A MATRIX TRANSFORM WITH INTERESTING SPECTRAL BEHAVIOUR

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The purpose of this note is to give a family of lower-triangular matrices, having prescribed diagonal entries $\lambda_0, \ldots, \lambda_{n-1}$, and such that their vertical reflections have eigenvalues λ_0 and $\pm\sqrt{\lambda_x \lambda_{n-x}}$ for $1 \le x \le n/2$.

To explain one motivation, let $J(n)$ be the $n \times n$ matrix having 1s on its anti-diagonal and 0s in all other positions. That is, $J(n)_{xy} = [x+y = n-1]$, where (as ever, unless otherwise specified) we number rows and columns of matrices and vectors from 0. The vertical reflection of a matrix H is then $HJ(n)$, and Theorem 1.1 below relates the spectra of H and $HJ(n)$ when H is lower-triangular. We ask, more generally:

Question. *How may the spectra of a lower-triangular matrix H and its vertical reflection $HJ(n)$ be related?*

To give one indication that this question has some depth, in [1] and [2] a different family of lower-triangular matrices H are considered in which the eigenvalues of $HJ(n)$ are $\lambda_0, -\lambda_1, \ldots, (-1)^{n-1}\lambda_{n-1}$. In §2 below we study a family of stochastic examples, also related to [1], but given by the construction in this note.

1. Construction

Fix a field F. All our matrices will have entries in an extension field of F. Given $r \in \mathbf{N}$, let $K(r)$ be the $r \times r$ lower-triangular matrix all of whose entries on or below the diagonal are 1. Our matrices are constructed using parameters m, n, L and v where:

- $m, n \in \mathbf{N}$ with $m \le \lfloor n/2 \rfloor$;
- L is an $m \times m$ lower-unitriangular matrix with entries in F such that every entry in the leftmost column of L is 1, i.e. $L_{x0} = 1$ and $L_{xx} = 1$ for $0 \le x < m$;
- $v \in \mathbb{R}^m$ has leftmost entry 1, i.e. $v_0 = 1$.

Given these data, let $Q_n(L, v)$ be the $n \times n$ matrix with the block structure shown below.

$$\begin{pmatrix} L & 0 \\ v & \\ \vdots & K(n-m) \\ v & \end{pmatrix}$$

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For example, if $n = 10$, $v = (1,2,3,4)$ and L is the 4×4 Pascal's Triangle matrix, then

$$Q_9(L,v) = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 2 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 3 & 3 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 2 & 3 & 4 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 2 & 3 & 4 & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & 2 & 3 & 4 & 1 & 1 & 1 & \cdot & \cdot & \cdot \\ 1 & 2 & 3 & 4 & 1 & 1 & 1 & 1 & \cdot & \cdot \\ 1 & 2 & 3 & 4 & 1 & 1 & 1 & 1 & 1 & \cdot \\ 1 & 2 & 3 & 4 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

where $\cdot$ indicates 0 entries implied by the lower-triangular structure of the Q-matrix. (We use this convention throughout.) Let $H_n(L,v)$ be the transform of the diagonal matrix $\mathrm{Diag}(\lambda_0, \lambda_1, \ldots, \lambda_{n-1})$ by $Q_n(L,v)$, defined so that the eigenvector of $H_n(L,v)$ with eigenvalue λ_y is column y of $Q_n(L,v)$. That is,

$$H_n(L,v) = Q_n(L,v)\mathrm{Diag}(\lambda_0, \lambda_1, \ldots, \lambda_{n-1})Q_n(L,v)^{-1}.$$

We give an example where $H_n(L,v)$ is stochastic in Example 2.2 below.

Theorem 1.1. *The eigenvalues of $H_n(L,v)J(n)$ are λ_0 and $\pm\sqrt{\lambda_x \lambda_{n-x}}$ for $1 \le x < n$.*

To prove this theorem it is most convenient to undo the matrix transform, so that it is applied instead to $J(n)$, by taking the conjugate

$$\begin{aligned} &Q_n(L,v)^{-1}\big(H_n(L,v)J(n)\big)Q_n(L,v) \\ &\qquad = \mathrm{Diag}(\lambda_0, \lambda_1, \ldots, \lambda_{n-1})Q_n(L,v)^{-1}J(n)Q_n(L,v). \end{aligned}$$

