Some answers to vacation questions.

If a question is pure bookwork then I have usually left you to look up the proof in notes or a book. I haven't given answers to the optional questions; please ask me if you want any help with them.

1. *True or false: give brief proofs or counterexamples as appropriate.*

(i) *If* $\lim_{x\to a} f(x)$ *exists then* f *is continuous at* a.

(ii) *If* $f : \mathbb{R} \to \mathbb{R}$ *is continuous and* (x_n) *is a sequence converging to* x *as* $n \to \infty$ *then* $f(x_n) \to f(x)$ *as* $n \to \infty$.

(iii) *If* $f : \mathbb{R} \to \mathbb{R}$ *is strictly increasing then* $f(x) \to \infty$ *as* $x \to \infty$.

(iv) *If* $f : [a, b] \to \mathbb{R}$ *is a continuous strictly increasing function with range* $[c, d]$ *then the inverse function* $f^{-1} : [c, d] \to [a, b]$ *is continuous.*

(v) *If* f *and* g *are real valued functions, both differentiable at* $a \in \mathbb{R}$*, then the product function* $(fg)(x) = f(x)g(x)$ *is differentiable at* a.

Parts (ii), (iv) and (v) are true (and the proofs are bookwork), and (i) and (iii) are false. If you think (i) is true you probably aren't using the correct definition of $\lim_{x\to a} f(x)$. For example, if

$$f(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}$$

then $\lim_{x\to 0} f(x)$ exists, with value 0. But f is not continuous at 0, as $\lim_{x\to 0} f(x) \neq f(0)$. For (iii) you might take as a counterexample the inverse function to a strictly increasing function with bounded domain. For instance $f(x) = \tan^{-1} x$ would be a suitable counterexample.

2. (i) *Prove that if* $f : [a, b] \to \mathbb{R}$ *is a continuous function then* f *is bounded and achieves its bounds.*

Bookwork. The quickest proofs use the Bolzano–Weierstrass theorem. For example, suppose f is not bounded. Then there must exist points $x_n \in [a, b]$ such that $|f(x_n)| \geq n$. By Bolzano–Weierstrass, the sequence (x_n) has a convergent subsequence, (x_{n_r}) say, converging to $x \in [a, b]$. By Q1(ii), $f(x_{n_r}) \to f(x)$ as $r \to \infty$. But the sequence $f(x_{n_1}), f(x_{n_2}), \ldots$ can't possibly converge because it isn't even bounded, so we have a contradiction.

(ii) *Suppose that* $g : \mathbb{R} \to \mathbb{R}$ *is differentiable. Let* $c < d$ *and suppose that* $g'(c) < r < g'(d)$. *Show that there exists* $\zeta \in (c, d)$ *such that* $f'(\zeta) = r$. [*Hint: Consider the function* $f(x) = g(x) - rx$ *on the interval* $[c, d]$.]

We argue that f must attain its upper bound somewhere in $[c, d]$, and that at this point, its deriviative will vanish.

By (i), f is bounded and attains its bounds on $[c, d]$. Suppose that $f(x_0) = \sup_{x\in[c,d]} f(x)$. Suppose first of all that $x_0 = c$. Then $(f(x_0 + h) - f(x_0))/h \leq 0$ for all $h > 0$, so $f'(c) \leq 0$. But $f'(c) = g'(c) - r > 0$, so this is impossible. Similarly we rule out $x_0 = d$, so x_0 must lie somewhere in the open interval (c, d). If $h > 0$ we get $(f(x_0 + h) - f(x_0))/h \leq 0$, so $f'(x_0) = \lim_{h\to 0}(f(x_0 + h) - f(x_0))/h \leq 0$. On the other hand, if $h < 0$ then $(f(x_0 + h) - f(x_0))/h \geq 0$, so $f'(x_0) \geq 0$. Hence $f'(x_0) = 0$. So at x_0 we have $g'(x_0) = r$, as required.

(iii) *Deduce Rolle's theorem from part (i).*

Bookwork using the same argument as the end of (ii).

3. *Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ are both solutions to the differential equation $y'' + y = 0$ satisfying the initial conditions $y(0) = 0$, $y'(0) = 1$.*

Show that $h(x) = f(x) - g(x)$ is infinitely differentiable and that $h^{(m)}(0) = 0$ for all $m \geq 0$. By bounding the error term in Taylor's theorem, prove that $h(x) = 0$ for all $x \in \mathbb{R}$. Conclude that $f(x) = g(x) = \sin x$.

By hypothesis, both f and g are twice differentiable. Using the relations $f'' = -f$ and $g'' = -g$, one can see (by induction) that in fact f and g are infinitely differentiable. Hence $h = f - g$ is also infinitely differentiable. Similarly one proves by induction that $h^{(m)}(0) = 0$ for all $m \geq 0$.

