AN INVOLUTIVE INTRODUCTION TO SYMMETRIC FUNCTIONS

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Preface

In Autumn 2015 I gave a 10 lecture course on symmetric functions at Royal Holloway, University of London, following a slightly unconventional path that emphasised bijective and involutive proofs. This is an expanded version of the lecture notes. Subsections marked $\star$ are not logically essential. A recurring theme is the combinatorial and algebraic meaning of the transition matrices between the various bases of the ring of symmetric functions. For example:

Elementary to monomial	Gale–Ryser Theorem	§1.4 and §5.2
Schur to monomial	Kostka Numbers	(1.11) and Q24
Schur to power sum	Symmetric group characters	§5.5
Power sum to monomial	Polya's Cycle Index Theorem	§5.7

Outline. In §1 the families of elementary, complete homogeneous and power sum symmetric functions are defined. Schur functions are defined combinatorially, using semistandard tableaux, and shown to be symmetric by the Bender–Knuth involution. Motivation comes from combinatorial results including the Gale–Ryser Theorem, MacMahon's Master Theorem, and the Cycle Index Formula for the symmetric group. The ring of symmetric functions is defined formally and then shown to be an inverse limit of the graded rings of symmetric polynomials.

In §2 the Jacobi–Trudi Identity is proved 'by sufficiently general example' using a special case of an involution due to Lindström and Gessel–Viennot. This is now a standard proof: it may be found in [14, §7.16] or [13, §4.5].

In §3 we switch focus to antisymmetric polynomials, and present the elegant involutive proofs in [8] of the Pieri, Young and Murnaghan–Nakayama Rules using Loehr's abacus model. A textbook account may be found in [9]. These are results on the Schur polynomials, defined as a quotient of two antisymmetric determinants, so do not obviously relate to the Schur functions already defined. In §4 we establish the equivalence of the two definitions using the Lascoux–Schützenberger involution (originally defined in [7]).

In §5 we unify the results so far using the Hall inner product and the ω involution on the ring of symmetric functions. We then prove the key properties of the characteristic isometry, relating class functions of symmetric groups and symmetric polynomials and apply it to prove Pólya's Cycle Index Formula.

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The Jacobi–Trudi formula and the Lascoux–Schützenberger involution extend to Schur functions labelled by skew partitions. This generality adds considerably to their utility, but appeared excessive in the early lectures. It was assumed in the final lecture, where I used the Lascoux–Schützenberger involution to give a proof of the Littlewood–Richardson Rule. The proof is given as a series of questions in §6, starting with Question 26. Most of these questions are on well-known results or proofs: possible exceptions are Question 7 (generalized derangements), Question 13 (an easy way to go wrong in the proof of the Jacobi–Trudi Identity) and Question 21 (an involutive proof of the Murnaghan–Nakayama Rule). Hints, references or solutions for all the questions are given in the final section.

Comments. Comments are very welcome. In particular I gratefully acknowledge extremely detailed comments and corrections from Darij Grinberg, sent over several years, to earlier versions of these notes. I also thank Eoghan McDowell for helpful comments and corrections. Of course I have full responsibility for the remaining errors.

1. INTRODUCTION: DEFINITIONS AND MOTIVATION

Lecture 1

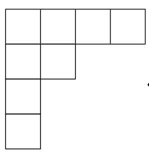
1.1. **Preliminary definitions.** The following definitions are standard. Note in particular that partitions have infinitely many parts, all but finitely many of which are zero. Permutations act on the right.

Compositions and partitions. A *composition of* $n \in \mathbf{N}_0$ is an infinite sequence $(\alpha_1, \alpha_2, \ldots)$ such that $\alpha_i \in \mathbf{N}_0$ for all i and $\alpha_1 + \alpha_2 + \cdots = n$. (The term 'weak composition' is also used in the literature.) The sequence elements are called *parts*. By this definition, there is a unique composition of 0, which we denote $\varnothing$. If $\alpha \neq \varnothing$, let $\ell(\alpha)$ be the maximum r such that $\alpha_r \neq 0$ and let $\ell(\varnothing) = 0$. If $\ell(\alpha) \leq N$ then we write $\alpha = (\alpha_1, \ldots, \alpha_N)$. A composition α is a *partition of* n if $\alpha_1 \geq \alpha_2 \geq \ldots$ and $\alpha_1 + \alpha_2 + \cdots = n$. We write $\alpha \models n$ to indicate that α is a composition of n and $\lambda \vdash n$ to indicate that λ is a partition of n.

It is often convenient to use exponents to indicate multiplicities of parts: thus $(m^{a_m}, \ldots, 2^{a_2}, 1^{a_1})$ denotes the partition with exactly a_j parts equal to j, for each $j \in \{1, \ldots, m\}$. For example,

$$(4, 4, 2, 1, 1, 1, 0, \ldots) = (4, 4, 2, 1, 1, 1) = (4^2, 3^0, 2^1, 1^3).$$

The *Young diagram* $[\lambda]$ of a partition λ is the set $\{(i, j) : i, j \in \mathbf{N}, 1 \leq i \leq \ell(\lambda), 1 \leq j \leq \lambda_i\}$. We represent Young diagrams by diagrams, such as the one shown below for $(4, 2, 1, 1)$:

 .

The *conjugate* of a partition λ is the partition λ' defined by

$$\lambda'_j = |\{i : \lambda_i \geq j\}|.$$

By definition $\lambda'_j = r$ if and only if λ has exactly r parts of size j or more, or equivalently, if and only if column j of $[\lambda]$ has length r. Thus $[\lambda']$ is obtained from $[\lambda]$ by reflection in the main diagonal. In particular $\lambda'' = \lambda$ for any partition λ.

Orders on partitions. The *dominance order* on partitions of n, denoted $\trianglerighteq$, is defined by $\lambda \trianglerighteq \mu$ if and only if

$$\lambda_1 + \cdots + \lambda_c \geq \mu_1 + \cdots + \mu_c$$

for all $c \in \mathbf{N}$. It is a partial order: for example $(3,1,1,1)$ and $(2,2,2)$ are incomparable. It is usually the correct order to use when working with symmetric functions or symmetric groups. The *lexicographic order*, denoted $>$, is defined by $\lambda > \mu$ if and only if $\lambda_1 = \mu_1, \ldots, \lambda_{c-1} = \mu_{c-1}$ and $\lambda_c > \mu_c$ for some c. It is a total order refining the dominance order. (The word 'refining' is mathematically correct, but may give the wrong impression: more precise information comes from using the dominance order.)

Symmetric group. Let $\mathrm{Sym}(X)$ denote the symmetric group on a set X. Let $\mathrm{Sym}_n = \mathrm{Sym}(\{1,\ldots,n\})$. In these notes permutations act on the right. For example, the composition of the cycles (12) and (123) in the symmetric group Sym_3 is $(12)(123) = (13)$, and the image of 1 under the permutation (12) is $1(12) = 2$.

1.2. **Gale–Ryser Theorem.** We begin with a combinatorial result. An $a \times b$ matrix X with entries in $\mathbf{C}$ has *row sums* $(\sum_{j=1}^{b} X_{ij})_{i\in\{1,\ldots,a\}}$ and *column sums* $(\sum_{i=1}^{a} X_{ij})_{j\in\{1,\ldots,b\}}$. A matrix is 0-1 if all its entries are either 0 or 1.

Theorem 1.1 (Gale–Ryser)**.** *Let λ and μ be partitions of n. There is a* 0-1 *matrix X with row sums λ and column sums μ if and only if $\lambda' \trianglerighteq \mu$.*

For example, take $\lambda = (4,1,1)$, so $\lambda' = (3,1,1,1)$. If $\mu = (2,2,1,1)$ then a suitable 0-1 matrix is

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

while if $\mu = (2,2,2)$ then no such matrix exists.

By Question 2, we have $\lambda' \trianglerighteq \mu$ if and only if $\lambda \trianglelefteq \mu'$. The condition in the Gale–Ryser Theorem is therefore symmetric with respect to λ and μ, as expected.

Proof that the Gale–Ryser condition is necessary. Let $a = \ell(\lambda)$ and let $b = \ell(\mu)$. Suppose that X is an $a \times b$ matrix with row sums λ and column sums μ. Think of $(\lambda_1, \ldots, \lambda_a)$ as the sizes of a vehicles, and $(\mu_1, \ldots, \mu_b)$ as the sizes of b families. Imagine putting someone from family j into vehicle i if and only if $X_{ij} = 1$. We then have a way to dispatch the families so that no members of the same family share a vehicle. Consider the $\mu_1 + \cdots + \mu_k$ people in the first k families. They occupy at most k seats in each vehicle, so, looking at the first k columns of the Young diagram of $[\lambda]$, we see that $\lambda'_1 + \cdots + \lambda'_k \geq \mu_1 + \cdots + \mu_k$, as required. $\square$

We will shortly use this result to relate elementary symmetric functions and monomial symmetric functions. We later use symmetric functions to prove that the Gale–Ryser condition is sufficient: see Lemma 5.4. Constructive proofs also exist: see Question 11.

1.3. **The ring of symmetric functions Λ.** Given a composition α of n, define the monomial $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots$ of *degree n*. These monomials should be regarded purely formally for the moment. For each $n \in \mathbf{N}_0$ define $\widehat{\mathbf{C}}[x_1, x_2, \ldots]_n$ to be the $\mathbf{C}$-vector space of all formal infinite $\mathbf{C}$-linear combinations of the x^α, for α a composition of n. In symbols

$$\widehat{\mathbf{C}}[x_1, x_2, \ldots]_n = \{ \sum_{\alpha \models n} c_\alpha x^\alpha : c_\alpha \in \mathbf{C} \}.$$

Note that $x^\varnothing = 1$, so $\widehat{\mathbf{C}}[x_1, x_2, \ldots]_0 = \mathbf{C}$. Let

$$\widehat{\mathbf{C}}[x_1, x_2, \ldots] = \bigoplus_{n=0}^{\infty} \widehat{\mathbf{C}}[x_1, x_2, \ldots]_n.$$

Then $\widehat{\mathbf{C}}[x_1, x_2, \ldots]$ is a ring, graded by degree, with product defined by formal bilinear extension of $x^\alpha x^\beta = x^{\alpha+\beta}$, where $\alpha + \beta$ is the composition defined by $(\alpha + \beta)_i = \alpha_i + \beta_i$ for each $i \in \mathbf{N}$. (The hat is included to distinguish $\widehat{\mathbf{C}}[x_1, x_2, \ldots]$ from the polynomial ring in the variables $x_1, x_2 \ldots$, which it properly contains. See §1.7 below.)

The symmetric group $\mathrm{Sym}(\mathbf{N})$ acts as a group of unital $\mathbf{C}$-algebra isomorphisms of $\widehat{\mathbf{C}}[x_1, x_2, \ldots]$ by $x_i \sigma = x_{i\sigma}$. That is, the action of each $\sigma \in \mathrm{Sym}(\mathbf{N})$ sends the unit element 1 is sent to itself, and its action respects products and all well-defined infinite linear combinations. We define

$$\Lambda = \widehat{\mathbf{C}}[x_1, x_2, \ldots]^{\mathrm{Sym}(\mathbf{N})}$$

to be the set of fixed points. Thus $f \in \Lambda$ if and only if $f\sigma = f$ for all $\sigma \in \mathrm{Sym}(\mathbf{N})$. If $f, g \in \widehat{\mathbf{C}}[x_1, x_2, \ldots]$ then $(fg)\sigma = (f\sigma)(g\sigma)$. Hence Λ is a subring of $\widehat{\mathbf{C}}[x_1, x_2, \ldots]$, again graded by degree. Setting $\Lambda_n = \Lambda \cap \widehat{\mathbf{C}}[x_1, x_2, \ldots]_n$, an equivalent statement of this grading is that $\Lambda = \bigoplus_{n=0}^{\infty} \Lambda_n$. The elements of Λ_n are called *symmetric functions of degree n*. For example, a basis for Λ_3 is

$$\begin{aligned}
&x_1^3 + x_2^3 + \cdots, \\
&x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + \cdots, \\
&x_1 x_2 x_3 + x_1 x_2 x_4 + \cdots.
\end{aligned}$$

Remark 1.2. Symmetric polynomials can be constructed more simply by taking the fixed points for the symmetric group S_n permuting the variables in $\mathbf{C}[x_1, \ldots, x_n]$. They are recovered by specializing symmetric functions at $x_{n+1} = \ldots = 0$: see §1.6. Remark 5.9 gives one reason why it is necessary to work with infinitely many variables. See also §1.7 for the construction of Λ using inverse limits.

Given a partition λ of $n \in \mathbf{N}_0$, let

$$\mathrm{mon}_\lambda = \sum_{\substack{\alpha \models n \\ P(\alpha)=\lambda}} x^\alpha,$$

where $P(\alpha)$ is the partition obtained by rearranging the parts of α into weakly decreasing order. We say that mon_λ is the *monomial symmetric function* labelled by λ. (The notation m_λ is more usual, but monomial symmetric functions are not so important, and m is a useful letter to have free.) For example, the basis of Λ_3 above is $\mathrm{mon}_{(3)}$, $\mathrm{mon}_{(2,1)}$, $\mathrm{mon}_{(1,1,1)}$. More generally,

$$\{\mathrm{mon}_\lambda : \lambda \vdash n\}$$

is a basis for Λ_n. Thus the dimension of Λ_n is the number of partitions of n. Note that $\mathrm{mon}_\varnothing = 1$ and (this is almost a triviality), the coefficient of x^λ in mon_λ is 1.

1.4. **Elementary symmetric functions.** While easily defined, the monomial basis is not the most useful for most computations. We shall now define several different bases, starting with the elementary symmetric functions, and see the first signs that the transition matrices between these bases are of combinatorial and algebraic interest.

For $n \in \mathbf{N}_0$ the *elementary symmetric function of degree* n is defined by $e_n = \mathrm{mon}_{(1^n)}$. For example $e_2 = x_1x_2 + x_1x_3 + x_2x_3 + x_1x_4 + \cdots$. For μ a composition of n define $e_\mu = e_{\mu_1} \ldots e_{\mu_{\ell(\mu)}}$. The e_μ are called *elementary symmetric functions*.

