Answers to first half of the vacation algebra problems

1. *For $\alpha \in \mathbb{R}$, let $T_\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map defined by*

$$T_\alpha \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + \alpha y \\ y + \alpha z \\ z \end{pmatrix}$$

(i) Find the matrix representing T_α with respect to the standard basis of $\mathbb{R}^3$ (as both initial and final basis).

Let $\mathcal{B} = (e_1, e_2, e_3)$ be the standard basis of $\mathbb{R}^3$. As $T_\alpha e_1 = e_1$, $T_\alpha e_2 = \alpha e_1 + e_2$ and $T_\alpha e_3 = \alpha e_2 + e_3$, the matrix representing T_α with respect to $\mathcal{B}$ is

$$\begin{pmatrix} 1 & \alpha & 0 \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{pmatrix}.$$

(ii) Find bases $\mathcal{E}$ and $\mathcal{F}$ for $\mathbb{R}^3$ such that the matrix representing T_α with respect to $\mathcal{E}$ as initial basis and $\mathcal{F}$ as final basis is the 3×3 identity matrix.

Being extremely unimaginative I took $\mathcal{E} = \mathcal{B}$ and $\mathcal{F} = (T_\alpha e_1, T_\alpha e_2, T_\alpha e_3)$. It's easy to check that $\mathcal{F}$ is a basis (for any α).

(iii) Prove that T_α is diagonalisable if and only if $\alpha = 0$.

The slick method is to argue that, by part (i), the only eigenvalue of T_α is 1. (Remember that the eigenvalues of an upper triangular matrix can be read off its diagonal.) So the only diagonal matrix that could possible represent T_α is the 3×3-identity matrix. But unless $\alpha = 0$, T_α is clearly not the identity map.

Any argument based on showing that the 1-eigenspace for T_α is only 1-dimensional is also more than acceptable.

2. *Let V and W be a finite dimensional real vector spaces of dimensions m and n respectively and let $T : V \to W$ be a linear map.*

(i) Show that there is a basis of V, $\mathcal{E} = (e_1, \ldots, e_m)$ and a basis of W, $\mathcal{F} = (f_1, \ldots, f_n)$ such that if $r = \operatorname{rank} T$,

$$Te_i = \begin{cases} f_i & : 1 \le i \le r \\ 0 & : r < i \le m \end{cases}.$$

[*Hint: Adapt the proof of the rank-nullity theorem.*]

Let $(v_1, \ldots, v_k)$ be a basis of $\ker T$. Extend this in any way to a basis of V, $\mathcal{E} = (u_1, \ldots, u_r, v_1, \ldots, v_k)$, where $r + k = m$. Then, by the proof of the rank-nullity theorem, $(Tu_1, \ldots, Tu_r)$ is a basis of $\operatorname{im} T$. Extend this to a basis of W, $\mathcal{F} = (Tu_1, \ldots, Tu_r, w_1, \ldots, w_s)$, where $r + s = n$.

(ii) Suppose now that $\mathcal{E}'$ is a basis of V and $\mathcal{F}'$ is a basis of W. Let A be the matrix representing T with respect to $\mathcal{E}'$ as initial basis and $\mathcal{F}'$ as final basis. Show that there exist invertible matrices P and Q such that

$$QAP^{-1} = J(r)$$

where $J(r)$ is the $n \times m$-matrix satisfying

$$J(r)_{ij} = \begin{cases} 1 & : i = j \text{ and } 1 \le i \le r \\ 0 & : \text{otherwise} \end{cases}.$$

The matrix representing T with respect to $\mathcal{E}$ as initial basis and $\mathcal{F}$ as final basis is $J(r)$. (This is a straight-forward check.)

Let P be the matrix of the identity transformation with respect to $\mathcal{E}'$ as initial basis and $\mathcal{E}$ as final basis. (So the columns of P are the coefficients expressing each element of $\mathcal{E}'$ in terms of the elements of $\mathcal{E}$.) We write this as $P = {}_{\mathcal{E}}[1_V]_{\mathcal{E}'}$. Similarly let $Q = {}_{\mathcal{F}}[1_V]_{\mathcal{F}'}$. We have

$$J(r) = {}_{\mathcal{F}}[T]_{\mathcal{E}} = {}_{\mathcal{F}}[1_V]_{\mathcal{F}'}\,{}_{\mathcal{F}'}[T]_{\mathcal{E}'}\,{}_{\mathcal{E}'}[1_V]_{\mathcal{E}} = QAP^{-1}.$$

Two remarks: 1) You should be prepared to produce a proof of the rank-nullity theorem without reference to your notes! I find that if I remember to always start 'Take a basis for $\ker T$ ...' the proof comes out fairly automatically.

2) We've shown that given any matrix A there exist invertible matrices R and S such that $RAS = J(r)$ for some r. In lectures you saw a more concrete proof of this result using row and column operations.