Fix some $x > 0$. By Taylor's Theorem (with one version of the remainder term), for any $m \geq 0$, there exists $u \in (0, x)$ such that

$$h(x) = \sum_{r=0}^{m-1} \frac{h^{(r)}(0)}{r!} x^r + \frac{h^{(m)}(u)}{m!} x^m = \frac{h^{(m)}(u)}{m!} x^m.$$

Set

$$K = \max\Big(\sup_{u \in [0,x]} h(u), \sup_{u \in [0,x]} h'(u) \Big).$$

(The suprema must exist as h and h' are continuous functions on the closed interval $[0, x]$.) By the previous equation,

$$|h(x)| \leq \frac{K x^m}{m!}.$$

Now as m tends to infinity, the right-hand-side tends to 0 (for example, because $x^m/m!$ is a term in the convergent power series defining the exponential function), so we must have $h(x) = 0$. The proof is similar if $x < 0$.

Finally, $x \mapsto \sin x$ is one solution to the differential equation, with the right initial conditions. So, by the uniqueness result just proved, we must have $f(x) = \sin x = g(x)$.

4. (i) *What does it mean to say that the power series $f(x) = \sum_{m=0}^{\infty} a_m x^m$ has* radius of convergence *R?*

Bookwork. Bear in mind that a power series need not converge at its radius of convergence. For example $\sum_{m=0}^{\infty} x^m$ has radius of convergence 1, but does not converge when $|x| = 1$.

(ii) *Suppose that $R > 0$. Fix $r \in \mathbb{R}$ such that $0 < r < R$. Show that the partial sums $\sum_{m=0}^{n} a_m x^m$ converge uniformly to f on $[-r, r]$ as $n \to \infty$. Deduce that f is continuous for all $x \in (-R, R)$.*

To show uniform convergence of the partial sums we must show that

$$\lim_{n \to \infty} \sup_{x \in [-r,r]} \left| f(x) - \sum_{m=0}^{n} a_m x^m \right| = 0.$$

More informally, this says that error in approximating f by taking a partial sum of its defining series can be made as small as we like, for all $x \in [-r, r]$ at once. We estimate this error by

$$\left| f(x) - \sum_{m=0}^{n} a_m x^m \right| = \left| \sum_{m=n+1}^{\infty} a_m x^m \right| < \sum_{m=n+1}^{\infty} |a_m x^m|.$$

The series $\sum_{m=0}^{\infty} |a_m x^m|$ is convergent (as a power series converges absolutely within its radius of convergence). Hence as n tends to infinity, its tail, $\sum_{m=n+1}^{\infty} |a_m x^m|$ tends to 0. So by the comparison test, $|f(x) - \sum_{m=0}^{n} a_m x^m|$ tends to 0 as $n \to \infty$.

Now use the theorem that a uniform limit of continuous functions is continuous to argue that f is continuous on $[-r, r]$. Given any $x \in (-R, R)$ we may choose r so that $x \in [-r, r] \subset (-R, R)$, so this shows that f is in fact continuous on all of $(-R, R)$.

Remark: The reason for first working in the closed intervals is that f might not converge *uniformly* on the whole open interval $(-R, R)$. For example, this is the case for $\sum_{m=0}^{\infty} x^m$.

5. [Taken from Q2 2002 analysis]. (i) *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable at a and $g : \mathbb{R} \to \mathbb{R}$ is differentiable at $f(a)$. Prove that $g(f(x))$ is differentiable at a with derivative $g'(f(a))f'(a)$.*

Bookwork.

(ii) *Consider the function f defined by*

$$f(x) = \begin{cases} x^2 \cos(1/x) & : x \neq 0 \\ 0 & : x = 0 \end{cases} .$$

Prove that f is differentiable for all x but f' is not continuous at $x = 0$. (Note that by 2(ii), f' does satisfy the intermediate value property.)

If $x \neq 0$ then we know $x \mapsto 1/x$ is differentiable, with derivative $-1/x^2$. We also know that $x \mapsto x^2$ and $x \mapsto \cos x$ are differentiable, and what their derivatives are. So by the chain and product rules, f is differentiable at x, with deriviative

$$f'(x) = 2x \cos(1/x) + \sin(1/x).$$

If $x = 0$ then we must work from first principles. For $h \neq 0$ we have

$$\frac{f(0+h) - f(0)}{h} = h \cos(1/h)$$

which tends to 0 as $h \to 0$ (for example, because $-h < h \cos 1/h < h$ for all $h \neq 0$). So f is differentiable at 0, and $f'(0) = 0$.

Let $x_n = 1/(\pi/2 + 2n\pi)$. We have

$$f'(x_n) = 2x \cos(\pi/2 + 2n\pi) + \sin(\pi/2 + 2n\pi) = +1.$$

Hence $\lim_{n\to\infty} f'(x_n) = +1$, which is not equal to $f'(\lim_{n\to\infty} x_n) = f'(0) = 0$. Therefore f' is not continuous at 0.

6. ($\star$) Draw a map of the analysis course.

Possibly a valuable exercise, but obviously of no use unless you do it yourself.