Lemma 1.3. *For μ a partition of n define coefficients $N_{\lambda\mu}$ by*

$$e_\mu = \sum_{\lambda \vdash n} N_{\lambda\mu} \mathrm{mon}_\lambda.$$

Then $N_{\lambda\mu}$ is the number of 0-1 *matrices of size $\ell(\lambda) \times \ell(\mu)$ with row sums λ and column sums μ.*

Proof. The coefficient of mon_λ in e_μ is the coefficient of x^λ in e_μ. This coefficient is the number of ways to choose one monomial from each bracket in the product

$$e_\mu = \prod_{j=1}^{\ell(\mu)} e_{\mu_j} = \prod_{j=1}^{\ell(\mu)} (x_1x_2 \ldots x_{\mu_j} + \cdots)$$

so that the product of all $\ell(\mu)$ monomials is x^λ. These choices correspond to 0-1 matrices Z of size $\ell(\lambda) \times \ell(\mu)$ with row sums λ and column sums μ: if we choose $x_{k_1} x_{k_2} \ldots x_{k_{\mu_j}}$ from the jth bracket we set $Z_{k_1 j} = \cdots = Z_{k_{\mu_j} j} = 1$, and all other entries in column j of Z to be 0. Thus column j of Z has sum μ_j and corresponds to a choice contributing 1 to the power of each $x^{k_1}, \ldots, x^{k_j}$. Conversely, such a matrix clearly determines a corresponding choice of monomials. $\square$

For example,

$$\begin{aligned} e_{(2,2)} &= (x_1x_2 + x_1x_3 + x_2x_3 + x_1x_4 + \cdots)(x_1x_2 + x_1x_3 + x_2x_3 + x_1x_4 + \cdots) \\ &= \mathrm{mon}_{(2,2)} + 2\mathrm{mon}_{(2,1,1)} + 6\mathrm{mon}_{(1,1,1,1)}. \end{aligned}$$

The 0-1 matrices with row sums $(2, 1, 1)$ and column sums $(2, 2)$ corresponding to the summand $2\mathrm{mon}_{(2,1,1)}$ are

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Proposition 1.4. *The set $\{e_\mu : \mu \vdash n\}$ is a basis for Λ_n.*

Proof. The 'necessity' direction of the Gale–Ryser Theorem says that if $N_{\lambda\mu} \neq 0$ then $\lambda' \trianglerighteq \mu$. Hence

$$e_\mu = \sum_{\substack{\lambda \\ \lambda' \trianglerighteq \mu}} N_{\lambda\mu} \mathrm{mon}_\lambda = \sum_{\nu \trianglerighteq \mu} N_{\nu'\mu} \mathrm{mon}_{\nu'} = \sum_{\nu \trianglerighteq \mu} X_{\nu\mu} \mathrm{mon}_{\nu'} \tag{1.1}$$

where $X_{\nu\mu} = N_{\nu'\mu}$. Note that $X_{\nu\mu} = 0$ unless $\nu \trianglerighteq \mu$. By Question 1(a), $X_{\mu\mu} = N_{\mu'\mu} = 1$ for all partitions μ. Hence, when partitions are ordered lexicographically the matrix X is unitriangular and so invertible. $\square$

For example, we have $e_{(n)} = \mathrm{mon}_{(1^n)}$ and

$$(e_{(4)}, e_{(3,1)}, \ldots, e_{(1^4)}) = (\mathrm{mon}_{(1^4)}, \mathrm{mon}_{(2,1,1)}, \ldots, \mathrm{mon}_{(4)})X$$

where

$$X = \begin{pmatrix} 1 & 4 & 6 & 12 & 24 \\ \cdot & 1 & 2 & 5 & 12 \\ \cdot & \cdot & 1 & 2 & 6 \\ \cdot & \cdot & \cdot & 1 & 4 \\ \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}.$$

Since X is upper triangular, so is X^{-1}. Thus $(X^{-1})_{\nu\mu} \neq 0$ only if $\nu \trianglerighteq \mu$. Multiplying (1.1) by X^{-1} we obtain $\sum_{\mu \trianglerighteq \kappa} (X^{-1})_{\mu\kappa} e_\mu = \mathrm{mon}_{\kappa'}$. Hence

$$\mathrm{mon}_\lambda = \sum_{\mu \trianglerighteq \lambda'} (X^{-1})_{\mu\lambda'} e_\mu. \tag{1.2}$$

In particular, $\mathrm{mon}_\lambda = e_{\lambda'} + g$ where g is a linear combination of e_μ for $\mu \triangleright \lambda'$. By the first equality in (1.1), when g is expressed in the monomial basis, each summand mon_ν with a non-zero coefficient satisfies $\nu' \triangleright$

λ', or equivalently, $\nu \lhd \lambda$. We have therefore recovered the traditional algorithm for writing a symmetric function f as a linear combination of elementary symmetric functions: find the lexicographically greatest monomial summand in f with a non-zero coefficient, say mon_λ, with coefficient c, then replace f with $f - ce_{\lambda'}$, and repeat.

Remark 1.5. Working with the dominance order rather than the lexicographic order gives more information. For example, to write $\mathrm{mon}_{(2,2,2)}$ as a linear combination of elementary symmetric functions, we take $g = \mathrm{mon}_{(2,2,2)} - e_{(3,3)}$ in the first step. In the lexicographic order we have $(4,1,1) > (2,2,2)'$, but since $(4,1,1) \not\trianglerighteq (2,,2,2)'$, it follows from (1.2) that, when expressed in the basis of elementary symmetric functions, g does not involve $e_{(4,1,1)}$.

1.5. **Complete homogeneous symmetric functions.** The *complete homogeneous symmetric function of degree n* is defined by $h_n = \sum_{\lambda \vdash n} \mathrm{mon}_\lambda$. By definition, $h_n = \sum_{\alpha \models n} x^\alpha$. For example, **Lecture 2**

$$h_1 = \mathrm{mon}_{(1)} = x_1 + x_2 + \cdots$$
$$h_2 = \mathrm{mon}_{(2)} + \mathrm{mon}_{(1^2)} = x_1^2 + x_2^2 + \cdots + x_1x_2 + x_1x_3 + x_2x_3 + \cdots .$$

For μ a composition of n define $h_\mu = h_{\mu_1} \ldots h_{\mu_{\ell(\mu)}} \in \Lambda_n$. The h_μ are called *complete homogeneous symmetric functions*.

We shall relate the h_μ to the elementary symmetric functions defined in §1.4. For this it will be useful to work in the ring $\widehat{\mathbf{C}}[x_1, x_2, \ldots][[t]]$ of formal power series $\sum_{n=0}^\infty f_n t^n$ with coefficients in $\widehat{\mathbf{C}}[x_1, x_2, \ldots]$. A subring is the ring $\Lambda[[t]]$ of formal power series with coefficients in Λ. Let

$$H(t) = \prod_{i=1}^\infty \frac{1}{1 - x_i t} \in \widehat{\mathbf{C}}[x_1, x_2, \ldots][[t]].$$

Observe that

$$H(t) = \prod_{i=1}^\infty (1 + x_i t + x_i^2 t^2 + \cdots) = \sum_{n=0}^\infty h_n t^n.$$

Hence $H(t) \in \Lambda[[t]]$. Let $E(t) = 1/H(t)$. We have

$$E(t) = \prod_{i=1}^\infty (1 - x_i t) = \sum_{n=0}^\infty (-1)^n e_n t^n.$$

Taking the coefficient of t^n in $1 = H(t)E(t)$ we obtain *Newton's Identity*

$$\sum_{k=0}^n (-1)^k e_k h_{n-k} = 0 \quad \text{if } n \in \mathbf{N}. \tag{1.3}$$

Proposition 1.6. *The set $\{h_\mu : \mu \vdash n\}$ is a basis of Λ_n.*

Proof. Newton's identity rearranges to $h_n = \sum_{k=1}^n (-1)^{k-1} e_k h_{n-k}$. Thus $h_n = (-1)^{n-1} e_n + f$ where f is in the polynomial algebra generated by

$e_1, \ldots, e_{n-1}, h_1, \ldots, h_{n-1}$. It follows by induction that $h_n \in (-1)^{n-1} e_n + \mathbf{C}[e_1, \ldots, e_{n-1}]$ and so $h_n = (-1)^{n-1} e_n + f$ where f is a linear combination of elementary symmetric functions e_ν for partitions ν of n; since $\deg f = n$, each ν has at least two parts. Multiplying we find that

$$h_\mu = \prod_{i=1}^{\ell(\mu)} \big((-1)^{\mu_i - 1} e_{\mu_i} + f_i \big)$$

where each f_i is a linear combination of elementary symmetric functions e_ν for partitions ν of μ_i with at least two parts. Therefore

$$h_\mu \in (-1)^{n-\ell(\mu)} e_\mu + \langle e_\nu : \nu \text{ is a proper refinement of } \mu \rangle. \tag{1.4}$$

(A partition ν is a proper refinement of a partition μ if $\nu \neq \mu$ and it is possible to add up parts of ν to obtain μ, using each part of ν exactly once. For example $(5, 3, 1, 1)$ is a proper refinement of $(6, 4)$ since $6 = 5 + 1$ and $4 = 3 + 1$.) The matrix expressing the h_μ in the basis $\{e_\nu : \nu \vdash n\}$ is therefore triangular with entries ± 1 on the main diagonal. $\square$

See Question 5 for further properties of this change of basis matrix.

1.6. **Specializations.** Let $N \in \mathbf{N}$. We define

$$\widehat{\mathrm{ev}}_N : \widehat{\mathbf{C}}[x_1, x_2, \ldots] \to \mathbf{C}[x_1, \ldots, x_N]$$

by linear extension of

$$\widehat{\mathrm{ev}}_N(x^\alpha) = \begin{cases} x^\alpha & \text{if } \ell(\alpha) \leq N \\ 0 & \text{otherwise} \end{cases}$$

to all finite linear combinations of monomials and all well-defined infinite linear combinations. It is clear from the definition of the product in $\widehat{\mathbf{C}}[x_1, x_2, \ldots]$ that $\widehat{\mathrm{ev}}_N$ is a ring homomorphism. Let $\mathrm{ev}_N : \Lambda \to \mathbf{C}[x_1, \ldots, x_N]$ be the restricted map. Thus if λ is a partition of n and $\ell(\lambda) \leq N$ then

$$\mathrm{ev}_N(\mathrm{mon}_\lambda) = \sum_{\substack{\alpha \models n \\ P(\alpha) = \lambda \\ \ell(\alpha) \leq N}} x^\alpha$$

while if $\ell(\lambda) > N$ then $\mathrm{ev}_N(\mathrm{mon}_\lambda) = 0$. (Recall from §1.3 that $P(\alpha)$ is the underlying partition of the composition α.) From this formula we see that the image of ev_N is contained in the ring of *symmetric polynomials in N variables* $\mathbf{C}[x_1, \ldots, x_N]^{\mathrm{Sym}_N}$. Just as Λ, this ring is graded by degree. It is clear from the monomial basis for Λ_n that $\mathrm{ev}_N : \Lambda_n \to \mathbf{C}[x_1, \ldots, x_N]_n^{\mathrm{Sym}_N}$ is always surjective. It is injective if and only if $N \geq n$.

We shall usually write $f(x_1, \ldots, x_N)$ rather than $\mathrm{ev}_N(f)$. For example,

$$\begin{aligned}
e_2(x_1, x_2, x_3) &= \mathrm{ev}_3(x_1x_2 + x_1x_3 + x_2x_3 + x_1x_4 + \cdots) \\
&= x_1x_2 + x_1x_3 + x_2x_3, \\
h_3(x_1, x_2) &= \mathrm{ev}_2(x_1^3 + \cdots + x_1^2x_2 + \cdots + x_1x_2x_3 + \cdots) \\
&= x_1^3 + x_2^3 + x_1^2x_2 + x_1x_2^2, \\
e_3(x_1, x_2) &= \mathrm{ev}_2(x_1x_2x_3 + x_1x_2x_4 + \cdots) \\
&= 0.
\end{aligned}$$

More generally, $\mathrm{ev}_N(e_n) = 0$ whenever $N < n$. We can specialize further by sending each x_i for $i \leq N$ to a given complex number $z_i \in \mathbf{C}$. For example $e_2(1,1,1) = 3$, $h_3(1,1) = 4$ and $e_2(1, \mathrm{i}, -1, -\mathrm{i}) = 0$, being the coefficient of z^2 in $z^4 - 1$.

Note that if $f, g \in \Lambda$ then

$$f = g \iff \mathrm{ev}_N(f) = \mathrm{ev}_N(g) \quad \text{for all } N \in \mathbf{N}. \tag{1.5}$$

Roughly put: a symmetric function is determined by its finite images.

1.7. **Inverse limits***. The previous comment suggests that Λ could somehow be constructed out of the rings $\mathbf{C}[x_1, \ldots, x_N]^{\mathrm{Sym}_N}$. Consider the commutative diagram below.

$$\begin{tikzcd}
 & \Lambda \arrow[d, two heads, "\mathrm{ev}_N"'] \arrow[dr, two heads, "\mathrm{ev}_{N-1}"] \arrow[drrr, two heads, "\mathrm{ev}_1"] & & & \\
{} \arrow[r, two heads, "q_{N+1}"'] & \mathbf{C}[x_1, \ldots, x_N]^{\mathrm{Sym}_N} \arrow[r, two heads, "q_N"'] & \mathbf{C}[x_1, \ldots, x_{N-1}]^{\mathrm{Sym}_{N-1}} \arrow[r, two heads, "q_{N-1}"'] & \cdots \arrow[r, two heads, "q_2"'] & \mathbf{C}[x_1]
\end{tikzcd}$$

The horizontal maps are determined by the commutativity: since ev_N kills x_{N+1}, but not x_N, whereas ev_{N-1} kills both, the map

$$q_N : \mathbf{C}[x_1, \ldots, x_N]^{\mathrm{Sym}_N} \twoheadrightarrow \mathbf{C}[x_1, \ldots, x_{N-1}]^{\mathrm{Sym}_{N-1}}$$

must be defined to be the restriction of the unique $\mathbf{C}$-algebra homomorphism $\mathbf{C}[x_1, \ldots, x_N] \to \mathbf{C}[x_1, \ldots, x_{N-1}]$ such that $x_N \mapsto 0$ and $x_k \mapsto x_k$ for $k < N$. We record the images $\mathrm{ev}_N(f) = f^{(N)}$ of $f \in \Lambda$ as a sequence

$$(f^{(N)})_{N=1}^{\infty} \in \prod_{N=1}^{\infty} \mathbf{C}[x_1, \ldots, x_N]^{\mathrm{Sym}_N}.$$

Since this sequence determines f, the ring Λ is isomorphic to a subring of $\prod_{N=1}^{\infty} \mathbf{C}[x_1, \ldots, x_N]^{\mathrm{Sym}_N}$. Imposing the compatibility condition from the commutative diagram, we can identify this subring explicitly: the degree n component is

$$\big\{(f^{(N)})_{N=1}^{\infty} \in \prod_{N=1}^{\infty} \mathbf{C}[x_1, \ldots, x_N]_n^{\mathrm{Sym}_N} : q_N(f^{(N)}) = f^{(N-1)} \text{ for all } N \geq 2\big\}.$$

This ring is the inverse limit $\varprojlim \mathbf{C}[x_1, \ldots, x_N]^{\mathrm{Sym}_N}$, defined using the maps q_N, taken in the category of graded rings.

