizontal border-strip n-diagram is simpler, using the definition of $\mathcal{D}^\kappa_{a,b}$ and the fact that (i_σ, j_σ) is the initial box of ϑ.)

Finally, we must check that if $\sigma \in D$ then there do not exist $\rho \in D$ and $\rho' \in E \cup \{\sigma\}$ with $(\rho', \rho) \in \mathcal{R}$. This is true because $(D, E) \in \mathcal{D}^\kappa_{a,b}$, and, by

the observation following Lemma 6.8, for all $\rho \in D$, $(\sigma, \rho) \notin \mathcal{R}$. (Again, the case $\sigma \in E$ is similar).

We have demonstrated that f and h are well-defined, and by their construction the maps are mutually inverse. $\qquad\square$

This completes the proof of Theorem 6.3 on deflation with respect to hook characters. We now give some results on $\mathrm{Defres}_{S_n}^{\vartheta} \chi^{\lambda}$ for an arbitrary irreducible character ϑ.

Proposition 4.1 combined with the Littlewood–Richardson rule yields the following corollary, which gives a useful sufficient condition for the deflation of an irreducible character of S_{mn} to vanish on an n-cycle.

Corollary 6.9. *Let m, $n \in \mathbb{N}$, let λ be a partition of mn. Let β be a partition of m and let $\vartheta = \chi^{\beta}$. Let $g \in S_n$ be an n-cycle. If $(\mathrm{Defres}_{S_n}^{\vartheta} \chi^{\lambda})(g) \neq 0$ then λ has empty n-core and moreover, $\lambda^{(i)} \subseteq \beta$ for each $i \in \{0, \ldots, n-1\}$, where $(\lambda^{(0)}, \ldots, \lambda^{(n-1)})$ is the n-quotient of λ. In this case*

$$(\mathrm{Defres}_{S_n}^{\vartheta} \chi^{\lambda})(g) = \varepsilon_n(\lambda/\varnothing) c_{\lambda^{(0)}\ldots\lambda^{(n-1)}}^{\beta}$$

where $c_{\lambda^{(0)}\ldots\lambda^{(n-1)}}^{\beta}$ denotes a generalized Littlewood–Richardson coefficient.

A related result is Proposition 6.10 below, which gives the degrees of the deflations of the irreducible characters of S_{mn} to S_n. It may be proved in the same way as Equation (2) in §1. Note that the right-hand side equals the generalized Littlewood–Richardson coefficient $c_{\beta\ldots\beta}^{\lambda}$.

Proposition 6.10. *Let m, $n \in \mathbb{N}$. Let β be a partition of m and let $\vartheta = \chi^{\beta}$. Then*

$$\mathrm{Defres}_{S_n}^{\vartheta}(\chi^{\lambda})(1_{S_n}) = \left\langle \mathrm{Ind}_{S_m \times \cdots \times S_m}^{S_{mn}} \chi^{\beta} \times \cdots \times \chi^{\beta}, \chi^{\lambda} \right\rangle$$

for any partition λ of mn. $\qquad\square$

We end with an example showing how Propositions 4.1, 4.5, and 6.10 and Corollary 6.9 may be used to calculate the values of an irreducible character of S_{mn} deflated with respect to an arbitrary character of S_m.

Example 6.11. Let $\vartheta = \chi^{(2,2)}$. We shall find $(\mathrm{Defres}_{S_4}^{\vartheta} \chi^{(6,4,4,2)})(g)$ in the cases where $g \in S_4$ is a transposition or a double transposition.

Firstly take g to be a transposition. By Proposition 4.5 we have

$$(\mathrm{Defres}_{S_4}^{\vartheta} \chi^{(6,4,4,2)})(g) = \sum_{\tau} (\mathrm{Defres}_{S_2}^{\vartheta} \chi^{\tau})(1_{S_2})(\mathrm{Defres}_{S_2}^{\vartheta} \chi^{(6,4,4,2)/\tau})(k)$$

where $k = (12) \in S_2$ and the sum is over all partitions τ of 8. Using Proposition 6.10 on the first term we obtain

$$(\mathrm{Defres}_{S_4}^{\vartheta} \chi^{(6,4,4,2)})(g) = \sum_{\tau} c_{(2,2)(2,2)}^{\tau} (\mathrm{Defres}_{S_2}^{\vartheta} \chi^{(6,4,4,2)/\tau})(k)$$

with the same conditions on the sum. By Proposition 4.1, we need only consider those partitions τ such that $(6, 4, 4, 2)/\tau$ is 2-decomposable. The 2-quotient of $(6, 4, 4, 2)$ is $\big((2, 1), (3, 2)\big)$, so the τ we must consider are the partitions $(6, 1^2)$, $(4, 3, 1)$, $(6, 2)$, $(4, 2^2)$, (4^2), $(3^2, 2)$, $(4, 2, 1^2)$, (2^4), $(3^2, 1^2)$. Calculation shows that $c^{\tau}_{(2,2)(2,2)} = 1$ when $\tau \in P$ where