Observe that $Q_n(L,v)^{-1}J(n)Q_n(L,v)$ is the matrix representing the involution $J(n)$ in the basis of columns of $Q_n(L,v)$. In the example above, this matrix is

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 1 & 1 & 1 & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & 2 & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & 3 & -6 & 2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & 0 & -3 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & -2 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

Lemma 1.2. *The non-zero entries of $Q_n(L,v)^{-1}J(n)Q_n(L,v)$ lie only in the marked regions in Figure 1.*

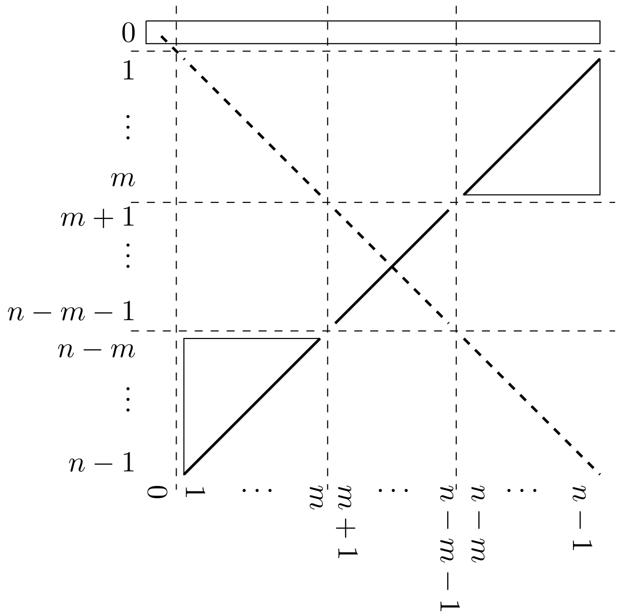

FIGURE 1. The non-zero entries of the matrix $Q_n(L,v)^{-1}J(n)Q_n(L,v)$ in Lemma 1.2 lie in the marked regions. If $m = n/2$ then the middle section is empty and the two triangular regions overlap in their top-right and bottom-left entries, as shown in Figure 2 below; if $m = (n+1)/2$ then the middle section is empty but there is no overlap to consider. The main diagonal and sub-antidiagonal, both important in the proof of Proposition 1.4, are shown by thick lines.

Proof. Let $q^{(y)}$ denote column y of $Q_n(L,v)$. Fix y and let $c = Jq^{(y)}$. By the remark before the proof, column y of $Q_n(L,v)^{-1}J(n)Q_n(L,v)$ records the coefficients expressing c as a linear combination of $q^{(0)}, q^{(1)} \ldots, q^{(n-1)}$. We consider three cases. Note that the second includes columns m and $n-m$ which lie just outside the middle region in Figure 1.

- If $0 \le y < m$ then since v is constant in positions $m, m+1, \ldots, n-1$, we have $c_0 = \ldots = c_{n-m-1}$. Hence $c = c_0 q^{(0)} + v$ where v is a linear combination of columns $q^{(n-m)}, \ldots, q^{(n-1)}$. There is a linear combination w of columns $q^{(n-m)}, \ldots, q^{(n-(y-1))}$ such that $c_0 q^{(0)} + w$ agrees with c in positions $0, 1, \ldots, n-m-1, n-m, \ldots, n-(y-1)$. Since c has the same entry in positions $n-y, \ldots, n-1$, and the same holds for $q^{(0)}$ and all the columns contributing to w, there exists $\alpha \in F$ such that $c_0 q^{(0)} + w + \alpha q^{(n-y)} = c$. Therefore column y of M has its only non-zero entries in row 0 and the rows $n-m, \ldots, n-y$.
- If $m \le y \le n-m$ then $q^{(y)} = (0, \ldots, 0, 1, \ldots, 1)^t$ where y entries are zero and the first 1 is in position y. Hence $c_0 = \ldots = c_{n-y-1} = 1$ and $c_{n-y} = \ldots = c_{n-1} = 0$ and so $q^{(0)} - c = (0, \ldots, 0, 1, \ldots, 1)^t$ where y entries are 1 and the first 1 is in position $n-y$. Since

$m \le n-y \le n-m$, we have $q^{(0)} - c = q^{(n-y)}$ and so $c = q^{(0)} - q^{(n-y)}$. Hence the non-zero entries in column y are 1 in the top row and -1 in row $n-y$.