Remark 1.7. It is essential to work with an inverse limit of graded rings. To see this, let

$$f^{(N)} = e_1(x_1, \ldots, x_N) + \cdots + e_N(x_1, \ldots, x_N)$$

for each $N \in \mathbf{N}$ and consider $(f^{(N)})_{N=1}^{\infty} \in \prod_{N=1}^{\infty} \mathbf{C}[x_1, \ldots, x_N]^{\mathrm{Sym}_N}$. This sequence satisfies the compatibility condition $q_N(f^{(N)}) = f^{(N-1)}$ but does not correspond to an element of Λ. I am grateful to Darij Grinberg for this example.

Completions. Let R be a ring and let $I_N \trianglelefteq R$ for $N \in \mathbf{N}$ be ideals of R such that $I_N \trianglerighteq I_{N+1}$ for each N. The *completion* of R with respect to $(I_N)_{N=1}^{\infty}$ is $\widehat{R} = \varprojlim R/I_N$, defined as a subring of $\prod_{N=1}^{\infty} R/I_N$ by

$$\widehat{R} = \big\{ (r_N + I_N)_{N=1}^{\infty} : r_N + I_{N-1} = r_{N-1} + I_{N-1} \text{ for all } N \geq 2 \big\}.$$

It is useful to think of $r_N + I_N$ as the 'Nth approximation' of the element $(r_N + I_N)_{N=1}^{\infty} \in \widehat{R}$: since $I_N \trianglerighteq I_{N+1}$, these approximations get better and better as N increases. If R and the ideals I_N are graded then, as suggested by the previous remark, one normally wants the graded completion, defined by $\widehat{R}_n = \varprojlim (R/I_N)_n$ for each n.

Example 1.8.

(1) The ring $\widehat{\mathbf{C}}[x_1, x_2, \ldots]$ defined in §1.3 is isomorphic to the graded completion of the polynomial ring $\mathbf{C}[x_1, x_2, \ldots]$ with respect to the ideals $I_N = \ker \widehat{\mathrm{ev}}_N = \langle x_{N+1}, x_{N+2}, \ldots \rangle$. An eventually constant sequence corresponds to an element of $\mathbf{C}[x_1, x_2, \ldots]$ and a sequence $(f^{(N)})_{N=1}^{\infty}$ where

$$f^{(N)} \in \mathbf{C}[x_1, x_2, \ldots]/I_N \cong \mathbf{C}[x_1, \ldots, x_N]$$

such that each $f^{(N)}$ is invariant under Sym_N and has homogeneous degree n corresponds to an element of Λ_n.

Note that $\widehat{\mathbf{C}}[x_1, x_2, \ldots]^{\mathrm{Sym}(\mathbf{N})} = \Lambda$, whereas $\mathbf{C}[x_1, x_2, \ldots]^{\mathrm{Sym}(\mathbf{N})} = \mathbf{C}$. Thus taking fixed points does not commute with taking completions and there is no sense in which Λ is a completion of the ring $\mathbf{C}[x_1, x_2, \ldots]^{\mathrm{Sym}(\mathbf{N})}$.

(2) The rings of formal power series $\mathbf{C}[[t]]$ and $\Lambda[[t]]$ are also examples of completions. Generally, for any ring S we have $S[[t]] \cong \varprojlim S[t]/\langle t^N \rangle$.

(3) Let p be prime. The ring of p-adic integers $\mathbf{Z}_p$ is isomorphic to the completion $\varprojlim \mathbf{Z}/\langle p^N \rangle$. The similarity to (2) is not coincidental: the rings $\mathbf{Z}_p$ and $F[[t]]$, where F is a field, have many properties in common.

1.8. **Counting compositions***. Let n, $N \in \mathbf{N}_0$. As an easy application of specializations we determine the number $C_n(N)$ of compositions α of n such that $\ell(\alpha) \le N$. Specialize h_n so that $x_i \mapsto 1$ if $i \le N$ and $x_i \mapsto 0$ if $i > N$ to get

$$\sum_{n=0}^{\infty} C_n(N)t^n = \sum_{n=0}^{\infty} h_n(1,\ldots,1)t^n = \frac{1}{(1-t)^N} = \sum_{n=0}^{\infty} \binom{-N}{n}(-t)^n.$$

Hence

$$\begin{aligned} C_n(N) = \binom{-N}{n}(-1)^n &= \frac{(-N)(-N-1)\ldots(-N-n+1)}{n!}(-1)^n \\ &= \frac{N(N+1)\ldots(N+n-1)}{n!} = \binom{N+n-1}{n}. \end{aligned}$$

(This trick of negating the top in a binomial coefficient is often useful when simplifying sums involving binomial coefficients.)

1.9. **MacMahon's Master Theorem***. The duality between the elementary and complete symmetric functions gives a slick proof of MacMahon's Master Theorem (his name for it). The presentation below is based on a question of Alexander Chervov: see MathOverflow 103919. Given a linear map $B : V \to V$, let $\mathrm{Sym}^n B : \mathrm{Sym}^n V \to \mathrm{Sym}^n V$ denote the map induced by B on the space $\mathrm{Sym}^n\, V$ of polynomials of degree n over V. For example, if V has basis u, v then $\mathrm{Sym}^2 V$ has basis u^2, uv, v^2 and we have

$$\mathrm{Sym}^2 \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha^2 & \alpha\beta & \beta^2 \\ 2\alpha\gamma & \alpha\delta + \beta\gamma & 2\beta\delta \\ \gamma^2 & \gamma\delta & \delta^2 \end{pmatrix}.$$

Lemma 1.9. *Let B be a square matrix with entries in a polynomial ring $\mathbf{C}[x_1,\ldots,x_N]$. Working in $\mathbf{C}[x_1,\ldots,x_N][[t]]$, we have*

$$\det(I - Bt)^{-1} = \sum_{n=0}^{\infty} \mathrm{Tr}\,\mathrm{Sym}^n(Bt).$$

Proof. Suppose that B is an $M \times M$-matrix. It suffices to prove the identity with the indeterminates $x_1, \ldots, x_N$ specialized to N arbitrary complex numbers. Let B' be the specialized matrix and let $\theta_1, \ldots, \theta_M \in \mathbf{C}$ be its eigenvalues. By specializing $H(t) = \prod_{i=1}^{\infty}(1 - x_i t)^{-1} = \sum_{n=0}^{\infty} h_n t^n$ at $\theta_1, \ldots, \theta_M$ we get

$$\det(I - B't)^{-1} = \prod_{i=1}^{M} \frac{1}{1 - \theta_i t} = \sum_{n=0}^{\infty} h_n(\theta_1, \ldots, \theta_M)t^n.$$

Now observe that the eigenvalues of $\mathrm{Sym}^n B'$ are exactly the monomials $\theta^\alpha = \theta_1^{\alpha_1} \ldots \theta_M^{\alpha_M}$ for α a composition of n with $\ell(\alpha) \le M$. □

Example 1.10 (Dixon's Identity)**.** We shall use Lemma 1.9 to show that

$$\sum_{k=0}^{2m} (-1)^k \binom{2m}{k}^3 = (-1)^m \frac{(3m)!}{m!^3}.$$

The left-hand side is

$$(\star) \qquad [x^{2m}y^{2m}z^{2m}](x-y)^{2m}(y-z)^{2m}(z-x)^{2m},$$

where square brackets denote taking a coefficient, since if we take x^k from $(x-y)^{2m}$ then we must also take y^k from $(y-z)^{2m}$ and z^k from $(z-x)^{2m}$. Consider the matrix

$$B = \begin{pmatrix} 0 & y & -z \\ -x & 0 & z \\ x & -y & 0 \end{pmatrix}$$

representing a linear transformation in a basis u, v, w of the $\mathbf{C}[x,y,z]$-module $\mathbf{C}[x,y,z]^3$. Let $i + j + k = n$. We have

$$\begin{aligned}(\operatorname{Sym}^n B)(u^i v^j w^k) &= B(u)^i B(v)^j B(w)^k \\ &= (-xv + xw)^i (yu - yw)^j (-zu + zv)^k \\ &= x^i y^j z^k (-v + w)^i (u - w)^j (-u + v)^k.\end{aligned}$$

Hence the coefficient of $x^{2m}y^{2m}z^{2m}$ in $\operatorname{Tr}\operatorname{Sym}^{6m} B$ is the coefficient of $u^{2m}v^{2m}w^{2m}$ in $(-v+w)^{2m}(u-w)^{2m}(-u+v)^{2m}$. The variables have changed, but the coefficient is the same as in $(\star)$. Hence

$$\sum_{k=0}^{2m} (-1)^k \binom{2m}{k}^3 = [x^{2m}y^{2m}z^{2m}] \operatorname{Tr}\operatorname{Sym}^{6m} B.$$

On the other hand, since $\det(I - Bt) = 1 + (xy + yz + zx)t^2$, Lemma 1.9 implies that the right-hand side is

$$\begin{aligned}[x^{2m}y^{2m}z^{2m}t^{6m}] \det(I - Bt)^{-1} &= [x^{2m}y^{2m}z^{2m}t^{6m}](1 + (xy + yz + zx)t^2)^{-1} \\ &= [x^{2m}y^{2m}z^{2m}](-1)^m (xy + yz + zx)^{3m}.\end{aligned}$$

If we take $(xy)^k$ from the multinomial expansion of $(xy + yz + zx)^{3m}$, we must then take $(yz)^{2m-k}$ to get the correct power of y, and then $(zx)^m$ to get a monomial of the correct degree. Therefore the only contribution to the coefficient of $x^{2m}y^{2m}z^{2m}$ comes from taking each of xy, yz and zx exactly m times. Hence the coefficient is $(-1)^m(3m)!/m!^3$, as claimed.

The general result is as follows.

Theorem 1.11 (MacMahon's Master Theorem)**.** *Let A be an $N \times N$ complex matrix and let $B = A \operatorname{diag}(x_1, \ldots, x_N)$. Thus the entries of B lie in*

$\mathbf{C}[x_1, \ldots, x_N]$. *Let* $y_1, \ldots, y_N$ *be* N *further indeterminates. If* $\alpha \models n$ *and* $\ell(\alpha) \leq N$ *then*

$$[x^\alpha t^n] \det(I - Bt)^{-1} = [y^\alpha] \prod_{i=1}^{N} (\sum_{j=1}^{N} A_{ji} y_j)^{\alpha_i}.$$

Proof. By Lemma 1.9, it is equivalent to show that

$$[x^\alpha] \operatorname{Tr} \operatorname{Sym}^n B = [y^\alpha] \prod_{i=1}^{N} (\sum_{j=1}^{N} A_{ji} y_j)^{\alpha_i}.$$

Let $v_1, \ldots, v_N$ be a basis of the $\mathbf{C}[x_1, \ldots, x_N]$-module $\mathbf{C}[x_1, \ldots, x_N]^N$. Since the only variable appearing in the ith column of B is x_i, the left-hand side is the coefficient of v^α in $\prod (Av_i)^{\alpha_i}$. Since $Av_i = \sum_{j=1}^{N} A_{ji} v_j$, this agrees with the right-hand side. □

See Questions 6 and 7 for some further applications of MacMahon's Master Theorem.

1.10. **Power sum symmetric functions.** For $n \in \mathbf{N}$, we define $p_n = \mathrm{mon}_{(n)} = \sum_{i=1}^{\infty} x_i^n \in \Lambda_n$. Let $p_0 = 1$. For α a composition of n define $p_\alpha = p_{\alpha_1} \ldots p_{\alpha_{\ell(\alpha)}}$. We say that p_α is the *power sum symmetric function* corresponding to α. To prove that the power sum symmetric functions labelled by partitions form a basis for Λ we adapt the argument using Newton's Identity seen earlier. **Lecture 3**

Let $Q(t) = \sum_{n=1}^{\infty} p_n t^n / n$. Observe that

$$Q(t) = \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \frac{x_i^n t^n}{n} = -\sum_{i=1}^{\infty} \log(1 - x_i t).$$

Hence

$$\exp Q(t) = \prod_{i=1}^{\infty} \frac{1}{1 - x_i t} = H(t). \tag{1.6}$$

By Question 3(c), we have the identity $tH'(t) = tQ'(t)H(t)$. Taking coefficients of t^n gives the Newton type identity

$$nh_n = \sum_{k=1}^{n} p_k h_{n-k}. \tag{1.7}$$

Hence $p_n = nh_n - \sum_{k=1}^{n-1} p_k h_{n-k}$ for each $n \in \mathbf{N}$. It follows as in §1.5 that $\mathbf{C}[p_1, \ldots, p_n] = \mathbf{C}[h_1, \ldots, h_n]$ for all n and that $\{p_\alpha : \alpha \vdash n\}$ is a basis of Λ_n.

We make the following remarks.

(1) Let $n \in \mathbf{N}$ and let $\Lambda_{\mathbf{Z},n} = \{\sum_{\alpha \vdash n} c_\alpha \mathrm{mon}_\alpha : c_\alpha \in \mathbf{Z}\}$. Since the transition matrix from $\{e_\mu : \mu \vdash n\}$ to $\{\mathrm{mon}_\alpha : \alpha \vdash n\}$ is unitriangular (see Proposition 1.4), we obtain the same set if we

replace each mon_α with e_λ. However, the subgroup

$$\{\sum_{\alpha \vdash n} c_\alpha p_\alpha : c_\alpha \in \mathbf{Z}\}$$

of Λ_n is properly contained in $\Lambda_{\mathbf{Z},n}$. For example, $e_2 = \frac{1}{2}(p_1^2 - p_2) = \frac{1}{2}p_{(1,1)} - \frac{1}{2}p_2$, so this subgroup contains $2e_2$, but not e_2. Informally, we say that the p_α do not form a $\mathbf{Z}$-basis for Λ_n.