$$P = \{(4, 3, 1), (4, 2^2), (4^2), (2^4), (3^2, 1^2)\}$$

and that $c^{\tau}_{(2,2)(2,2)}$ is zero in the other four cases. Hence by Proposition 4.1 we have

$$(\mathrm{Defres}^{\vartheta}_{S_4} \chi^{(6,4,4,2)})(g) =$$
$$\sum_{\tau \in P} \varepsilon_2\big((6, 4, 4, 2)/\tau\big) \left\langle \mathrm{Ind}^{S_4}_{H} \chi^{(2,1)/\tau^{(0)}} \times \chi^{(3,2)/\tau^{(1)}}, \chi^{(2,2)} \right\rangle$$

where $H = S_{|(2,1)/\tau^{(0)}|} \times S_{|(3,2)/\tau^{(1)}|}$. The contributions to the sum from the elements of P, in the order given above, are -1, $+2$, $+1$, $+1$ and $+1$ respectively. For example, the 2-quotient of $(6, 4, 4, 2)/(4, 2, 2)$ is

$$\big((2, 1)/(1), (3, 2)/(2, 1)\big)$$

and $\varepsilon_2\big((6, 4, 4, 2)/(4, 2, 2)\big) = 1$, so the contribution from $(4, 2, 2)$ is

$$\left\langle \mathrm{Ind}^{S_4}_{S_2 \times S_2} \chi^{(2,1)/(1)} \times \chi^{(3,2)/(2,1)}, \chi^{(2,2)} \right\rangle = \left\langle \mathrm{Ind}^{S_4}_{1} 1, \chi^{(2,2)} \right\rangle = 2.$$

Therefore $(\mathrm{Defres}^{\vartheta}_{S_4} \chi^{(6,4,4,2)})(g) = 4$.

Similar arguments can be used in the case where g is a double transposition. By Proposition 4.5 we have

$$(\mathrm{Defres}^{\vartheta}_{S_4}\chi^{(6,4,4,2)})(g) = \sum_{\tau}(\mathrm{Defres}^{\vartheta}_{S_2} \chi^{\tau})(k)(\mathrm{Defres}^{\vartheta}_{S_2} \chi^{(6,4,4,2)/\tau})(h)$$

where k and h are transpositions, and the sum is over all partitions τ of 8. Proposition 4.1 and Corollary 6.9 restrict the possible partitions τ to be considered and show that $(\mathrm{Defres}^{\vartheta}_{S_4} \chi^{(6,4,4,2)})(g)$ equals

$$\sum_{\tau \in P} \varepsilon_2\big(\tau/\varnothing\big) c^{(2,2)}_{\tau^{(0)}\tau^{(1)}} \varepsilon_2\big((6, 4, 4, 2)/\tau\big) \left\langle \mathrm{Ind}^{S_4}_{H} \chi^{(2,1)/\tau^{(0)}} \times \chi^{(3,2)/\tau^{(1)}}, \chi^{(2,2)} \right\rangle$$

where

$$P = \{(4, 3, 1), (4, 2^2), (4^2), (2^4), (3^2, 1^2)\}$$

and $H = S_{|(2,1)/\tau^{(0)}|} \times S_{|(3,2)/\tau^{(1)}|}$. The contributions to the sum from the elements of P, in the order given above, are $+1$, $+2$, $+1$, $+1$ and $+1$ respectively. Hence $(\mathrm{Defres}^{\vartheta}_{S_4}\chi^{(6,4,4,2)})(g) = 6$.

7. Symmetric functions

Finally, we discuss the translation of our results into the language of symmetric functions. The following well-known facts can be found in [12, Chapter I]. Let Λ be the ring of symmetric functions with integer coefficients in variables $x_1, x_2, \ldots$, and let $R = \bigoplus_{n \geq 0} \mathcal{C}(S_n)$. There is a well-known canonical ring isomorphism $\mathrm{ch} \colon R \to \Lambda$, where the ring structure on R is given, for $f \in \mathcal{C}(S_m)$ and $g \in \mathcal{C}(S_n)$, by $fg = \mathrm{Ind}_{S_m \times S_n}^{S_{mn}}(f \times g)$. Moreover, the map ch is an isometry with respect to the standard inner products $\langle \cdot, \cdot \rangle$ on R and Λ. If λ/μ is any skew-partition, then $\mathrm{ch}(\chi^{\lambda/\mu}) = s_{\lambda/\mu}$, the skew Schur function corresponding to λ/μ. Furthermore, suppose that β and ν are partitions, with $|\beta| = m$ and $|\nu| = n$. Denote by $s_\nu \circ s_\beta$ the plethysm of s_ν and s_β (see [12, §I.8]). Then

$$(12) \qquad \mathrm{ch}\left(\mathrm{Ind}_{S_m \wr S_n}^{S_{mn}}(\widetilde{(\chi^\beta)^{\times n}}\, \mathrm{Inf}_{S_n}^{S_m \wr S_n} \chi^\nu)\right) = s_\nu \circ s_\beta$$