3. *(i) Suppose that U and W are vector subspaces of a vector space V. Show that there is a basis of V containing bases for $U \cap W$, U and W.*
[*You may assume that if X is a vector subspace of the vector space Y then any basis of X can be extend to a basis of Y.*]

Let $(e_1, \ldots, e_l)$ be a basis for the vector subspace $U \cap W$. We may extend this basis to a basis $(e_1, \ldots, e_l, f_1, \ldots, f_m)$ of U. Independently we may extend it to a basis $(e_1, \ldots, e_l, g_1, \ldots, g_n)$ of W. We expect that $\mathcal{B} = (e_1, \ldots, e_l, f_1, \ldots, f_m, g_1, \ldots, g_n)$ will be a basis for $U + W$, the vector subspace of V spanned by U and W. The next two paragraphs prove this.

First of all we check that $\mathcal{B}$ spans $U + W$. Given $u \in U$ we can write u as a linear combination of the e_i and f_j. Similarly given $w \in W$ we can write w as a linear combination of the e_i and g_k. So $u + w$ is a linear combination of elements of $\mathcal{B}$.

Linear independence is a bit more fiddly. Suppose that

$$\sum_{i=1}^{l} \lambda_i e_i + \sum_{j=1}^{m} \mu_j f_j + \sum_{k=1}^{n} \nu_k g_k = 0.$$

Subtracting the third sum we find that

$$\sum_{i=1}^{l} \lambda_i e_i + \sum_{j=1}^{m} \mu_j f_j = -\sum_{k=1}^{n} \nu_k g_k \in U \cap W.$$

Accordingly there exist some further scalars ζ_i such that $\sum_k \nu_k g_k = \sum_i \zeta_i e_i$. But together the e_i and g_k form a basis (the basis of W found above), so they are linearly independent. Hence $\nu_k = 0$ for all k. Substituting back we get $\sum_i \lambda_i e_i + \sum_j \mu_j f_j = 0$, and now a similar argument shows that the λ_i and μ_j are also all 0.

If we extend $\mathcal{B}$ to a basis of V, we get a basis of V with the required properties.

(ii) Deduce that $V = U \oplus W$ if and only if $U \cap W = 0$ and $\dim U + \dim W = \dim V$.

Recall that, by definition, V is the direct sum of U and W (written $V = U \oplus W$) if $U \cap W = 0$ and $V = U + W$.

if: Part (i) proves the identity

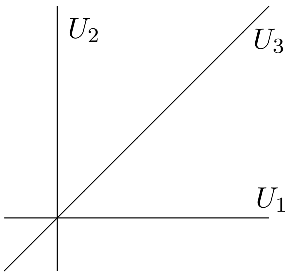
$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W).$$

So if $U \cap W = 0$ and $\dim U + \dim W = \dim V$ then we have $\dim(U + W) = \dim V$, hence $U + W = V$. So $V = U \oplus W$.

only if: Given that $V = U \oplus W$, we have $U \cap W = 0$ and $U + W = V$. So $\dim V = \dim(U + W)$ as required.

(iii) If U_1, U_2 and U_3 are vector subspaces of a vector space V, must there be a basis of V containing bases for each of U_1, U_2 and U_3?

There need not. The diagram below shows 3 different 1-dimensional subspaces of $\mathbb{R}^2$. Any basis of $\mathbb{R}^2$ containing bases for each of U_1, U_2 and U_3 would have (at least) 3 elements, but the plane is 2-dimensional.



This can often be a useful example/counterexample to bear in mind.

4. *Let a, b and c be any 3 complex numbers, and let* $A = \begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix}$.

(i) Let $\omega = \exp(2\pi i/3)$. Show that A has eigenvalues $a + b + c$, $a + \omega b + \omega^2 c$ and $a + \omega^2 b + \omega c$.

One can find eigenvectors for each of the given eigenvalues by inspection:

$$A \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = (a + b + c) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$A \begin{pmatrix} 1 \\ \omega \\ \omega^2 \end{pmatrix} = (a + \omega b + \omega^2 c) \begin{pmatrix} 1 \\ \omega \\ \omega^2 \end{pmatrix}$$

$$A \begin{pmatrix} 1 \\ \omega^2 \\ \omega \end{pmatrix} = (a + \omega^2 b + \omega c) \begin{pmatrix} 1 \\ \omega^2 \\ \omega \end{pmatrix}.$$

(ii) Let α, β and γ be complex numbers. By diagonalising A, or otherwise, give necessary and sufficient conditions for the following system of linear equations over $\mathbb{C}$ to have a solution:

$$\begin{aligned} ax + by + cz &= \alpha \\ cx + ay + bz &= \beta \\ bx + cy + az &= \gamma. \end{aligned}$$

First of all, I apologise for setting such a nasty question. Let P be the matrix whose columns are the eigenvectors of A found above, so

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}$$

and $AP = PD$ where D is the diagonal matrix with entries $a+b+c$, $a+\omega b+\omega^2 c$ and $a+\omega^2 b+\omega c$.