(2) Suppose that $f(p_1, \dots, p_m) = 0$, where $f \in \mathbf{C}[y_1, \dots, y_m]$. Let

$$f(y_1, \dots, y_m) = \sum_\beta c_\beta y_1^{\beta_1} \dots y_m^{\beta_m}$$

where the sum is over all compositions β with $\ell(\beta) \leq m$. Substituting p_k for y_k for each $k \in \{1, \dots, m\}$, we get

$$\sum_\beta c_\beta p_{(m^{\beta_m}, \dots, 1^{\beta_1})} = 0.$$

Since the p_λ for λ a partition form a basis for Λ, we see that $c_\beta = 0$ for all β. Hence $f = 0$ and $p_1, p_2, \dots$ are algebraically independent. An analogous result holds for $e_1, e_2, \dots$ and $h_1, h_2, \dots$.

1.11. **The cycle index of the symmetric group*.** We have

$$\exp Q(t) = \exp \sum_{k=1}^\infty p_k \frac{t^k}{k} = \prod_{k=1}^\infty \exp \frac{p_k t^k}{k} = \prod_{k=1}^\infty \sum_{m=0}^\infty \frac{p_k^m}{k^m} \frac{t^{km}}{m!}.$$

Let α be a partition of n having exactly a_k parts of size k for each $k \in \{1, \dots, n\}$. There is a unique way to obtain $p_\alpha t^n$ by multiplying out the right-hand side above; the coefficient of $p_\alpha t^n$ is $\prod_{k=1}^n 1/(k^{a_k} a_k!)$. Therefore

$$\exp Q(t) = \sum_{n=0}^\infty \sum_{\alpha \vdash n} \frac{p_\alpha}{z_\alpha} t^n \tag{1.8}$$

where, by definition,

$$z_\alpha = \prod_{k=1}^n k^{a_k} a_k!.$$

This proves the following result.

Theorem 1.12 (Cycle Index Formula)**.**

$$\begin{aligned}
\exp \sum_{k=1}^\infty \frac{p_k}{k} t^k &= \sum_{n=0}^\infty \sum_{\alpha \vdash n} \frac{p_\alpha}{z_\alpha} t^n \\
&= \sum_{n=0}^\infty \sum_{(n^{a_n}, \dots, 1^{a_1}) \vdash n} \frac{p_1^{a_1} \dots p_n^{a_n}}{1^{a_1} \dots n^{a_n} a_1! \dots a_n!} t^n.
\end{aligned}$$

Note that by (1.6), the left-hand side is $\prod_{i=1}^{\infty}(1-x_i t)^{-1} = \sum_{n=0}^{\infty} h_n t^n$.

By Question 8, z_α is the order of the centralizer in Sym_n of an element of cycle-type α. Thus there are $n!/z_\alpha$ such elements in Sym_n, and $1/z_\alpha$ is the probability of picking one, when permutations are chosen uniformly at random. Therefore a restatement of the Cycle Index Formula is

$$\begin{aligned}\exp \sum_{k=1}^{\infty} \frac{p_k}{k} t^k &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\sigma \in \mathrm{Sym}_n} p_{\rho(\sigma)} t^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\sigma \in \mathrm{Sym}_n} p_1^{\mathrm{cyc}_1(\sigma)} \ldots p_n^{\mathrm{cyc}_n(\sigma)} t^n \qquad (1.9)\end{aligned}$$

where the partition $\rho(\sigma)$ is the cycle-type of $\sigma \in \mathrm{Sym}_n$ and $\mathrm{cyc}_m(\sigma)$ is the number of m-cycles in σ.

Many appealing results can be obtained by specializing the Cycle Index Formula by setting the p_n to particular complex numbers or indeterminates. This is justified by the algebraic independence of the p_n.

In the following result, recall that $\sigma \in \mathrm{Sym}_n$ is a *derangement* if σ has no fixed points.

Corollary 1.13.

(i) *Let d_n be the number of derangements in* Sym_n. *Then*

$$\frac{d_n}{n!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{(-1)^n}{n!}.$$

(ii) *Let O_n be the number of derangements in* Sym_n *that have an odd number of cycles, and let E_n be the number of derangements in* Sym_n *that have an even number of cycles. Then $O_n - E_n = n - 1$.*

(iii) *Let o_n be the number of derangements in* Sym_n *that are odd permutations, and let e_n be the number of derangements in* Sym_n *that are even permutations. Then $o_n - e_n = (-1)^n(n-1)$.*

Proof. Let $\mathcal{D}_n$ be the set of derangements in Sym_n. For (i) and (ii) specialize the version of the Cycle Index Formula in (1.9) by setting $p_1 = 0$ and $p_k = z$ for $k \geq 2$, where z is an indeterminate. We get

$$\exp \sum_{k=2}^{\infty} \frac{zt^k}{k} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{D}_n} z^{\mathrm{cyc}(\sigma)}$$

where $\mathrm{cyc}(\sigma)$ is the number of cycles in σ. Setting $z = 1$ we get

$$\frac{\exp(-t)}{1-t} = \exp\bigl(-t - \log(1-t)\bigr) = \sum_{n=0}^{\infty} \frac{d_n}{n!} t^n.$$

Taking the coefficient of t^n on the left-hand side gives (i). For (ii) set $z = -1$ to get

$$(1-t)\exp t = \exp\bigl(t + \log(1-t)\bigr) = \sum_{n=0}^{\infty} \frac{E_n - O_n}{n!} t^n$$

and again take the coefficient of t^n. Part (iii) is left as an exercise: see Question 9. $\square$

1.12. **Asymptotics of cycles*.** Treat the p_k in (1.9) as formal indeterminates. Observe that if we differentiate (1.9) with respect to p_m we obtain

$$\frac{t^m}{m}\exp\sum_{k=1}^{\infty}\frac{p_k}{k}t^k = \sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{\sigma\in\mathrm{Sym}_n}\mathrm{cyc}_m(\sigma)p_m^{\mathrm{cyc}_m(\sigma)-1}\prod_{k\neq m}p_k^{\mathrm{cyc}_k(\sigma)}.$$

Specialize by setting p_k to 1 for all k to get

$$\frac{t^m}{m}\frac{1}{1-t} = \sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{\sigma\in\mathrm{Sym}_n}\mathrm{cyc}_m(\sigma).$$

Taking the coefficient of t^n, we see that the mean number of m-cycles in a permutation in Sym_n is $1/m$ if $n\geq m$, and otherwise zero.

More generally, let β be a composition and set $b=\beta_1+2\beta_2+\cdots$. Differentiate the identity above β_m times with respect to p_m, for each m, to obtain

$$\begin{aligned}\frac{t^b}{\prod_{m\in\mathbf{N}}m^{\beta_m}}&\exp\sum_{k=1}^{\infty}\frac{p_k}{k}t^k\\ &= \sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{\sigma\in\mathrm{Sym}_n}\prod_{m\in\mathbf{N}}\beta_m!\binom{\mathrm{cyc}_m(\sigma)}{\beta_m}\prod_{k\in\mathbf{N}}p_k^{\mathrm{cyc}_k(\sigma)-\beta_k}.\end{aligned}$$

Specialize by setting $p_k=1$ for all k to get

$$\frac{t^b}{\prod_{m\in\mathbf{N}}m^{\beta_m}}\frac{1}{1-t} = \sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{\sigma\in\mathrm{Sym}_n}\prod_{m\in\mathbf{N}}\beta_m!\binom{\mathrm{cyc}_m(\sigma)}{\beta_m}.$$

Taking the coefficient of t^n it follows that

$$\mathbf{E}\Big[\prod_{m\in\mathbf{N}}\binom{\mathrm{cyc}_m(\sigma)}{\beta_m}\Big] = \frac{1}{z_{(1^{\beta_1},2^{\beta_2},\ldots)}}\quad\text{for all } n\geq b \tag{1.10}$$

where the expected value is taken over all permutations $\sigma\in\mathrm{Sym}_n$, chosen uniformly at random. (Here $(1^{\beta_1},2^{\beta_2},\ldots)$ denotes the partition with exactly β_m parts of size m, for each $m\in\mathbf{N}$.)

Below we write $\mathrm{cyc}_m^{(n)}(\sigma)$ for $\mathrm{cyc}_m(\sigma)$ when $\sigma\in\mathrm{Sym}_n$.

Theorem 1.14 (Goncharov). *As $n\to\infty$ the random variables $\mathrm{cyc}_m^{(n)}$ converge to independent Poisson random variables with mean $1/m$.*

Outline proof. The binomial moments of a Poisson random variable X with mean γ are $\mathbf{E}[\binom{X}{r}]=\gamma^r/r!$. Let C_m for $m\in\mathbf{N}$ be independent Poisson random variables with mean $1/m$. By independence we have

$$\mathbf{E}\Big[\prod_{m\in\mathbf{N}}\binom{C_m}{\beta_m}\Big] = \frac{1}{z_{(1^{\beta_1},2^{\beta_2},\ldots)}}$$

for all compositions β of n. Comparing with (1.10) we see that if β is a composition of b then

$$\mathbf{E}\Big[\prod_{m\in\mathbf{N}} C_m^{\beta_m}\Big] = \mathbf{E}\Big[\prod_{m\in\mathbf{N}} (\mathrm{cyc}_m^{(n)})^{\beta_m}\Big] \quad \text{for all } n \geq b.$$

It follows that the moments of the $\mathrm{cyc}_m^{(n)}(\sigma)$ converge to the moments of the C_m as $n \to \infty$. This implies convergence in joint distribution. $\square$

1.13. **Schur functions.** Recall that $[\lambda]$ denotes the Young diagram of a partition λ: formally this is the set of *boxes* $\{(i,j) : 1 \leq i \leq \ell(\lambda), 1 \leq j \leq \lambda_i\}$.

Definition 1.15. Let λ be a partition of n. A *λ-tableau* (or *tableau of shape λ*) is a function $[\lambda] \to \mathbf{N}$. To draw a λ-tableau t, draw the Young diagram of λ and, for each $(i,j) \in [\lambda]$, put $t(i,j)$ inside the corresponding box. A λ-tableau is *semistandard* if its rows are weakly increasing (from left to right) and its columns are strictly increasing (from top to bottom). Let $\mathrm{SSYT}(\lambda)$ be the set of all semistandard λ-tableaux. Given a λ-tableau t, let $c_k(t) = \big|\{(i,j) \in [\lambda] : t(i,j) = k\}\big|$. The *content* of t is the composition $(c_1(t), c_2(t), \ldots)$. For α a composition of n, let $\mathrm{SSYT}(\lambda,\alpha)$ be the set of semistandard λ-tableaux with content α. Define

$$x^t = x_1^{c_1(t)} x_2^{c_2(t)} \ldots$$

The *Schur function s_λ* is then defined by

$$s_\lambda = \sum_{t\in\mathrm{SSYT}(\lambda)} x^t.$$

For example,

$$\begin{aligned} s_{(3)} &= x^{\ytableaushort{111}} + x^{\ytableaushort{112}} + x^{\ytableaushort{122}} + x^{\ytableaushort{123}} + \cdots \\ &= x_1^3 + x_1^2x_2 + x_1x_2^2 + x_1x_2x_3 + \cdots \\ &= \mathrm{mon}_{(3)} + \mathrm{mon}_{(2,1)} + \mathrm{mon}_{(1,1,1)}. \end{aligned}$$

Notice that if we specialize to three variables then $x^t = 0$ whenever t has an entry strictly greater than 3. So

$$\begin{aligned} &s_{(2,1)}(x_1,x_2,x_3) \\ &= x^{\ytableaushort{11,2}} + x^{\ytableaushort{11,3}} + x^{\ytableaushort{12,2}} + x^{\ytableaushort{12,3}} + x^{\ytableaushort{13,2}} + x^{\ytableaushort{13,3}} + x^{\ytableaushort{22,3}} + x^{\ytableaushort{23,3}} \\ &= x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_1x_2x_3 + x_1x_2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 \\ &= \mathrm{mon}_{(2,1)}(x_1,x_2,x_3) + 2\mathrm{mon}_{(1,1,1)}(x_1,x_2,x_3). \end{aligned}$$

Since $\mathrm{ev}_3 : \Lambda_3 \to \mathbf{C}[x_1,x_2,x_3]^{\mathrm{Sym}_3}$ is injective, it follows, on the assumption that $s_{(2,1)}$ is a symmetric function, that $s_{(2,1)} = \mathrm{mon}_{(2,1)} + 2\mathrm{mon}_{(1,1,1)}$.

Lemma 1.16. *Let $n \in \mathbf{N}$. Then $s_{(n)} = h_n$ and $s_{(1^n)} = e_n$.*

Proof. This follows easily from the definition of semistandard. $\square$

Note that in every case seen so far s_λ is a symmetric function. This is not obvious from our definition, but has a beautiful short proof due to Bender and Knuth.

Lecture 4

Proposition 1.17. *The Schur functions are symmetric functions.*

Proof. Let λ be a partition and let $i \in \mathbf{N}$. It suffices to define an involution $\mathrm{BK} : \mathrm{SSYT}(\lambda) \to \mathrm{SSYT}(\lambda)$ such that $x^{\mathrm{BK}(t)} = x^t(i, i+1)$. (More explicitly, if $x^t = g x_i^c x_{i+1}^d$ where g is a monomial not involving x_i or x_{i+1}, then $x^{\mathrm{BK}(t)} = g x_i^d x_{i+1}^c$.) For it then follows that

$$s_\lambda = \sum_{t \in \mathrm{SSYT}(\lambda)} x^t = \sum_{t \in \mathrm{SSYT}(\lambda)} x^{\mathrm{BK}(t)} = \sum_{t \in \mathrm{SSYT}(\lambda)} x^t(i, i+1) = s_\lambda(i, i+1)$$

and so s_λ is invariant under all transpositions, and so invariant under all finitary permutations (i.e. permutations fixing all but finitely many elements of $\mathbf{N}$). It is an easy exercise to show that if $f \in \widehat{\mathbf{C}}[x_1, x_2, \ldots]$ is fixed by all finitary permutations then $f \in \Lambda$. Therefore $s_\lambda \in \Lambda$.