(see [12, §I.8 and §I.A.6]). By first using Frobenius reciprocity, then the inflation-deflation reciprocity relation of Equation (11), we have

$$\left\langle \mathrm{Ind}_{S_m \wr S_n}^{S_{mn}}(\widetilde{(\chi^\beta)^{\times n}}\, \mathrm{Inf}_{S_n}^{S_m \wr S_n} \chi^\nu), \chi^{\lambda/\mu} \right\rangle = \left\langle \chi^\nu, \mathrm{Defres}_{S_n}^{\chi^\beta} \chi^{\lambda/\mu} \right\rangle$$

for any skew-partition λ/μ with $|\lambda/\mu| = mn$. Comparing with Equation (12) shows that if λ/μ is such a skew-partition then

$$(13) \qquad \mathrm{Defres}_{S_n}^{\chi^\beta} \chi^{\lambda/\mu} = \sum_\nu \langle s_{\lambda/\mu}, s_\nu \circ s_\beta \rangle \chi^\nu$$

where the sum is over all partitions ν of n.

Let $p_l = \sum_i x_i^l$ be the power-sum symmetric function, and write $p_\gamma = p_{\gamma_1} \cdots p_{\gamma_d}$ for any composition $\gamma = (\gamma_1, \ldots, \gamma_d)$. If γ is a composition of n, then $\chi^\nu(g_\gamma) = \langle p_\gamma, s_\nu \rangle$ (see [12, Equation (I.7.8)]). Using Equation (13), we obtain

$$(14) \qquad \begin{aligned} (\mathrm{Defres}_{S_n}^{\chi^\beta} \chi^{\lambda/\mu})(g_\gamma) &= \sum_\nu \langle s_{\lambda/\mu}, s_\nu \circ s_\beta \rangle \chi^\nu(g_\gamma) \\ &= \sum_\nu \langle s_{\lambda/\mu}, s_\nu \circ s_\beta \rangle \langle s_\nu, p_\gamma \rangle = \langle s_{\lambda/\mu}, p_\gamma \circ s_\beta \rangle \end{aligned}$$

where the sums are over all partitions of n.

In the case when $\beta = (m)$, we have $s_\beta = h_m$, where h_m is the complete symmetric function of degree m (see [12, Section I.2]). Thus Equation (14) shows that Theorem 1.5 is equivalent to the identity

$$(15) \qquad \langle s_{\lambda/\mu}, p_\gamma \circ h_m \rangle = a_{\lambda/\mu,\gamma}.$$

In [3, Section 9], Désarménien, Leclerc, and Thibon obtain a formula which implies that

$$s_\mu(p_n \circ h_m) = \sum_\lambda a_{\lambda/\mu,(n)} s_\lambda, \tag{16}$$

where the sum is over the partitions λ of $|\mu| + mn$ such that $\lambda \supseteq \mu$. (Our definition of an m-border-strip tableau of type (n) is equivalent to the definition of a horizontal n-ribbon tableau of weight m in [3].) In fact, the formula in [3] is more general, giving a combinatorial description of $s_\mu(p_n \circ s_\kappa)$ for any partition κ of n. Clearly, Equation (16) is equivalent to Equation (15) in the case $\gamma = (n)$; this leads, after some work, to an alternative proof of Theorem 1.5.

Furthermore, in the special case when $\mu = \varnothing$, a combinatorial description of the left-hand side of Equation (15) is given by Macdonald in [12, §I.8, Example 8]. Using Lemma 4.2, one can see that this description is equivalent to the definition of $a_{\lambda,\gamma}$.

Acknowledgements

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ANTON EVSEEV, SCHOOL OF MATHEMATICS, UNIVERSITY OF BIRMINGHAM, EDGBASTON, BIRMINGHAM B15 2TT, UK,
E-mail address: `a.evseev@bham.ac.uk`

ROWENA PAGET, SCHOOL OF MATHEMATICS, STATISTICS & ACTUARIAL SCIENCE, UNIVERSITY OF KENT, CANTERBURY, KENT CT2 7NF, UK,
E-mail address: `r.e.paget@kent.ac.uk`

MARK WILDON, MATHEMATICS DEPARTMENT, ROYAL HOLLOWAY, UNIVERSITY OF LONDON, EGHAM TW20 0EX, UK,
E-mail address: `mark.wildon@rhul.ac.uk`