MT181 Number Systems: Revision Questions

For further questions and questions on congruences and relations see Sheet 10 and *How to think like a mathematician* by Kevin Houston (Cambridge University Press, 2009), or the other books in the recommended reading.

You should also practice by doing past January Tests.

FUNCTIONS. Let X and Y be sets. Recall from Definitions 2.5 and 2.11 that a function $f : X \to Y$ is

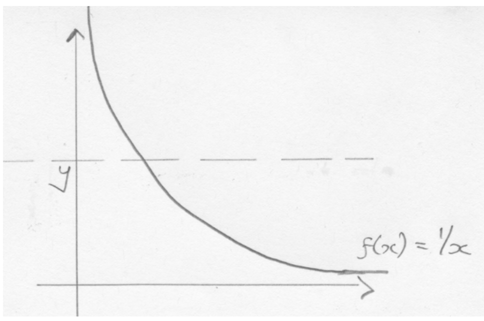
(i) *bijective* if for all $y \in Y$ there exists a unique x such that $f(x) = y$,
(ii) *injective* if $f(x) = f(x') \implies x = x'$,
(iii) *surjective* if for each $y \in Y$ there exists some $x \in X$ such that $f(x) = y$,

Equivalently

- f is injective if for all $y \in Y$ there exists at most one x such that $f(x) = y$,
- f is bijective if and only if f is injective and surjective.

You are welcome to use these as alternative definitions. **Definitions must be accurately stated to get marks in an exam.**

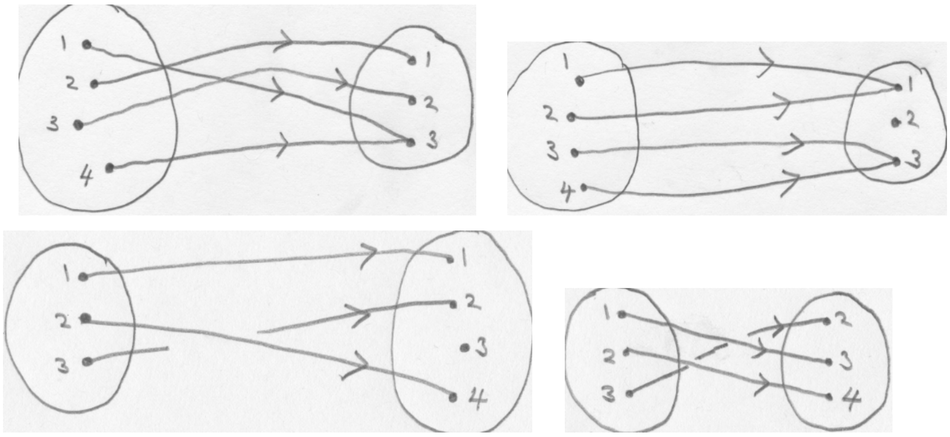
These properties can be recognized from the graph of a function. For example, let $\mathbb{R}^{>0} = \{x \in \mathbb{R} : x > 0\}$ and consider $f : \mathbb{R}^{>0} \to \mathbb{R}^{>0}$ defined by $f(x) = 1/x$. The graph is shown below.



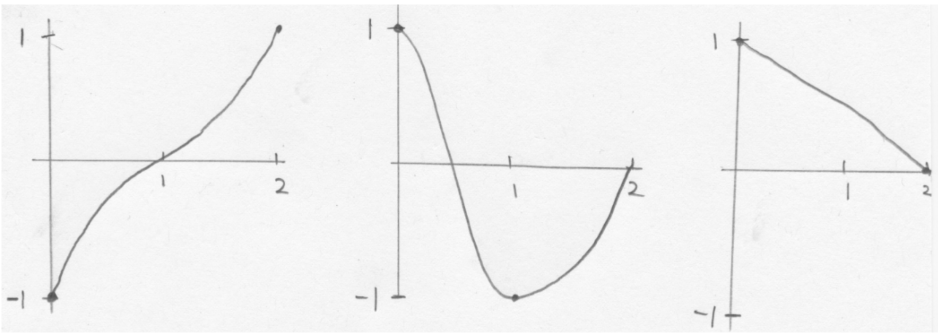
Since for each $y \in \mathbb{R}^{>0}$ the horizontal line of height y meets the graph at a unique point, the function is bijective.

If instead we define $g : \mathbb{R}^{>0} \to \mathbb{R}$ by $g(x) = 1/x$ then g has the same graph as f, but g is no longer surjective. For instance, -1 is in the codomain of g and $g(x) \neq -1$ for any $x \in \mathbb{R}^{>0}$. Correspondingly, the horizontal line of height -1 does not meet the graph above.

1. For each of the diagrams below decide whether the function it represents is (1) injective, (2) surjective, (3) bijective.



2. Let $X = \{x \in \mathbb{R} : 0 \le x \le 2\}$ and let $Y = \{y \in \mathbb{R} : -1 \le y \le 1\}$. The graphs below show three functions $f : X \to Y$. Decide for each each graph whether the function it shows is (1) injective, (2) surjective, (3) bijective.



3. Let $X = \{x \in \mathbb{R} : x \neq -1\}$ and let $Y = \{y \in \mathbb{R} : y \neq 0\}$. Let $g : X \to Y$ be the function defined by

$$g(x) = \frac{1}{x+1}$$

Show that g is bijective and find a formula for $g^{-1} : Y \to X$.