Let $t \in \mathrm{SSYT}(\lambda)$. With minor changes for the top and bottom rows, the is and $(i+1)$s in t in each row of t are arranged as shown below:

$$\begin{array}{cccccccccccc}
 & & & \multicolumn{3}{c}{\overset{a}{\longleftrightarrow}} & \multicolumn{3}{c}{< i} & \multicolumn{1}{|c|}{i} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{i} \\ \hline
\multicolumn{1}{|c|}{i} & \cdots & \multicolumn{1}{|c|}{i} & i & \multicolumn{1}{|c|}{\cdots} & i & \multicolumn{1}{|c|}{i+1} & \cdots & \multicolumn{1}{|c|}{i+1} & i+1 & \multicolumn{1}{|c|}{\cdots} & \multicolumn{1}{c|}{i+1} \\ \hline
\multicolumn{1}{|c|}{i+1} & \cdots & \multicolumn{1}{|c|}{i+1} & \multicolumn{3}{c}{> i+1} & \multicolumn{3}{c}{\underset{b}{\longleftrightarrow}} & & & \\ \cline{1-3}
\end{array}$$

Define $\mathrm{BK}(t)$ by replacing the a is and b $(i+1)$s in the middle part of each row with b is and a $(i+1)$s. The new tableau clearly has weakly increasing rows, and the columns are strictly increasing because the entries below the middle part are $> i+1$ and the entries above the middle part are $< i$. $\square$

It follows that if λ is a partition of n then

$$s_\lambda = \sum_{\mu \vdash n} |\mathrm{SSYT}(\lambda, \mu)| \mathrm{mon}_\mu. \tag{1.11}$$

By Question 11, $\mathrm{SSYT}(\lambda, \mu) = \varnothing$ unless $\lambda \trianglerighteq \mu$. Moreover $|\mathrm{SSYT}(\lambda, \lambda)| = 1$. By the standard argument it follows that $\{s_\lambda : \lambda \vdash n\}$ is a basis for Λ_n. An alternative proof of this fundamental result is given in §2. We remark that the *Kostka Numbers* $K_{\lambda\mu}$ are defined by $K_{\lambda\mu} = |\mathrm{SSYT}(\lambda, \mu)|$.

2. The Jacobi–Trudi Identity

In this section we shall see how to express the Schur functions in the basis of complete homogeneous symmetric functions.

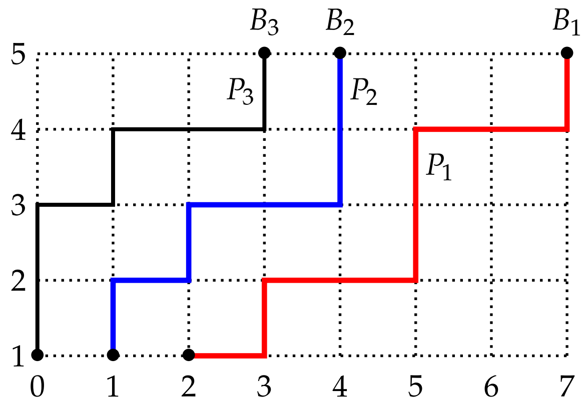
Theorem 2.1 (Jacobi–Trudi Identity). *Let λ be a partition of n and let $M \geq \ell(\lambda)$. Then*

$$s_\lambda = \det \begin{pmatrix} h_{\lambda_1} & h_{\lambda_1+1} & \cdots & h_{\lambda_1+(M-1)} \\ h_{\lambda_2-1} & h_{\lambda_2} & \cdots & h_{\lambda_2+(M-2)} \\ \vdots & \vdots & \ddots & \vdots \\ h_{\lambda_M-(M-1)} & h_{\lambda_M-(M-2)} & \cdots & h_{\lambda_M} \end{pmatrix}$$

where we set $h_r = 0$ if $r < 0$.

We offer a 'proof by example' that shows all the main features of the general proof, outlined in Question 12.

Example 2.2. Take $\lambda = (5,3,3)$ and $M = 3$. We prove that the Jacobi–Trudi Identity holds for s_λ when both sides are evaluated at $x_1, \ldots, x_5$. Let $B_i = (3 - i + \lambda_i, 5)$ for $i \in \{1,2,3\}$. Consider the diagram below.

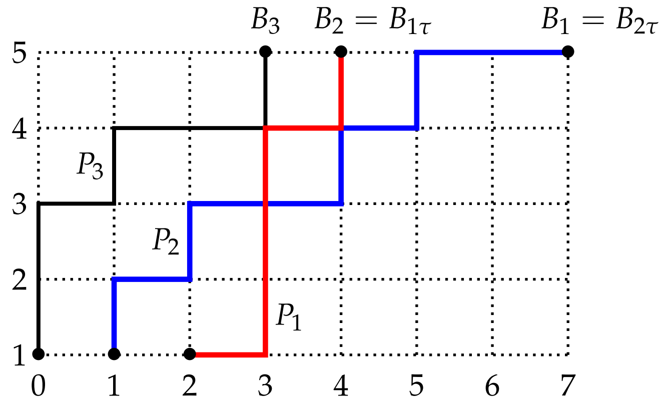


By a *path* we mean a sequence of unit steps, each either east or north. Marked on the grid is a triple of paths (P_3, P_2, P_1) such that P_i goes from $(3-i, 1)$ to B_i for each i. (Note that B_3 is strictly left of B_2, which is strictly left of B_1; this remains true in the general case where $B_i = (M - i + \lambda_i, 5)$, for each i.) Let $\mathcal{S}$ be the set of all such triples, like the one shown above, where no two paths intersect.

Claim. The map sending the $(5,3,3)$-tableau t with entries $a_1 \leq \ldots \leq a_{\lambda_i}$ in row i to the triple (P_3, P_2, P_1) where P_i has horizontal steps at heights $a_1, \ldots, a_{\lambda_i}$ is a bijection between the set of semistandard $(5,3,3)$-tableaux with entries from $\{1,2,3,4,5\}$ and $\mathcal{S}$.

Proof. Let (P_3, P_2, P_1) correspond to the tableau t. Suppose paths P_i and P_{i+1} meet, for the first time, at (a,b). Then P_{i+1} goes from $(a-1,b)$ to (a,b), say in its rth rightward step, and P_i goes from $(a, b-1)$ to (a,b) and makes its rth rightward step at (a,c) for some $c \geq b$. Hence $t(i,r) = c \geq b = t(i+1,r)$, so t is not semistandard. The converse is similar and is left as an exercise. $\square$

Let $\mathcal{A}$ be the set of all triples of paths (P_3, P_2, P_1) such that P_i goes from $(3-i, 1)$ to $B_{i\tau}$ for some permutation $\tau \in \mathrm{Sym}_3$. We now allow paths to intersect: if $\tau \neq \mathrm{id}$ then at least one intersection is inevitable. An example where $\tau = (1,2)$ is shown below.



Define the *weight* of a path P_i with horizontal steps at heights $h_1, \ldots, h_r$ by $\mathrm{wt}(P_i) = x_{h_1} \ldots x_{h_r}$. Thus the path P_2 from $(1,1)$ to $(7,5)$ drawn in blue above has weight $x_2 x_3^2 x_4 x_5^2$. Define the *weight* of a triple (P_3, P_2, P_1) by $\mathrm{wt}(P_3, P_2, P_1) = \mathrm{wt}(P_3)\,\mathrm{wt}(P_2)\,\mathrm{wt}(P_1)$. Define the *sign* of a triple (P_3, P_2, P_1) where each P_i ends at $B_{i\tau}$ to be $\mathrm{sgn}(\tau)$. For the triple shown above we have $\mathrm{wt}(P_3, P_2, P_1) = x_1 x_2 x_3^3 x_4^4 x_5^2$ and $\mathrm{sgn}(P_3, P_2, P_1) = -1$.

The previous claim shows that

$$\mathrm{ev}_5 s_\lambda = \sum_t x^t = \sum_{(P_3,P_2,P_1)\in\mathcal{S}} \mathrm{wt}(P_3, P_2, P_1) \tag{2.1}$$

where the middle sum is over all semistandard λ-tableaux with entries from $\{1, 2, 3, 4, 5\}$.

Claim. Let $P \in \mathcal{A}\backslash\mathcal{S}$. Choose i minimal so that P_i meets another path. Follow P_i, starting at $(M-i, 1)$ and stepping up and right, until it meets another path, at a point (a, b). Let j be maximal such that P_j passes through (a, b). Let $\overline{P}_i$ agree with P_i until this intersection and then follow P_j. Similarly let $\overline{P}_j$ agree with P_j until this intersection and then follow P_i. Let $\overline{P}_k = P_k$ if $k \neq i, j$. Define $J : \mathcal{A} \to \mathcal{A}$ by

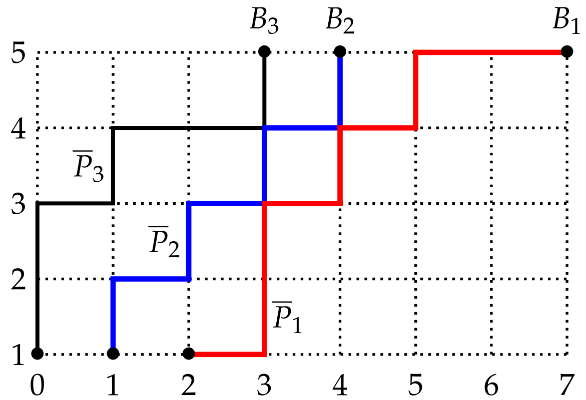
$$J(P_3, P_2, P_1) = \begin{cases} (\overline{P}_3, \overline{P}_2, \overline{P}_1) & \text{if } (P_3, P_2, P_1) \in \mathcal{A}\backslash\mathcal{S} \\ (P_3, P_2, P_1) & \text{otherwise.} \end{cases}.$$

Then J is a weight-preserving involution on $\mathcal{A}$ fixing exactly the triples in $\mathcal{S}$. Moreover, J is sign-reversing on $\mathcal{A}\backslash\mathcal{S}$.

Proof. Let $(P_3, P_2, P_1) \in \mathcal{A}\backslash\mathcal{S}$ and let i and j be as above. In $(\overline{P}_3, \overline{P}_2, \overline{P}_1)$, the positions of intersections do not change; the paths involved in each intersection change only when one or both was originally P_i or P_j and afterwards $\overline{P}_i$ or $\overline{P}_j$. Therefore i is still minimal such that $\overline{P}_i$ meets another path, the first intersection point is still (a,b), and j is still maximal such that $\overline{P}_j$ passes through (a,b).

Hence J is an involution. If τ is the permutation for (P_3, P_2, P_1) then, since the final destinations of the paths starting at $(M-i,1)$ and $(M-j,1)$ are swapped, $(i,j)\tau$ is the permutation for $(\overline{P}_3, \overline{P}_2, \overline{P}_1)$. Hence J is sign-reversing on $\mathcal{A}\backslash\mathcal{S}$. Since the right steps in P_3, P_2, P_1 and $\overline{P}_3, \overline{P}_2, \overline{P}_1$ occur at the same heights, J is weight-preserving. □

For example, let (P_3, P_2, P_1) be the intersecting paths above. The image $(\overline{P}_3, \overline{P}_2, \overline{P}_1)$ under J is shown below. We have $i = 1$, $j = 2$ and $(a,b) = (3,3)$.



By pairing up (P_3, P_2, P_1) and $J(P_3, P_2, P_1)$ for each $(P_3, P_2, P_1) \in \mathcal{A}\backslash\mathcal{S}$, and observing that their contributions to the left-hand side of the equation below cancel, we get

$$\sum_{(P_3,P_2,P_1)\in\mathcal{A}} \mathrm{wt}(P_3, P_2, P_1)\,\mathrm{sgn}(P_3, P_2, P_1) = \sum_{(P_3,P_2,P_1)\in\mathcal{S}} \mathrm{wt}(P_3, P_2, P_1).$$

The right-hand side is $\mathrm{ev}_5 s_\lambda$ by (2.1). So to complete the proof, we need to show that

$$\begin{aligned}\sum_{(P_3,P_2,P_1)\in\mathcal{A}} \mathrm{wt}(P_3, P_2, P_1)\,\mathrm{sgn}(P_3, P_2, P_1) &= \mathrm{ev}_5 \det \begin{pmatrix} h_5 & h_6 & h_7 \\ h_2 & h_3 & h_4 \\ h_1 & h_2 & h_3 \end{pmatrix} \\ &= \mathrm{ev}_5\big(h_{(5,3,3)} - h_{(5,2,4)} + h_{(1,6,4)} - h_{(2,6,3)} + h_{(2,2,7)} - h_{(1,3,7)}\big).\end{aligned}$$

This is easily checked: for example, the summand $h_{(2,6,3)} = h_{(6,3,2)}$ enumerates the weighted paths from $(2,1)$ to $(4,5)$, $(1,1)$ to $(7,5)$ and $(0,1)$ to $(3,5)$, with 2, 6 and 3 steps right, respectively. □

The general result on how summands in the expansion of the determinant correspond to path tuples can be guessed from this example: see Question 12. See also Lemma 4.9 and the following remark for the connection with the dot action.

An immediate corollary of Theorem 2.1 is a proof (independent of Proposition 1.17) that Schur functions are symmetric functions. We also get the following result.

Lecture 5

Corollary 2.3. *Let $n \in \mathbf{N}_0$. There exist $L_{\lambda\mu} \in \mathbf{Z}$ for $\lambda, \mu \vdash n$ such that*

$$s_\lambda = \sum_{\mu \vdash n} L_{\lambda\mu} h_\mu.$$

Moreover $L_{\lambda\mu} = 0$ unless $\mu \trianglerighteq \lambda$ and $L_{\lambda\lambda} = 1$ for all λ and μ.