- If $n - m < y < n$ then, as seen in the second case, $q^{(0)} - c = (0,\ldots,0,1,\ldots,1)^t$ where y entries are 1 and the first 1 is in position $n-y$. Since $y > n-m$, we have $n-y < m$, as shown diagrammatically below where the bottom numbers show positions:

$$q^{(0)} - c = (\overbrace{0,\ldots,0}^{n-y},\ \overbrace{\underset{n-y}{1},\ldots,\underset{m-1}{1}}^{m-(n-y)},\ \overbrace{\underset{m}{1},\ldots,1}^{n-m}).$$

  Now, arguing as in the first case, there exists a linear combination w of columns $q^{(n-y)},\ldots,q^{(m-1)}$ such that $q^{(0)} + w$ agrees with c in positions $0,1,\ldots,m-1$. Moreover, since c has the same entry (namely 0) in positions $m,\ldots,n-1$, there exists $\beta \in F$ such that $q^{(0)} + w + \beta q^{(m)} = c$. Therefore column y of M has its only non-zero entries in row 0 and the rows $n-y,\ldots,m-1,m$. □

We now use this lemma to find the characteristic polynomial of the matrix $\mathrm{Diag}(\lambda_0,\ldots,\lambda_{n-1})Q_n(L,v)^{-1}J(n)Q_n(L,v)$. To make the inductive step as transparent as possible, we isolate it in the following lemma. The hypotheses states that the non-zero elements of the matrix M lie in the marked regions in Figure 2 below. This is the matrix from Figure 1, defined in the extreme cases $n = 2m$ and $n = 2m+1$, with row 0 and column 0 deleted.

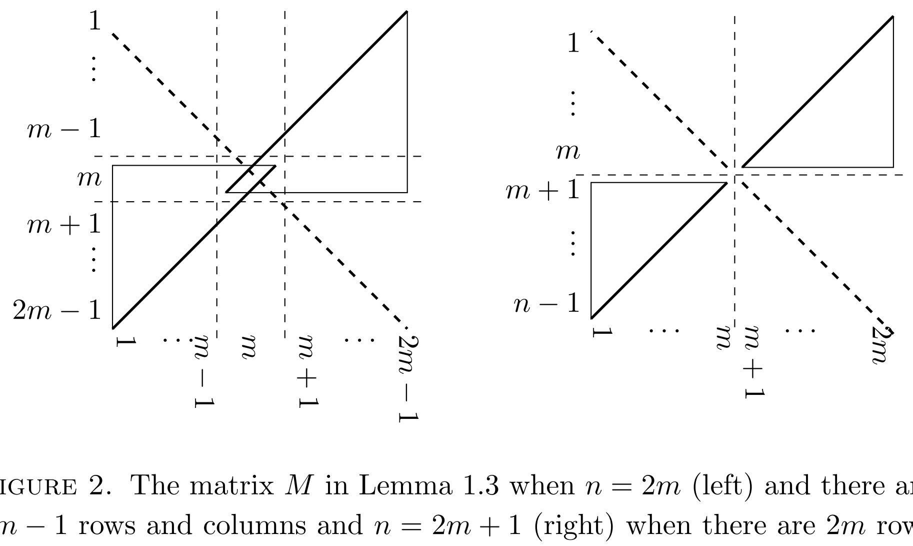

Figure 2. The matrix M in Lemma 1.3 when $n = 2m$ (left) and there are $2m-1$ rows and columns and $n = 2m+1$ (right) when there are $2m$ rows and columns.

Lemma 1.3. *Let $n \ge 2$ and let M be an $(n-1)\times(n-1)$ matrix with rows and columns labelled by $\{1,\ldots,n-1\}$ such that if $M_{xy} \neq 0$ then one of:*

- *$x = y$;*

- $1 \leq x \leq \lfloor n/2 \rfloor$ *and* $x + y \geq n$*;*
- $\lceil n/2 \rceil \leq x \leq n-1$ *and* $x + y \leq n$*.*

Then the determinant of M agrees with the determinant of the matrix obtained from M by setting to zero all entries M_{xy} except the diagonal entries (those with $x = y$) and the antidiagonal entries (those with $x + y = n$).

