Some analysis questions to motivate vacation revision.

You should regard any question marked ($\star$) or ($\star\star$) as optional. These are harder questions that are included in the hope that they may be found interesting; you should not worry if you find them difficult.

1. True or false: (give brief proofs or counterexamples as appropriate)

(i) A convergent sequence is bounded.

(ii) A bounded sequence is convergent.

(iii) If the sequence (a_n) does not tend to infinity then there is a constant K such that $|a_n| < K$ for all $n \geq 1$.

(iv) If X and Y are non-empty sets of real numbers and $x < y$ for all $x \in X$ and $y \in Y$ then $\sup X$ and $\inf Y$ exist and $\sup X \leq \inf Y$.

(v) A subsequence of a convergent sequence is convergent, and has the same limit as the original sequence.

(vi) If $a_n > 0$ for all n and $a_n \to 0$ as $n \to \infty$ then $1/a_n \to \infty$ as $n \to \infty$.

2. (i) Let (a_n) be a bounded monotone sequence of real numbers. Prove that (a_n) is convergent.

(ii) Let $b > 1$ be a fixed real number. We define a sequence (a_n) inductively by taking

$$a_0 = b, \quad a_{n+1} = \frac{a_n}{2} + \frac{b}{2a_n} \quad \text{for } n \geq 0.$$

Prove that (a_n) converges. Show that if $\beta = \lim_{n\to\infty} a_n$ then $\beta > 0$ and $\beta^2 = b$.

(iii) ($\star$) Show that if $a_n \geq 1$ and $|a_n - \beta| < \epsilon$ then $|a_{n+1} - \beta| < \epsilon^2/2$. Deduce that if for our initial guess a_0 we pick the natural number whose square is nearest to b then $|a_n - \beta| < 1/2^{2^n-1})$. Can $a_n = \beta$ for any n?

[*Before the days of pocket calculators, people were forced to use algorithms such as the one above if they wanted to calculate square roots accurately. The rapid convergence of this algorithm was therefore of some practical importance. To understand why it works so well, try reading about the Newton–Raphson method.*]

3. (i) State and prove the Bolzano–Weierstrass Theorem concerning sequences of real numbers.

(ii) What does it mean to say that a sequence is a *Cauchy sequence*? Prove that a sequence of real numbers is a Cauchy sequence if and only if it converges.

(iii) Deduce that if the series $\sum_{r=1}^{\infty} a_r$ converges absolutely then it converges. Give an example to show that the converse of this result is false.

4. [Based on Q1 1999 Mods Analysis.] Let (a_n) be a sequence of real numbers. What is meant by the statement that (a_n) is *convergent*?

Let (a_n) and (b_n) be sequences converging to the limits l and m respectively. Show that:

(i) The sequence $(a_n + b_n)$ converges to $l + m$.

(ii) The sequence $(a_n b_n)$ converges to lm.

(iii) If $a_n \leq b_n$ for all n then $l \leq m$.

Give an example to show that if $a_n < b_n$ for all n then it is not neccesarily true that $l < m$.

Now suppose that $l = 0$. Define a new sequence (c_n) by $c_n = \frac{1}{n}\sum_{r=1}^{n} a_r$. Show that (c_n) also converges to 0.

5. [Based on Q4 2001 Mods Analysis] State the comparison test and derive the integral test for a series of real numbers $\sum_{r=1}^{\infty} a_r$.

Prove that the series $\sum_{r=1}^{\infty} r^{-\alpha}$ converges if $\alpha > 1$ and diverges if $\alpha \leq 1$. Determine whether the following series converge or diverge:

$$\sum_{r=2}^{\infty} \frac{1}{r \log r}, \quad \sum_{r=1}^{\infty} \frac{1}{r} \sin \frac{1}{r}.$$

Further questions for enthusiasts.

6. Suppose that the power series $\sum_{n=0}^{\infty} a_n x^n$ converges for all x such that $|x| < 1$. Let $c > 1$ be a fixed real number. Prove that there exists some $N \in \mathbb{N}$ (depending on c) such that $|a_n| \leq c^n$ for all $n \geq N$.

7. Show that given a divergent series $\sum_{r=1}^{\infty} a_r$ with positive terms there exists a sequence of positive real numbers b_r such that

(i) $\sum_{r=1}^{\infty} b_r$ diverges

(ii) $\lim_{r \to \infty} \frac{b_r}{a_r} = 0$.

[*Hint: show first that there are numbers* $N_1 < N_2 < \ldots$ *with the property that for every* r, $\sum_{n=N_r}^{N_{r+1}-1} a_n > 1$.] Formulate an analogous result for convergent series.

8. Three questions connecting countability with analysis.

(i) ($\star$) Is it possible to draw uncountably many circles in the plane, with no 2 circles intersecting, and no circle lying entirely within another?

(ii) ($\star$) Is there an uncountable family of subsets of $\mathbb{N}$ with the property that any 2 sets in this family have finite intersection? [*Hint: one solution uses the fact that given any* $r \in \mathbb{R}$ *there is a sequence of rational numbers that converges to* r.]

(iii) ($\star\star$) Is it possible to draw uncountably many non-intersecting 'Y' shapes in the plane?

9. ($\star$) Let $\alpha > 0$ be an irrational number. Show that given $N > 0$ there exist natural numbers m and n such that $n \leq N$ and

$$\left| \alpha - \frac{m}{n} \right| < \frac{1}{n^2}$$

[*Hint: let* $\{x\}$ *denote the fractional part of* $x \in \mathbb{R}$. *Apply the pigeonhole principle to the* $N+1$ *numbers* $0, \{\alpha\}, \{2\alpha\}, \ldots, \{N\alpha\}$. *For more about the pigeonhole principle see* `www.cut-the-knot.org/do_you_know/pigeon.shtml`.] Deduce that for any $\epsilon > 0$ there exist points $(m, n) \in \mathbb{N} \times \mathbb{N}$ lying within a distance ϵ of the line $y = \alpha x$.

10. ($\star$) There are 10 pirates who have recently acquired a bar of gold. The leader, number 1, must propose a way to divide up the bar. For instance he might say 'I'll take 1/2 of it, and the remainder will be divided equally amongst the rest of you'. A vote is then taken. If the leader gets *half or more* of the votes (the leader getting one vote himself), the bar is so divided. Otherwise he is made to walk the plank by his dissatisfied subordinates, and number 2 takes over, with the same responsibility to propose an acceptable division.

Assuming that the pirates are all greedy, untrustworthy, and capable mathematicians, what happens?