Proof. Let $\ell(\lambda) = M$. Suppose we expand the determinant of the $M \times M$ matrix in the Jacobi–Trudi Identity for s_λ by taking $h_{\lambda_i - i + c_i}$ from row i. This gives a contribution to the coefficient of h_μ where μ is the partition obtained by rearranging the entries of $(\lambda_1 - 1 + c_1, \ldots, \lambda_M - M + c_M)$ into decreasing order. We have

$$\sum_{i=1}^{j} \mu_i \geq \sum_{i=1}^{j} (\lambda_i - i + c_i) = \sum_{i=1}^{j} \lambda_i + \sum_{i=1}^{j} (c_i - i) \geq \sum_{i=1}^{j} \lambda_i$$

where the final inequality holds because $\{c_1, \ldots, c_M\} = \{1, \ldots, M\}$. Hence $\mu \trianglerighteq \lambda$. Moreover, if $\mu = \lambda$ then $c_i = i$ for each i (since if j is minimal such that $c_j > j$ then we have strict inequality above) and so $L_{\lambda\lambda} = 1$. □

The matrix expressing the s_λ for λ a partition of n in the basis $\{h_\mu : \mu \vdash n\}$ of Λ_n is therefore triangular with diagonal entries 1. It follows, by the usual argument, that $\{s_\lambda : \lambda \vdash n\}$ is a basis for Λ_n. (A more standard proof using the monomial basis was indicated at the end of §1.13.)

Corollary 2.4. *Let $0 \leq m \leq n/2$. Then*
(i) $s_{(n-m,m)} = h_{(n-m,m)} - h_{(n-m+1,m-1)}$,
(ii) $h_{(n-m,m)} = s_{(n-m,m)} + \cdots + s_{(n-1,1)} + s_{(n)}$.

Proof. By the Jacobi–Trudi Identity we have

$$s_{(n-m,m)} = \det \begin{pmatrix} h_{n-m} & h_{n-m+1} \\ h_{m-1} & h_m \end{pmatrix} = h_{(n-m,m)} - h_{(n-m+1,m-1)}.$$

This proves (i). Part (ii) then follows by induction, using $s_{(n)} = h_{(n)}$ for the base case. □

3. Antisymmetric functions and the abacus

Fix $N \in \mathbf{N}$ throughout this section. We define an *N-strict partition* to be a partition $(\beta_1, \ldots, \beta_N)$ such that $\ell(\beta) \leq N$ and $\beta_1 > \ldots > \beta_N$. Thus the partition $(4, 2, 1, 0, \ldots)$ is 3-strict and 4-strict. We write $\beta \vdash_N n$ to denote that β is an N-strict partition of n.

3.1. **Antisymmetric functions.** Let

$$\Gamma = \{f \in \mathbf{C}[x_1, \ldots, x_N] : f\sigma = f \operatorname{sgn}(\sigma) \text{ for all } \sigma \in \mathrm{Sym}_N\}.$$

Let $\Gamma_n = \{f \in \Gamma : f \text{ is homogeneous of degree } n\}$ and note that $\Gamma = \bigoplus_{n \in \mathbf{N}_0} \Gamma_n$. Given an N-strict partition β of n, we define

$$a_\beta = \det \begin{pmatrix} x_1^{\beta_1} & \ldots & x_N^{\beta_1} \\ \vdots & \ddots & \vdots \\ x_1^{\beta_N} & \ldots & x_N^{\beta_N} \end{pmatrix}.$$

Observe that

$$a_\beta = \sum_{\sigma \in \mathrm{Sym}_N} x_1^{\beta_1} \ldots x_N^{\beta_N} \sigma \operatorname{sgn}(\sigma) \tag{3.1}$$

so $a_\beta \in \Gamma_n$. Since the leading term (i.e. the lexicographically greatest monomial summand) in a_β is x^β, it follows that $\{a_\beta : \beta \vdash_N n\}$ is a basis for Γ_n. Define

$$\delta = (N-1, N-2, \ldots, 1, 0). \tag{3.2}$$

Observe that if $\beta \vdash_N n$ then β has N distinct parts (including perhaps 0) and so $\beta - \delta$ is a partition. Hence the basis of Γ_n above is $\{a_{\lambda+\delta} : |\lambda + \delta| = n, \ell(\lambda) \leq N\}$.

If β is an N-strict partition then a_β specializes to 0 if we set $x_i = x_j$ for distinct i and j. By unique factorization, a_β is divisible by $\prod_{i<j}(x_i - x_j)$. Then, as a special case of this remark,

$$a_\delta = \det \begin{pmatrix} x_1^{N-1} & \ldots & x_N^{N-1} \\ \vdots & \ddots & \vdots \\ x_1 & \ldots & x_N \\ 1 & \ldots & 1 \end{pmatrix}$$

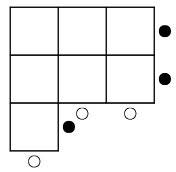
is divisible by $\prod_{i<j}(x_i - x_j)$. Since each x_i has degree $N-1$ in the product, and also in a_δ, and both the product and a_δ have leading term x^δ, equality holds. Therefore each a_β is divisible by a_δ and $a_{\lambda+\delta}/a_\delta$ is a non-zero polynomial for each λ with $\ell(\lambda) \leq N$.

Remark 3.1. We show in Theorem 4.10 that

$$\frac{a_{\lambda+\delta}}{a_\delta} = s_\lambda(x_1, \ldots, x_N) \tag{3.3}$$

for any partition λ with $\ell(\lambda) \leq N$. For a slightly direct proof, in the spirit of this section, see Question 24.

3.2. **The abacus.** Let λ be a partition. The *rim* of λ consists of all boxes $(i,j) \in [\lambda]$ such that $(i+1,j+1) \notin [\lambda]$. Imagine walking along the rim of λ (as drawn by geometric boxes), starting at the SW corner of the box $(\lambda'_1, 1)$ and ending at the NE corner of the box $(1, \lambda_1)$. For each step right, put a *gap* $\circ$, and for each step up, put a *bead* $\bullet$. For example, the rim of $(3,3,1)$ is encoded by $\circ \bullet \circ \circ \bullet \bullet$, as shown below. (The corresponding walk from SW corner to NE corner is $\rightarrow\uparrow\rightarrow\rightarrow\uparrow\uparrow$.) **Lecture 6**



Definition 3.2. An *abacus* representing λ is any sequence obtained by taking the sequence of beads and gaps encoding the rim of λ, then prepending any number of beads, and appending any number of gaps. Beads before the first gap are said to be *initial*; gaps after the final bead are said to be *final*.

We number the positions in abaci from 0. For example the abacus $\bullet \circ \bullet \circ \circ \bullet \bullet \circ \circ$ represents $(3,3,1)$, and has beads in positions $0,2,5,6$. The bead in position 0 is initial and the gaps in positions 7 and 8 are final. The abacus $\bullet \bullet \circ \circ$ represents the empty partition $\varnothing$. Clearly a partition can be reconstructed from any of its abaci. The abacus representation for partitions is due to G. D. James: see [5, page 78] for a textbook account.

The following definition is key to the approach in this section. It is due to Loehr [8].

Definition 3.3. Let λ be a partition with $\ell(\lambda) \leq N$. Fix the unique abacus for λ with exactly N beads and no final gaps. A *labelled abacus* for λ is any sequence obtained from this abacus by replacing each gap with 0 and each bead with a unique element of $\{1, \ldots, N\}$. Let $\mathrm{Abc}(\lambda)$ be the set of labelled abaci for λ.

The N relevant to $\mathrm{Abc}(\lambda)$ will always be clear from the context. We write labelled abaci as words, as shown in Example 3.4 immediately below. Let λ be a partition. The symmetric group Sym_N acts transitively on $\mathrm{Abc}(\lambda)$ by

$$(A\sigma)_i = \begin{cases} 0 & \text{if } A_i = 0 \\ A_i\sigma & \text{if } A_i \neq 0. \end{cases}$$

This action is free, i.e. the stabiliser of each labelled abacus is trivial. Let $A(\lambda) \in \mathrm{Abc}(\lambda)$ be the unique labelled abacus whose non-zero entries are decreasing.

Example 3.4. Let $N = 3$. The unique 3 bead abacus for $(3,1)$ with no final gap is $\bullet \circ \bullet \circ \circ \bullet$. Thus $A(3,1) = 302001$ and

$$\mathrm{Abc}(3,1) = \{302001, 301002, 203001, 201003, 103002, 102003\}.$$

If $\sigma = (1,2,3) \in \mathrm{Sym}_3$ then $302001\sigma = 103002$. Observe that $x_3^0 x_2^2 x_1^5$ is the leading term in $a_{(3,1)+(2,1,0)}$, and that

$$\begin{aligned} a_{(3,1)+(2,1,0)} &= x_2^2 x_1^5 - x_1^2 x_2^5 + x_3^2 x_1^5 - x_1^2 x_3^5 + x_2^2 x_3^5 - x_3^2 x_2^5 \\ &= \sum_{\sigma \in \mathrm{Sym}_3} x_2^2 x_1^5 \sigma \,\mathrm{sgn}(\sigma) \end{aligned}$$

The example shows why the freedom to have initial beads in abaci is important. It also motivates the following definition and lemma.

Definition 3.5. Let $A \in \mathrm{Abc}(\lambda)$. The *weight* of A is $\prod_{i=1}^N x_i^{\alpha_i}$ where α_i is the position of A containing i. We write $\mathrm{wt}(A)$ or x^A for the weight of A. The *sign* of A is $\mathrm{sgn}(\sigma)$ where $\sigma \in \mathrm{Sym}_N$ is the unique permutation such that $A(\lambda)\sigma = A$. We write $\mathrm{sgn}(A)$ for the sign of A.

Observe that if $A \in \mathrm{Abc}(\lambda)$ and $\sigma \in \mathrm{Sym}_N$ then $x^{A\sigma} = x^A \sigma$.

Lemma 3.6. *Let λ be a partition with $\ell(\lambda) \leq N$. Then*

$$a_{\lambda+\delta} = \sum_{A \in \mathrm{Abc}(\lambda)} x^A \,\mathrm{sgn}(A).$$

Proof. An unlabelled N bead abacus for λ has beads in positions $\lambda_i + N - i$ for each $i \in \{1, \ldots, N\}$; these are the entries of $\lambda + \delta$. Hence

$$\begin{aligned} a_{\lambda+\delta} &= \sum_{\sigma \in \mathrm{Sym}_N} x_1^{\lambda_1 + (N-1)} \ldots x_N^{\lambda_N} \sigma \,\mathrm{sgn}(\sigma) \\ &= \sum_{\sigma \in \mathrm{Sym}_N} x^{A(\lambda)} \sigma \,\mathrm{sgn}(\sigma) \\ &= \sum_{A \in \mathrm{Abc}(\lambda)} x^A \,\mathrm{sgn}(A) \end{aligned}$$

as required. □

3.3. **Pieri's Rule.** Given a partition λ of n and $r \in \mathbf{N}_0$, we define $P_r(\lambda) = \{\mu \vdash n + r \,:\, [\mu]$ is obtained from $[\lambda]$ by adding r boxes, no two in the same row$\}$.

Theorem 3.7 (Pieri's Rule)**.** *Let λ be a partition of n with $\ell(\lambda) \leq N$ and let $r \in \mathbf{N}_0$. Then*

$$a_{\lambda+\delta} e_r(x_1, \ldots, x_N) = \sum_{\substack{\mu \in P_r(\lambda) \\ \ell(\mu) \leq N}} a_{\mu+\delta}.$$

For example, if $N = 3$ we have

$$a_{(1,1)+\delta} e_2(x_1, x_2, x_3) = a_{(2,2)+\delta} + a_{(2,1,1)+\delta}.$$

Like the proof of the Jacobi–Trudi Identity, the proof uses a sign-reversing involution.

Proof. Let

$$\mathcal{P} = \{(A, S) : A \in \mathrm{Abc}(\lambda), S \subseteq \{1, \ldots, N\}, |S| = r\}.$$

For $(A, S) \in \mathcal{P}$ define $\mathrm{wt}(A, S) = x^A \prod_{i \in S} x_i$ and $\mathrm{sgn}(A, S) = \mathrm{sgn}(A)$. Thus the left-hand side of the claimed identity is

$$\sum_{(A,S) \in \mathcal{P}} \mathrm{wt}(A, S)\, \mathrm{sgn}(A, S).$$

This sum usually involves some cancellation. For example, if $\lambda = (1, 1)$ and $r = 2$ then the summands for $(3021, \{2, 3\})$ and $(3012, \{1, 3\})$, namely $x_2^2 x_1^3 x_2 x_3$ and $-x_1^2 x_2^3 x_1 x_3$, cancel. To capture this we define a sign-reversing involution $J : \mathcal{P} \to \mathcal{P}$. Given $(A, S) \in \mathcal{P}$, read A right to left, and try to move each bead labelled by an element of S one step right in A. (To move the rightmost bead, it may be necessary to first extend the abacus by a new space at the far right.) There are two cases:

- If there are no collisions, a new labelled abacus B is obtained. With labels removed, B represents a partition $\mu \in P_r(\lambda)$. Since B has no final gaps, we have $B \in \mathrm{Abc}(\mu)$. Define $J(A, S) = (A, S)$ and $K(A, S) = B$.
- Suppose that the first collision occurs when bead j is moved onto bead k. Then $j \in S$ and $k \notin S$. Define $J(A, S) = (A(j, k), S(j, k))$.

The following claims are fairly easy to check:

(1) J is an involution. *Proof:* suppose $J(A, S) = \big(A(j, k), S(j, k)\big)$ as above. Then when we move beads in $A(j, k)$ the first collision occurs when bead $k \in S(j, k)$ bumps bead j. $||$

(2) J is weight-preserving. *Proof:* suppose $J(A, S) = \big(A(j, k), S(j, k)\big)$ as in (1) and that the bead labelled j is in position a. Then the bead labelled k is in position $a + 1$. Let $g = \prod_i x_i^{\alpha_i}$ where the product is over all $i \in \{1, \ldots, N\}$ such that $i \neq j, k$ and α_i is the position of A containing i. We have

$$\begin{aligned}
\mathrm{wt}(A, S) &= x_j^a x_k^{a+1} g \prod_{i \in S} x_i \\
&= x_j^{a+1} x_k^{a+1} g \prod_{\substack{i \in S \\ i \neq j}} x_i \\
&= x_j^{a+1} x_k^{a+1} g \prod_{\substack{i \in S(j,k) \\ i \neq k}} x_i \\
&= x_j^{a+1} x_k^a g \prod_{i \in S(j,k)} x_i \\
&= \mathrm{wt}\big(A(j, k), S(j, k)\big). \;||
\end{aligned}$$

(3) J is sign-reversing on its non-fixed points.