Proof. If $n = 2$ then the matrix is 1×1 and there is nothing to prove; if $n = 3$ then the matrix is 2×2 with all entries potentially non-zero and again there is nothing to prove.

Suppose that $n \geq 4$. Let σ be a permutation of $\{1, \ldots, n-1\}$ such that $\prod_{x=1}^{n-1} M_{x\sigma(x)} \neq 0$. It suffices to show that $\sigma(x) \in \{x, n-x\}$ for all x. By the hypotheses, $\{\sigma(1), \sigma(n-1)\} = \{1, n-1\}$. If $\sigma(1) = 1$ and $\sigma(n-1) = n-1$ we may delete rows 1 and $n-1$ and columns 1 and $n-1$ to reach an inductive case. Otherwise $\sigma(1) = n-1$ and $\sigma(n-1) = 1$ and again we may delete these rows and columns to reach an inductive case. □

Proposition 1.4. *The matrix* $\mathrm{Diag}(\lambda_0, \ldots, \lambda_{n-1}) Q_n(L, v)^{-1} J(n) Q_n(L, v)$ *has characteristic polynomial* $(z - \lambda_0) \prod_{x=0}^{n/2} (z^2 - \lambda_x \lambda_{n-1})$.

Proof. Let $N = \mathrm{Diag}(\lambda_0, \ldots, \lambda_{n-1}) Q_n(L, v)^{-1} J(n) Q_n(L, v) - zI$ where I is the $n \times n$ identity matrix. Since the only non-zero entry of M in column 0 is $\lambda_0 - z$ in row 0, we have $\det N = (\lambda_0 - z) \det M$ where M is the matrix obtained from N by deleting row 0 and column 0. By Lemma 1.2, increasing the size m of the matrix L defining $Q_n(L, v)$ only introduces new positions where N may be non-zero. Hence we may assume that $n = \lfloor n/2 \rfloor$. But now, by Lemma 1.3, we may assume all the entries not on the main diagonal or sub-antidiagonal of M are zero; equivalently, we are in the case $m = 1$.

If σ is a permutation of $\{1, \ldots, n-1\}$ such that $\prod_{x=1}^{n-1} M_{x\sigma(x)} \neq 0$, then since $m = 1$ we have $\{\sigma(x), \sigma(n-x)\} = \{x, n-x\}$ for each x. Hence, setting $J = \{x : \sigma(x) \neq x\}$, we find that

$$\det M = \sum_{J \subseteq \{1, \ldots, \lfloor n/2 \rfloor\}} (-1)^{|J|} z^{2|J|} \prod_{x \in J} \lambda_x \lambda_{n-x}.$$

This is the expansion of $\prod_{x=0}^{n/2} (\lambda_x \lambda_{n-x} - z^2)$, as required. □

Theorem 1.1 follows at once.

2. The involutive random walk

An interesting family of examples is obtained by taking $L = B(m)$ where $B(m)$ is the $m \times m$ Pascal's Triangle matrix with entries $B(m)_{xy} = \binom{x}{y}$ and $v = v(m)$ where $v(m)_x = \binom{m}{x}$. Thus $v(m)$ is the first m entries in the bottom row of $B(m+1)$ and $H_n\big(B(m), v\big)$ has $B(m+1)$ as its top-left $(m+1) \times (m+1)$-submatrix.

For $d \in \mathbf{N}_0$ and $y \in N_0$ with $d + y < n$, define $\Delta^d \lambda_y = \sum_{k=0}^{d} \binom{d}{k} \lambda_{y+k}$. Given $x < n$, let $x^\dagger = \min(x, m)$. It follows by a routine computation (see

[1, Lemma 7.1] for the case $y < x \le m$) that the matrix $H_n\bigl(B(m), v(m)\bigr)$ has entries

$$H_n\bigl(B(m), v(m)\bigr)_{xy} = \begin{cases} 0 & \text{if } x < y \\ \binom{x^\dagger}{y}\Delta^{x^\dagger - y}\lambda_y & \text{if } y < m \text{ and } x \ge y \\ \lambda_x & \text{if } y \ge m \text{ and } x = y \\ \lambda_y - \lambda_{y+1} & \text{if } y \ge m \text{ and } x > y. \end{cases} \tag{1}$$