Complex Numbers.

4. (a) Write $-1 - i$ in polar and exponential forms.

(b) Let $\phi = \tan^{-1} 2$. Plot $1+2i$, $2+i$, $-2+i$ and $-1-2i$ on an Argand diagram, and convert these numbers to polar form, writing your answers in terms of ϕ and multiples of π.

(c) Let $z = \frac{1}{2} - i\frac{\sqrt{3}}{2}$. Write z in polar and exponential forms.

(d) What are $\mathrm{Arg}(i)$ and $\mathrm{Arg}(-i)$? Put i and $-i$ in exponential form.

(e) Convert $\mathrm{e}^{-\pi i/6}$ to Cartesian form.

5. Draw the sets $\{z \in \mathbb{C} : |z| = 2\}$ and $\{w \in \mathbb{C} : |w - 2| = 2\}$ on the same Argand diagram.

6. Let T be the set of $z \in \mathbb{C}$ such that $|z| = 1$ and $0 \leq \operatorname{Arg} z \leq \pi/2$. Draw T on an Argand diagram.

7. Solve the following equations, giving the solutions in Cartesian form. Be sure to give all solutions.

(a) $(1 + i)z - (3 + i) = 6$,

(b) $(z + 3)^3 = -2$,

(c) $\exp z = 10 + 10i$,

(d) $z + \bar{z} = 3$

Induction and natural numbers.

8. (a) Prove by induction that

$$\sum_{k=1}^{n} k^2 = \frac{1}{3} n(n + \tfrac{1}{2})(n + 1)$$

for all $n \in \mathbb{N}$.

(b) Prove by induction (or using congruences if you prefer) that $7^n - 1$ is divisible by 6 for all $n \in \mathbb{N}$.

(c) Prove by induction that $1 + 1/2 + \cdots + 1/2^n = 2 - 1/2^n$ for all $n \in \mathbb{N}_0$.

9. Work through Euclid's proof that there are infinitely many primes. Then, a day later, try to write out your own version of the proof, or explain it to a friend.

10. (a) Express 79 in base 2.

(b) Let $0 \leq a, b, c \leq 9$. Show that the base 10 number *abccba* is divisible by 11.

(c) Let $0 \leq a_k \leq 9$ for each $k \in \{0, 1, \ldots, d - 1\}$. Prove that

$$\sum_{k=0}^{d-1} a_k 10^k \text{ is divisible by } 9 \iff \sum_{k=0}^{d-1} a_k \text{ is divisible by } 9.$$

(d) Is 123456789987654321 a multiple of 9?

Propositions.

11. Show that the following propositions formed from propositions P, Q and R are logically equivalent:

(a) $(P \implies Q)$ and $(\neg Q \implies \neg P)$,
(b) $\neg(P \vee Q \vee R)$ and $\neg P \wedge \neg Q \wedge \neg R$,
(c) $P \implies (Q \vee R)$ and $\neg P \vee Q \vee R$.

12. Negate each of the following propositions. Decide which are true and which are false. Justify your answers.

(a) $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(y^2 = x)$
(b) $(\forall x \geq 0)(\exists y \in \mathbb{R})(y^2 = x)$
(c) $(\forall x \in \mathbb{R})(\exists n \in \mathbb{N})(n \geq x)$
(d) $(\exists n \in \mathbb{N})(\forall x \in \mathbb{R})(n \geq x)$

13. Which of the following are tautologies? Justify your answers. (If you use truth tables, make it clear which feature of the truth table you use.)

(a) $(P \iff Q) \iff \big((P \wedge Q) \vee (\neg P \wedge \neg Q)\big)$,
(b) $(P \implies Q) \implies (Q \implies P)$,
(c) $\big(P \implies (Q \implies R)\big) \implies \big(Q \implies (P \implies R)\big)$.

Sets.

14. Let X be the set $\big\{1, \pi, \{42, \sqrt{2}\}, \{\{1,3\}\}\big\}$. Decide which of the following statements are true and which are false.

(i) $\pi \in X$;
(ii) $\{\pi\} \notin X$;
(iii) $\{42, \sqrt{2}\} \in X$;
(iv) $\{1\} \subseteq X$;
(v) $\{1, \sqrt{2}\} \subseteq X$;
(vi) $\{1, \pi\} \subseteq X$;
(vii) $(\exists A \in X)(1 \in A)$;
(viii) $\{1,3\} \subseteq X$;
(ix) $\{1,3\} \in X$
(x) $(\exists A \in X)(\{1,3\} \in A)$;

15. Define subsets X, Y and Z of the natural numbers as follows:

$$X = \{n \in \mathbb{N} : 6 \text{ divides } n-1\}$$
$$Y = \{n \in \mathbb{N} : 3 \text{ divides } n-1\}$$
$$Z = \{n \in \mathbb{N} : 3 \text{ divides } n^2-1.\}$$

(a) Let $n \in \mathbb{N}$. Show that $n \in X \implies n \in Y$.
(b) Is $X \subseteq Y$ true? Is $Y \subseteq X$ true?
(c) Let $n \in \mathbb{N}$. Show that $n \in Y \implies n \in Z$.
(d) Is $X \subseteq Z$ true?