(4) K is a bijection between the set of fixed points of J and the set $\bigcup_{\mu \in P_r(\lambda) : \ell(\mu) \leq N} \mathrm{Abc}(\mu)$. *Proof:* each such partition μ can be obtained in a unique way by moving r beads on an (unlabelled) N-bead abacus for λ one step to the right. (Note that only partitions with at most N parts can be obtained, because only N beads are present.) Hence, passing to labelled abaci, K is bijective.

(5) K is weight-preserving. *Proof:* if $K(A, S) = B$ then each bead in A labelled by $j \in S$ is moved one step to the right in B, so $x^B = x^A \prod_{i \in S} x_i = \mathrm{wt}(A, S)$. $||$

(6) K is sign-preserving. *Proof:* If $K(A, S) = B$ then the bead labels appear in the same order in A and B. $||$

Hence

$$\begin{aligned} a_{\lambda+\delta} e_r(x_1, \ldots, x_N) &= \sum_{(A,S) \in \mathcal{P}} \mathrm{wt}(A, S) \operatorname{sgn}(A, S) \\ &= \sum_{\substack{(A,S) \in \mathcal{P} \\ J(A,S) = (A,S)}} \mathrm{wt}(A, S) \operatorname{sgn}(A, S) \\ &= \sum_{\substack{\mu \in P_r(\lambda) \\ \ell(\mu) \leq N}} \sum_{B \in \mathrm{Abc}(\mu)} \mathrm{wt}(B) \operatorname{sgn}(B) \\ &= \sum_{\substack{\mu \in P_r(\lambda) \\ \ell(\mu) \leq N}} a_{\mu+\delta}, \end{aligned}$$

where the second equality holds by (1), (2), (3) and the third by (4), (5) and (6). $\square$

3.4. **Young's Rule.** Given a partition λ of n and $r \in \mathbf{N}_0$, let $Y_r(\lambda) = \{\mu \vdash n + r : [\mu] \text{ is obtained from } [\lambda] \text{ by adding } r \text{ boxes, no two in the same column}\}$.

Theorem 3.8 (Young's Rule). *Let λ be a partition of n with $\ell(\lambda) \leq N$ and let $r \in \mathbf{N}_0$. Then*

$$a_{\lambda+\delta} h_r(x_1, \ldots, x_N) = \sum_{\substack{\mu \in Y_r(\lambda) \\ \ell(\mu) \leq N}} a_{\mu+\delta}.$$

A highly recommended exercise (see Question 16) is to prove Young's Rule in a similar way to Pieri's Rule, replacing the sets S with multisets. The key idea is to define $J(A, S)$ and the sequence of bead moves so that we have the analogues of (1) and (2) above: if bead j bumps bead k in the moves for (A, S) then bead k should bump bead j in the moves for $J(A, S)$, and $\mathrm{wt}(A, S) = \mathrm{wt}(J(A, S))$.

Lecture 7

Corollary 3.9. *Let α be a composition of n. Then*

$$h_\alpha(x_1,\ldots,x_N) = \sum_{\substack{\lambda \vdash n \\ \ell(\lambda)\leq N}} |\mathrm{SSYT}(\lambda,\alpha)| \frac{a_{\lambda+\delta}}{a_\delta}.$$

Proof. Multiplying by a_δ it is equivalent to prove that the coefficient of $a_{\lambda+\delta}$ in the antisymmetric polynomial $a_\delta h_\alpha(x_1,\ldots,x_N)$ is $|\mathrm{SSYT}(\lambda,\alpha)|$. By Theorem 3.8, this coefficient is the number of ways to obtain λ from $\varnothing$ by performing Young's Rule additions of α_1 boxes, then α_2 boxes, and so on. Labelling the boxes added in step number i by i we see this is $|\mathrm{SSYT}(\lambda,\alpha)|$. $\square$

For example, if $\alpha = (3,2,1)$ and $N = 3$ then, abusing notation by writing h_r for $h_r(x_1,x_2,x_3)$, we have

$$\begin{aligned} a_\delta \mathrm{ev}_3(h_{(3,2,1)}) &= a_\delta h_3 h_2 h_1 \\ &= (a_{(3)+\delta}) h_2 h_1 \\ &= (a_{(5)+\delta} + a_{(4,1)+\delta} + a_{(3,2)+\delta}) h_1 \\ &= (a_{(6)+\delta} + a_{(5,1)+\delta}) + (a_{(5,1)+\delta} + a_{(4,2)+\delta} + a_{(4,1,1)+\delta}) \\ &\qquad + (a_{(4,2)+\delta} + a_{(3,3)+\delta} + a_{(3,2,1)+\delta}). \end{aligned}$$

Thus the coefficient of $a_{(4,2)+\delta}$ is 2, corresponding to the semistandard tableau

$$\begin{array}{|c|c|c|c|}\hline 1&1&1&2\\\hline 2&3 \\\cline{1-2}\end{array}\,, \quad \begin{array}{|c|c|c|c|}\hline 1&1&1&3\\\hline 2&2 \\\cline{1-2}\end{array}\,.$$

Remark 3.10. Recall that $P(\alpha)$ is the partition obtained from the composition α by rearranging the parts into decreasing order. By Proposition 1.17, which was proved using the Bender–Knuth involution, if λ is a partition of n and α, β are compositions of n such that $P(\alpha) = P(\beta)$ then $|\mathrm{SSYT}(\lambda,\alpha)| = |\mathrm{SSYT}(\lambda,\beta)|$. Since $h_\alpha = h_\beta$, Corollary 3.9 gives an alternative proof of this fact.

3.5. **Rim-hooks.** Let λ be a partition. Fix an unlabelled abacus representing λ. Moving a bead in position b to a gap in position $b+r$ corresponds to adding r boxes to the rim of λ. For an example see Figure 1 above. If the new partition is μ, then we say that μ/λ is an *r-rim-hook* of μ. The *height* of the r-rim-hook μ/λ, denoted $\mathrm{ht}(\mu/\lambda)$, is one less than the number of rows in the Young diagram of $[\mu]$ that have a non-empty intersection with the set $[\mu]\backslash[\lambda]$. Equivalently $\mathrm{ht}(\mu/\lambda)$ is the number of beads strictly between positions b and $b+r$. We define the *sign* of μ/λ by $\mathrm{sgn}(\mu/\lambda) = (-1)^{\mathrm{ht}(\mu/\lambda)}$.

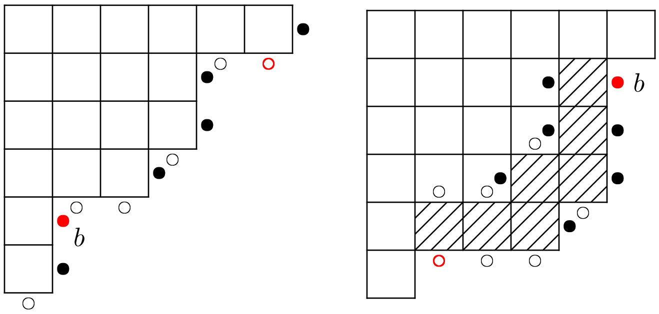


FIGURE 1. Adding a 7-rim-hook to $(6,4,4,3,1,1)$ by moving the bead b in the abacus $\circ \bullet \bullet^b \circ \circ \bullet \circ \bullet \bullet \circ \circ \bullet$ from position 2 to position 9 gives an abacus representing the partition $(6,5,5,5,4,1)$. The new gap in position 2 means we step right (adding a new box) rather than up; we then follow the rim of λ (making the same steps as before, but in the process adding a new box each time) until the new bead in position 9 means we step up rather than right, after walking around the rim of 7 added boxes.

3.6. **Murnaghan–Nakayama Rule.**

Theorem 3.11. *Let λ be a partition of n with $\ell(\lambda) \leq N$ and let $r \in \mathbf{N}$. Then*

$$a_{\lambda+\delta} p_r(x_1,\ldots,x_N) = \sum_{\substack{\mu \vdash n+r \\ \ell(\mu)\leq N \\ \mu/\lambda \text{ } r\text{-rim-hook}}} \operatorname{sgn}(\mu/\lambda) a_{\mu+\delta}.$$

Proof. The proof follows the same model as the proof of Pieri's rule. Let

$$\mathcal{P} = \big\{(A,j) : A \in \operatorname{Abc}(\lambda), j \in \{1,\ldots,N\}\big\}.$$

For $(A,j) \in \mathcal{P}$ define $\operatorname{wt}(A,j) = x^A x_j^r$ and $\operatorname{sgn}(A,j) = \operatorname{sgn}(A)$. By construction

$$a_{\lambda+\delta} p_r(x_1,\ldots,x_N) = \sum_{(A,j)\in\mathcal{P}} \operatorname{sgn}(A,j) \operatorname{wt}(A,j).$$

As before, this sum involves some cancellation. For example, let $\lambda = (3,1)$, $r = 3$ and $N = 3$; the summands for $(302001,2)$ and $(301002,1)$, namely $x_2^2 x_1^5 x_2^3$ and $-x_1^2 x_2^5 x_1^3$, cancel. We capture this cancellation by a sign-reversing involution J and a bijection K.

Let $(A,j) \in \mathcal{P}$ and suppose the bead labelled j is in position b.

- If there is a gap in position $b+r$ then define $J(A,j) = (A,j)$ and $K(A,j) = B$ where B is the labelled abacus obtained from A by moving the bead labelled j to position $b+r$. (This may require some 0s to be appended to A.)
- Otherwise define $J(A,j) = (A(j,k),k)$ where k is the label of the bead in position $b+r$.

Then (1) J is an involution, (2) J is weight-preserving (the example above shows the idea) and (3) J is sign-reversing on its non-fixed points.

Moreover we have the analogues of (4) and (5): K is a weight-preserving bijection

$$\{(A,j) \in \mathcal{P} : J(A,j) = (A,j)\} \to \bigcup_{\substack{\mu \vdash n+r \\ \ell(\mu) \leq N \\ \mu/\lambda \text{ } r\text{-rim-hook}}} \mathrm{Abc}(\mu).$$

Finally suppose that $K(A,j) = B$ where $B \in \mathrm{Abc}(\mu)$. If $\mathrm{ht}(\mu/\lambda) = h$ then there are exactly h beads between the bead labelled j in A and the gap r positions to its right. As noted above, $\mathrm{sgn}(\mu/\lambda) = (-1)^h$. If these beads are labelled $i_1, \ldots, i_h$ then the bead labels, read from left to right, are

$$\begin{array}{ll} \ldots j, i_1, \ldots, i_h \ldots & \text{in } A \\ \ldots i_1, \ldots, i_h, j \ldots & \text{in } B. \end{array}$$

Hence if σ, $\tau \in \mathrm{Sym}_N$ are the permutations such that $A(\lambda)\sigma = A$ and $A(\mu)\tau = B$ then

$$\tau = \sigma(j, i_1, \ldots, i_h).$$

Since an $(h+1)$-cycle has sign $(-1)^h$, we have $\mathrm{sgn}(A) = (-1)^h \,\mathrm{sgn}(B)$. Hence $\mathrm{sgn}(B) = \mathrm{sgn}(A)(-1)^h = \mathrm{sgn}(A)\,\mathrm{sgn}(\mu/\lambda)$. Applying K to the fixed points of J we get

$$\begin{aligned} a_{\lambda+\delta} p_r(x_1, \ldots, x_N) &= \sum_{\substack{(A,j)\in\mathcal{P} \\ J(A,j)=(A,j)}} \mathrm{sgn}(A,j)\,\mathrm{wt}(A,j) \\ &= \sum_{\substack{\mu \\ B \in \mathrm{Abc}(\mu)}} \mathrm{sgn}(\mu/\lambda)\,\mathrm{sgn}(B) x^B \\ &= \sum_{\mu} \mathrm{sgn}(\mu/\lambda) a_{\mu+\delta} \end{aligned}$$

where the sums over μ are as in the statement of the theorem. □

As for Young's Rule there is an equivalent version of the Murnaghan–Nakayama rule for multiplication of an antisymmetric polynomial by a general power sum symmetric polynomial. We need the following definition. In it we abuse notation slightly by writing $\lambda \subseteq \mu$ to mean $[\lambda] \subseteq [\mu]$.

Definition 3.12. Let $\lambda \vdash n$ and let $\alpha \models n$. Let $\alpha = (\alpha_1, \ldots, \alpha_k)$. A *border-strip tableau* of *shape* λ and *type* α is a sequence of partitions

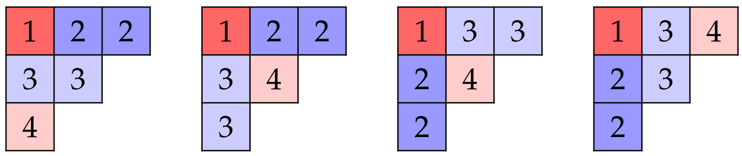
$$\varnothing = \lambda^{(0)} \subseteq \lambda^{(1)} \subseteq \ldots \subseteq \lambda^{(k)} = \lambda$$

such that $\lambda^{(i)}/\lambda^{(i-1)}$ is an α_i-rim-hook for each i. The *sign* of this border-strip tableau is $\prod_{i=1}^k \mathrm{sgn}(\lambda^{(i)}/\lambda^{(i-1)})$. Define

$$\mathrm{c}^\lambda(\alpha) = \sum_T \mathrm{sgn}(T)$$

where the sum is over all border-strip tableaux T of shape λ and type α.

For example, there are four border-strip tableaux of shape $(3,2,1)$ and type $(1,2,2,1)$. They are shown below with the rim-hook corresponding to the ith part of α labelled i. The signs are $1, -1, -1, 1$, so we have $\mathrm{c}^{(3,2,1)}(1,2,2,1) = 0$.