Hence $H_n\bigl(B(m), v(m)\bigr)$ is non-negative if and only if $\Delta^d \lambda_y \ge 0$ for all $d, y \in \mathbf{N}_0$ with $d + y \le m$ and $\lambda_m \ge \ldots \ge \lambda_{n-1} \ge 0$. Moreover, all the rows have sum λ_0 so the matrix is stochastic if and only if, in addition, $\lambda_0 = 1$. The vertical reflection $H_n\bigl(B(m), v(m)\bigr)J(n)$ is then the transition matrix of a random walk on $\{0, 1, \ldots, n-1\}$ in which, starting at $x \in \{0, 1, \ldots, n-1\}$, an element $y \in \{0, 1, \ldots, n-1\}$ is chosen with probability $H_n\bigl(B(m), v(m)\bigr)_{xy}$ and the walk then steps to $x^\star$, where $\star$ is the involution on $\{0, 1, \ldots, n-1\}$ defined by $x^\star = n - 1 - x$. This is an instance of the involutive random walk studied in detail in [1]. In particular, by [1, Theorem 1.3], provided $\lambda_1 < 1$, the walk is irreducible, recurrent and ergodic with a unique invariant distribution. By Theorem 1.1 its eigenvalues are 1 and $\pm\sqrt{\lambda_x \lambda_{x^\star+1}}$ for $1 \le x \le \lfloor n/2 \rfloor$.

Remark 2.1. The $m \times m$ matrices $B(m)\mathrm{Diag}(\lambda_0, \ldots, \lambda_{m-1})B(m)^{-1}$ appearing in the top-left corner of $H_n\bigl(B(m), v(m)\bigr)$ are studied in [2], also in the context of stochastic processes. That the entries of $H_n\bigl(B(m), v(m)\bigr)$ are as claimed when $y < x \le m$ also follows from [2, Lemma 2.30].

Example 2.2. If $m = 3$ and $n = 6$ then the matrices $Q_8\bigl(B(4), (1, 4, 6, 4)\bigr)$ and $H_4\bigl(B(4), (1, 4, 6, 4)\bigr)$ are as shown below.

$$\begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & 2 & 1 & \cdot & \cdot & \cdot \\ 1 & 3 & 3 & 1 & \cdot & \cdot \\ 1 & 3 & 3 & 1 & 1 & \cdot \\ 1 & 3 & 3 & 1 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} \lambda_0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \lambda_0 - \lambda_1 & \lambda_1 & \cdot & \cdot & \cdot & \cdot \\ \lambda_0 - 2\lambda_1 + 2\lambda_2 & 2(\lambda_1 - \lambda_2) & \lambda_2 & \cdot & \cdot & \cdot \\ \lambda_0 - 3\lambda_1 + 3\lambda_2 + \lambda_3 & 3(\lambda_1 - 2\lambda_2 + \lambda_3) & 3(\lambda_2 - \lambda_3) & \lambda_3 & \cdot & \cdot \\ \lambda_0 - 3\lambda_1 + 3\lambda_2 + \lambda_3 & 3(\lambda_1 - 2\lambda_2 + \lambda_3) & 3(\lambda_2 - \lambda_3) & \lambda_3 - \lambda_4 & \lambda_4 & \cdot \\ \lambda_0 - 3\lambda_1 + 3\lambda_2 + \lambda_3 & 3(\lambda_1 - 2\lambda_2 + \lambda_3) & 3(\lambda_2 - \lambda_3) & \lambda_3 - \lambda_4 & \lambda_4 - \lambda_5 & \lambda_5 \end{pmatrix}$$

As claimed, the entries are non-negative if and only if $\lambda_0 \ge \lambda_1 \ge \ldots \ge \lambda_5$ and in addition, $\lambda_0 - 2\lambda_1 + \lambda_2 \ge 0$, $\lambda_1 - 2\lambda_2 + \lambda_3 \ge 0$ and $\lambda_0 - 3\lambda_1 + 3\lambda_2 - \lambda_3 \ge 0$. In fact the first of these additional inequalities follows from the final two, so can be omitted. Moreover, the row sums are all λ_0 and so $H_8\bigl(B(4), v(4)\bigr)$ is stochastic if and only if $\lambda_0 = 1$.