Let $z = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$. Suppose there is a solution $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ to $Av = z$. Then $w = P^{-1}v$ is a solution to $Dw = P^{-1}APw = P^{-1}z$. And conversely, given a solution w to $Dw = P^{-1}z$, we get a solution v to $Av = z$.

Now $P^{-1}z = \frac{1}{3}\begin{pmatrix} \alpha+\beta+\gamma \\ \alpha+\omega^2\beta+\omega\gamma \\ \alpha+\omega\beta+\omega^2\gamma \end{pmatrix}$. So by the previous paragraph, the equation $Av = z$ has a solution if and only if there exists scalars λ, μ and ν such that

$$D\begin{pmatrix} \lambda \\ \mu \\ \nu \end{pmatrix} = \begin{pmatrix} \alpha+\beta+\gamma \\ \alpha+\omega^2\beta+\omega\gamma \\ \alpha+\omega\beta+\omega^2\gamma \end{pmatrix}$$

or equivalently,

$$\begin{aligned}(a+b+c)\lambda &= \alpha+\beta+\gamma, \\ (a+\omega b+\omega^2 c)\mu &= \alpha+\omega^2\beta+\omega\gamma, \\ (a+\omega^2 b+\omega c)\nu &= \alpha+\omega\beta+\omega^2\gamma.\end{aligned}$$

Hence a necessary and sufficient condition for there to be a solution is that

$$\begin{aligned} a+b+c=0 &\implies \alpha+\beta+\gamma=0, \\ a+\omega b+\omega^2 c=0 &\implies \alpha+\omega^2\beta+\omega\gamma=0, \\ a+\omega^2 b+\omega c=0 &\implies \alpha+\omega\beta+\omega^2\gamma=0. \end{aligned}$$

5. *Let $n \geq 1$ and let A be the $n \times n$ matrix such that $A_{ij} = 1$ if $i \neq j$ and $A_{ij} = 0$ if $i = j$,*

$$A = \begin{pmatrix} 0 & 1 & \dots & 1 & 1 \\ 1 & 0 & \dots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \dots & 0 & 1 \\ 1 & 1 & \dots & 1 & 0 \end{pmatrix}.$$

(i) Show that $n-1$ and -1 are eigenvalues of A and find bases of the associated eigenspaces.

Let $(e_1, \dots, e_n)$ be the standard basis of $\mathbb{R}^n$. One finds that $A(e_1 + \dots + e_n) = (n-1)(e_1 + \dots + e_n)$ and $A(e_1 - e_i) = -(e_1 - e_i)$. So the -1 eigenspace is $n-1$ dimensional, with basis $(e_1-e_2, \dots, e_1-e_n)$ and the $(n–1)$-eigenspace is 1-dimensional.

(ii) Find an invertible $n \times n$ matrix P such that

$$P^{-1}AP = \begin{pmatrix} n-1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix}.$$

As usual we take for P a matrix whose columns are a basis of eigenvectors for A.

(iii) (This part may be regarded as optional.) One says that a permutation of the numbers $\{1, 2, \ldots, n\}$ *is a* derangement *if it has no fixed points. So for permutations of* $\{1, 2, 3, 4\}$, $(12)(34)$ *and* (1234) *are derangements, but* (123) *is not, as* $4(123) = 4$.

Let e_n *be the number of derangements of* $\{1, 2, \ldots, n\}$ *that are even permutations and let* o_n *be the number of derangements of* $\{1, 2, \ldots, n\}$ *that are odd permutations. By evaluating the determinant of* A *in 2 different ways prove that*

$$e_n - o_n = (-1)^{n-1}(n-1).$$

(You might first check this holds for small n*, e.g.* $n = 2$*,* $n = 3$*, …)*

If $n = 2$ then there is just 1 derangement, (12) and it is odd. So $e_2 - o_2 = -1$. If $n = 3$ then there are 2 derangements, (123) and (132) both even, so $e_3 - o_3 = 2$. If $n = 4$ there are 9 derangements, the 6 4-cycles and the 3 double-transpositions. So $e_4 - o_4 = 3 - 6 = -3$.

The determinant of a diagonal matrix is just the product of the diagonal entries. Hence $\det A = \det P^{-1}AP = (-1)^{(n-1)}(n-1)$.

On the other hand, we can attempt to evaluate the determinant of A by working directly from the definition:

$$\det A = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) A_{1\,1\sigma} \ldots A_{n\,n\sigma}.$$

If $\sigma \in S_n$ is not a derangement, i.e. there is some m such that $m\sigma = m$, the contribution from σ in this sum is 0 (because $A_{mm} = 0$). If σ is a derangement then its contribution is just $\operatorname{sgn}(\sigma)$. So we count $+1$ for each even derangement and -1 for each odd derangement. Hence

$$(-1)^{(n-1)}(n-1) = \det A = e_n - o_n.$$