2.1. **Eigenvectors when** $m = 1$**.** When $m = 1$ the matrix $H_n\big(B(1), v(1)\big)$ admits the easier definition

$$H_{xy} = \begin{cases} \lambda_x & \text{if } y = x \\ \lambda_y - \lambda_{y+1} & \text{if } y < x \\ 0 & \text{if } y > x. \end{cases}$$

Let $P = HJ(n)$ be the corresponding transition matrix of the involutive random walk. In this case it is possible to write down the eigenvectors of P explicitly. For $x \in \{0, 1, \ldots, n-1\}$, let $e^{(x)} \in \mathbf{R}^n$ be the row vector with 1 in position x.

Proposition 2.3. *The matrix P is diagonalizable with eigenvalues* 1 *and* $\pm\sqrt{\lambda_x \lambda_{x^\star+1}}$ *for* $0 < x \leq n/2$*. If* $x < n/2$ *then the* $\pm\sqrt{\lambda_x \lambda_{x^\star+1}}$ *eigenspace contains*

$$\sqrt{\lambda_{x^\star+1}}\big(e^{(x)} - e^{(x-1)}\big) \pm \sqrt{\lambda_x}\big(e^{(x^\star+1)} - e^{(x^\star)}\big).$$

When $n = 2m$*, there is an eigenvalue* $-\lambda_m$ *and the* $-\lambda_m$*-eigenspace contains* $e^{(m)} - e^{(m-1)}$.

The proof is a fairly routine calculation by considering the action of H on $e^{(x)} - e^{(x-1)}$ and is omitted.

2.2. **Reversibility when** $m = 1$**.** There is also an interesting characterisation of when the walk is reversible. To prove it, we require the version of Kolmogorov's Criterion, as stated below.

Lemma 2.4. *Let P be the transition matrix of a random walk on* $\{0, 1, \ldots, n-1\}$ *such that if* $P_{xy} \neq 0$ *then* $x + y \geq n - 1$*. Suppose that P has a unique invariant distribution. The walk is reversible if and only if*

$$P_{x_0x_1}P_{x_1x_2}\ldots P_{x_{\ell-1}x_0} = P_{x_0x_{\ell-1}}\ldots P_{x_2x_1}P_{x_1x_0}$$

for all distinct $x_0, x_1, \ldots, x_{\ell-1} \in n$ *with* $\ell \geq 3$*, such that* $x_i + x_{i+1} \geq n - 1$ *for all* $i \in n$*, taking indices modulo* ℓ*.*

Proposition 2.5. *The involutive walk with transition matrix P is reversible if and only if*

$$\lambda_1\lambda_{n-1} = \lambda_2\lambda_{n-2} = \ldots = \lambda_{n-1}\lambda_1.$$

Proof. Suppose that the walk is reversible. Let $1 \leq x < (n-1)/2$. Consider the 3-cycle $n-1 \mapsto x \mapsto x^\star \mapsto n-1$ and its reverse $n-1 \mapsto x^\star \mapsto x \mapsto n-1$. Since $x + x^\star = n - 1$, the positions $(x, x^\star)$ and $(x^\star, x)$ are on the anti-diagonal of P, while the other two relevant positions are strictly below the anti-diagonal. By (1) and Kolmogorov's Criterion we have

$$(\lambda_{x^\star} - \lambda_{x^\star+1})\lambda_x(1 - \lambda_1) = (\lambda_x - \lambda_{x+1})\lambda_{x^\star}(1 - \lambda_1).$$

Simplifying, this becomes $\lambda_x\lambda_{x^\star+1} = \lambda_{x+1}\lambda_{x^\star}$ as required.

Conversely, suppose that this condition holds whenever $1 \leq x < n-1$. Let $x_0 \mapsto x_1 \mapsto \ldots \mapsto x_{\ell-1} \mapsto x_0$ be a cycle (with distinct vertices). Denote this cycle by C and let C' denote the reversed cycle $x_0 \mapsto x_{\ell-1} \mapsto \ldots \mapsto x_1 \mapsto x_i$. Throughout, all indices are to be regarded modulo p. Using Lemma 2.4, we may assume that $\ell \geq 3$ and $x_{i-1} + x_i \geq n-1$ for each i; it then suffices to show that the product of transition probabilities is the same for C and C'. Let $I = \{i : x_{i-1} + x_i = n-1\}$ be the set of indices i of those steps $x_{i-1} \mapsto x_i$ that contribute $\lambda_{x_i^\star}$ (rather than $\lambda_{x_i^\star} - \lambda_{x_i^\star+1}$) to the product for C. Now i' appears in the analogous set for C', of those indices i' such that the step $x_{i'+1} \mapsto x_{i'}$ contributes $\lambda_{x_{i'}^\star}$ (rather than $\lambda_{x_{i'}^\star} - \lambda_{x_{i'}^\star+1}$) to the product of C', if and only if $x_{i'} + x_{i'+1} = n-1$, so if and only if $i'-1 \in I$. Let $I-1 = \{i-1 : i \in I\}$ be the set of such indices i'. Observe that if $i \in I \cap (I-1)$ then the step $x_{i-1} \mapsto x_i$ in C is $x_i^\star \mapsto x_i$, and the step $x_{i+1} \mapsto x_i$ in C' is also $x_i^\star \mapsto x_i$. Therefore C has a subcycle of length 2, contrary to our assumption that the vertices are distinct. Hence I and $I-1$ are disjoint. If $i \not\in I \cup (I-1)$ then the step to x_i contributes $\lambda_{x_i^\star} - \lambda_{x_i^\star+1}$ to both products. Hence the two products are equal if and only if

$$\prod_{i \in I} \lambda_{x_i^\star} \prod_{i \in I-1} (\lambda_{x_i^\star} - \lambda_{x_i^\star+1}) = \prod_{i \in I} (\lambda_{x_i^\star} - \lambda_{x_i^\star+1}) \prod_{i \in I-1} \lambda_{x_i^\star}.$$

Equivalently

$$\prod_{i \in I} \lambda_{x_i^\star} (\lambda_{x_{i-1}^\star} - \lambda_{x_{i-1}^\star+1}) = \prod_{i \in I} (\lambda_{x_i^\star} - \lambda_{x_i^\star+1}) \lambda_{x_{i-1}^\star}.$$

If $i \in I$ then $x_{i-1} + x_i = n-1$, and so $x_{i-1}^\star = x_i$. Therefore a final equivalent form is

$$\prod_{i \in I} \lambda_{x_i^\star} (\lambda_{x_i} - \lambda_{x_i+1}) = \prod_{i \in I} (\lambda_{x_i^\star} - \lambda_{x_i^\star+1}) \lambda_{x_i}.$$

This holds term-by-term, since $\lambda_{x_i^\star} \lambda_{x_i+1} = \lambda_{x_i^\star+1} \lambda_{x_i}$. □

We remark that if $\lambda_x = r^x$ then the detailed balance equations have the explicit solution $\pi_x = (r^{x+1} - r^x)/(r^n - 1)$ and, as expected from the theorem just proved, the involutive random walk is reversible. In general the invariant distribution is π where

$$\pi_x = \begin{cases} \dfrac{\lambda_{n-1}(1-\lambda_1)}{1-\lambda_1\lambda_{n-1}} & \text{if } x = 0 \\ \dfrac{(\lambda_{x^\star} - \lambda_{x^\star+1})(1-\lambda_{x+1}) + (\lambda_x - \lambda_{x+1})(1-\lambda_{x^\star})\lambda_{x^\star+1}}{(1-\lambda_x\lambda_{x^\star+1})(1-\lambda_{x+1}\lambda_{x^\star})} & \text{if } 0 < x < n-1 \\ \dfrac{1-\lambda_1}{1-\lambda_1\lambda_{n-1}} & \text{if } x = n-1. \end{cases}$$

The author's proof is an explicit calculation most conveniently performed by computer algebra.

Corollary 2.6. *The involutive walk with transition matrix P is reversible if and only if it has exactly 3 distinct eigenvalues.*

Proof. By Theorem 2.5, the walk is reversible if and only if $\lambda_x \lambda_{x^\star+1}$ is a constant, α say. By Theorem 1.1 this is the case if and only if the eigenvalues of P are 1 and $\pm\sqrt{\alpha}$. □

2.3. **Question.** It would be interesting to know if these results generalize to larger m.

References

1. John R. Britnell and Mark Wildon, *The involutive random walk on total orders and the anti-diagonal eigenvalue property*, (2020, in preparation).
2. Hiroyuki Ochiai, Makiko Sasada, Tomoyuki Shirai, and Takashi Tsuboi, *Eigenvalue problem for some special class of anti-triangular matrices*, arXiv:1403.6797 (March 2014), 23 pages.