Corollary 3.13. *Let α be a composition of n.*

$$p_\alpha(x_1, \ldots, x_N) = \sum_{\substack{\lambda \vdash n \\ \ell(\lambda) \leq N}} \mathrm{c}^\lambda(\alpha) \frac{a_{\lambda+\delta}}{a_\delta}.$$

Proof. This follows in the same way as Corollary 3.9 by multiplying a_δ by p_α using Theorem 3.11. $\square$

It follows, again as in Young's Rule, that if λ is a partition of n and α and β are compositions of n such that $P(\alpha) = P(\beta)$ then $\mathrm{c}^\lambda(\alpha) = \mathrm{c}^\lambda(\beta)$. This gives a quicker way to see that $\mathrm{c}^{(3,2,1)}(1,2,2,1) = 0$: just observe that $P(1,2,2,1) = (2,2,1,1)$ and that $(3,2,1)$ has no 2-rim-hooks.

On the symmetric group side $\mathrm{c}^\lambda(\alpha)$ is the value of the irreducible character of Sym_n labelled by λ on elements of cycle-type α: see §5.5.

For a further corollaries of the Murnaghan–Nakayama rule see Question 17. An alternative proof is outlined in Questions 20 and 21. A related result on cores is in Question 19.

4. The Lascoux–Schützenberger involution

The coplactic maps in this section were defined in [10, §5.5] (where they are called 'coplactic operations'). For further background see [7]. The exposition here is essentially a greatly expanded version of [12]. The proofs of Lemmas 4.2 and 4.4 are adapted from [18]. **Lecture 8**

4.1. **Words.** A *word* is an element of $\mathbf{N}^\star$, the free monoid on $\mathbf{N}$. The *word of a tableau t*, denoted $\mathrm{w}(t)$, is obtained by reading the rows of t from bottom to top, left to right in each row. For example

$$\mathrm{w}\left(\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 3 & 3 \\ \hline 2 & 3 & 3 & 4 & \multicolumn{1}{c}{} \\ \cline{1-4} 4 & 5 & \multicolumn{3}{c}{} \\ \cline{1-2} \end{array}\right) = 45233411233.$$

Note that if t is semistandard then $\mathrm{w}(t)$ determines the shape of t, and so t itself.

Definition 4.1. Let $k \in \mathbf{N}$ and let $w \in \mathbf{N}^\star$. A k in w is *k-unpaired* if when w is read from left to right, this k sets a new record for the excess of ks over $(k+1)$s. Dually, a $k+1$ in w is *k-unpaired* if when w is read right to left, this $k+1$ sets a new record for the excess of $(k+1)$s over ks. The *k-unpaired* subword of w is the subword of k-unpaired entries. A k or $k+1$ in w is *k-paired* if it is not k-unpaired.

We may omit the 'k-' in 'k-paired' or 'k-unpaired' if it will be clear from the context.

A useful way to identify the k-unpaired entries of a word w is to replace each $k+1$ in w with a left parenthesis '(' and each k in w with a right parenthesis ')'. Then a k or $k+1$ is paired if and only if its parenthesis has a pair, according to the usual rules of bracketing. That this method works can be proved by strong induction on the length of w. If every k is to the left of every $k+1$ then all parentheses are unpaired, otherwise w has a subword of the form $(k+1)vk$ where v contains no k or $k+1$; then $k+1$ and k correspond to paired parentheses and removing the subword $(k+1)vk$ from w gives a shorter word to which the inductive hypothesis applies. For example, if $w = 45233411233$ is the word above, then underlining unpaired parentheses:

- If $k = 1$ the parenthesised word is $45(334)\underline{)}\underline{(}33$. The unpaired subword is 12 in positions 8 and 9 and removing 23341 gives a word with only unpaired parentheses.
- If $k = 2$ the parenthesised word is $45\underline{)}(\underline{(}411)\underline{(}\underline{(}$. The unpaired subword is 2333, taken from positions 3, 5, 10 and 11.

Lemma 4.2. *Let $w \in \mathbf{N}^\star$ and let $k \in \mathbf{N}$. The k-unpaired subword of w is of the form $k^c(k+1)^d$ for some $c, d \in \mathbf{N}_0$. Changing the letters of this subword so that it becomes $k^{c'}(k+1)^{d'}$ where $c' + d' = c + d$ gives a new word whose k-unpaired entries are in the same positions as the k-unpaired entries of w.*

Proof. There cannot be an unpaired k to the right of an unpaired $k+1$ in w. Therefore all ks after the rightmost unpaired $k+1$ are paired. Similarly there cannot be an unpaired $k+1$ to the left of an unpaired k in w, so each $k+1$ before the leftmost unpaired k are paired. Hence the unpaired subword is as claimed.

When $d \geq 1$, changing the letters of the unpaired subword from $k^c(k+1)^d$ to $k^{c+1}(k+1)^{d-1}$ replaces the leftmost unpaired $k+1$, in position i say, with a k; since every $k+1$ to the left of position i is paired, this new k is unpaired. The dual result holds when $c \geq 1$; together these imply the lemma. □

For an immediate application of Lemma 4.2 see Question 23, which gives a short proof of Theorem 15.14 in [6].

Definition 4.3. Let $w \in \mathbf{N}^\star$, let $k \in \mathbf{N}$ and suppose that w has k-unpaired subword $k^c(k+1)^d$. Let $S_k(w)$ be the word obtained from w by changing the letters of the k-unpaired subword to $k^d(k+1)^c$. If $d > 0$, let $E_k(w)$ be defined similarly, changing the subword to $k^{c+1}(k+$

$1)^{d-1}$. If $c > 0$, let $F_k(w)$ be defined similarly, changing the subword to $k^{c-1}(k+1)^{d+1}$.

By Lemma 4.2, E_k and F_k are mutual inverses and S_k is self-inverse.

4.2. **Coplactic operations on tableaux.** Using the word map w we can extend the definition of k-unpaired and the maps E_k, F_k and S_k to tableaux of a fixed shape. Thus an entry of a μ-tableau t is *k-unpaired* if and only if it corresponds to a k-unpaired letter in $\mathrm{w}(t)$, $E_k(t)$ is the unique μ-tableau with word $E_k(\mathrm{w}(t))$, and so on. For example, if

$$t = \begin{array}{|c|c|c|c|c|}\hline 1&1&1&2&3\\ \hline 2&4&4&4 \\ \cline{1-4}\end{array}$$

then the 3-unpaired entries of t are in positions $(2,2)$ and $(2,3)$ and

$$\begin{tikzcd}
\begin{array}{|c|c|c|c|c|}\hline 1&1&1&2&3\\ \hline 2&4&4&4 \\ \cline{1-4}\end{array} \arrow[r, shift right, "E_3"'] \arrow[d, leftrightarrow, "S_3"'] & \begin{array}{|c|c|c|c|c|}\hline 1&1&1&2&3\\ \hline 2&3&4&4 \\ \cline{1-4}\end{array} \arrow[l, shift right, "F_3"'] \arrow[r, shift right, "E_3"'] \arrow[d, leftrightarrow, "S_3"'] & \begin{array}{|c|c|c|c|c|}\hline 1&1&1&2&3\\ \hline 2&3&3&4 \\ \cline{1-4}\end{array} \arrow[l, shift right, "F_3"'] \arrow[d, leftrightarrow, "S_3"'] \\
\begin{array}{|c|c|c|c|c|}\hline 1&1&1&2&3\\ \hline 2&3&3&4 \\ \cline{1-4}\end{array} \arrow[r, shift right, "F_3"'] & \begin{array}{|c|c|c|c|c|}\hline 1&1&1&2&3\\ \hline 2&3&4&4 \\ \cline{1-4}\end{array} \arrow[l, shift right, "E_3"'] \arrow[r, shift right, "F_3"'] & \begin{array}{|c|c|c|c|c|}\hline 1&1&1&2&3\\ \hline 2&4&4&4 \\ \cline{1-4}\end{array} \arrow[l, shift right, "E_3"']
\end{tikzcd}$$

To motivate part of the next lemma, observe that S_3E_3 acts as an involution on the four tableaux with a 3-unpaired 4. Let

$$\epsilon(k) = (0, \ldots, 1, -1, \ldots, 0) \in \mathbf{Z}^N,$$

where the unique non-zero entries are in positions k and $k+1$.

Lemma 4.4. *Let μ be a partition and let α be a composition with $\ell(\alpha) \leq N$. Fix $k \in \{1, \ldots, N-1\}$. Let $\mathrm{SSYT}_k(\mu, \alpha)$ and $\mathrm{SSYT}_{k+1}(\mu, \alpha)$ be the sets of all semistandard tableaux of shape μ and content α that have a k-unpaired k or a k-unpaired $k+1$, respectively. Let $\alpha' = (\alpha_1, \ldots, \alpha_{k-1}, \alpha_{k+1}, \alpha_k, \alpha_{k+2}, \ldots, \alpha_N)$. The maps*

$$\begin{aligned}
E_k &: \mathrm{SSYT}_{k+1}(\mu, \alpha) \to \mathrm{SSYT}_k\big(\mu, \alpha + \epsilon(k)\big) \\
F_k &: \mathrm{SSYT}_k(\mu, \alpha) \to \mathrm{SSYT}_{k+1}\big(\mu, \alpha - \epsilon(k)\big) \\
S_k &: \mathrm{SSYT}_k(\mu, \alpha) \to \mathrm{SSYT}_{k+1}\big(\mu, \alpha'\big)
\end{aligned}$$

are bijections and the composition $S_kE_k : \mathrm{SSYT}_{k+1}(\mu, \alpha) \to \mathrm{SSYT}_{k+1}(\mu, \alpha' - \epsilon(k))$ has inverse $S_kE_k : \mathrm{SSYT}_{k+1}(\mu, \alpha' - \epsilon(k)) \to \mathrm{SSYT}_{k+1}(\mu, \alpha)$.

Proof. The main thing to check is that if $t \in \mathrm{SSYT}_{k+1}(\mu, \alpha)$ then $t' = E_k(t)$ is semistandard. Suppose that the leftmost unpaired $k+1$ in $\mathrm{w}(t)$ corresponds to the entry $t(a, b)$ in position (a, b) of t; this entry is changed to k in t'. If t' is not semistandard then $t(a-1, b) = k$. (It is easy to rule out that $t(a, b-1) = k+1$.) This k is to the right of the unpaired $k+1$ in $\mathrm{w}(t)$, so by Lemma 4.2 it is paired, necessarily with a $k+1$ in row a and some column $b' > b$ of t. Since

$$k = t(a-1,b) \leq t(a-1,b') < t(a,b') = k+1$$

we have $t(a-1,b') = k$. Thus $t(a,c) = k+1$ and $t(a-1,c) = k$ for all $c \in \{b,\ldots,b'\}$. It follows that the entries in positions $(a-1,b+j)$ and $(a,b'-j)$ of t are paired for each $j \in \{0,1,\ldots,b'-b\}$.

	col b		col b'
row $a-1$	$k)$	$\cdots$	$k)$
row a	$k+1($	$\cdots$	$k+1($

In particular, the $k+1$ in position (a,b) of t is paired, a contradiction. Hence $E_k(t)$ is semistandard. The proof is similar for F_k, in the case when t has an unpaired k.

For each $t \in \mathrm{SSYT}_k(\mu,\alpha)$, the tableau $S_k(t)$ is of the form $E_k^r(t)$ or $F_k^r(t)$ for some $r \in \mathbf{N}_0$. Therefore it is semistandard; since the operation S switches the number of unpaired ks and $(k+1)$s, it is in $\mathrm{SSYT}(\mu,\alpha')$.

It is now routine to check that E_kF_k and F_kE_k are the identity maps on their respective domains, so E_k and F_k are bijective. If the unpaired subword of $\mathrm{w}(t)$ is $k^c(k+1)^d$ then $S_k(t) = E_k^{d-c}(t)$ if $d \geq c$ and $S_k(t) = F_k^{c-d}(t)$ if $c \geq d$. Hence S_k and S_kE_k are involutions. It is easily checked that the image of S_kE_k is as claimed. $\square$

The final part of the claim gives some motivation for the next definition.

4.3. **Dot action.** The symmetric group Sym_N acts on $\mathbf{Z}^N$ by place permutation: thus if $\alpha \in \mathbf{Z}^N$ and $\sigma \in \mathrm{Sym}_N$ then $(\alpha\sigma)_k = \alpha_{k\sigma^{-1}}$. Note that the entry α_k in position k of α appears in position $k\sigma$ of $\alpha\sigma$. This shows why the inverse appears in the definition; without it, the action is not a well-defined right action.

We define the *dot action* of Sym_N on $\mathbf{Z}^N$ by

$$\alpha \cdot (k,k+1) = \alpha(k,k+1) - \epsilon(k) \tag{4.1}$$

for $k \in \{1,\ldots,N-1\}$. An equivalent definition is

$$\alpha \cdot \sigma = (\alpha+\delta)\sigma - \delta$$

where $\delta = (N-1,\ldots,1,0)$, as in (3.2) in §3. Since $\alpha \cdot \sigma$ has ith entry equal to $(\alpha+\delta)_{i\sigma^{-1}} - \delta_i = \alpha_{i\sigma^{-1}} + (N - i\sigma^{-1}) - (N-i)$, it follows that

$$(\alpha \cdot \sigma)_i = \alpha_{i\sigma^{-1}} - i\sigma^{-1} + i. \tag{4.2}$$

Definition 4.5. For α a composition with $\ell(\alpha) \leq N$ and $\sigma \in \mathrm{Sym}_N$ we define $\alpha \cdot \sigma$ to be the infinite integer sequence obtained from $(\alpha_1,\ldots,\alpha_N)\cdot\sigma$ by appending zeros.

Note that $\alpha \cdot \sigma$ may have negative parts, so is not in general a composition. By Question 22, if λ is a partition with $\ell(\lambda) \leq N$ and $\sigma \in \mathrm{Sym}_N$, then $\lambda \cdot \sigma$ is a partition if and only if $\sigma = \mathrm{id}_{\mathrm{Sym}_N}$. Note that, by Lemma 4.4, if $t \in \mathrm{SSYT}(\mu, \alpha)$ and $E_k(t)$ is defined then

$$S_k E_k(t) \in \mathrm{SSYT}(\mu, \alpha \cdot (k, k+1)). \tag{4.3}$$

Lemma 4.9 below gives a close connection between the dot action and the Jacobi–Trudi Identity. See MathOverflow 80150 for some other appearances of the dot action.

4.4. **Latticed tableaux and the J involution.** Say that a word w is *latticed* if it has no k-unpaired $(k+1)$s, for any k. Equivalently, by Definition 4.1, for each k, when w is read from right to left, the number of ks is always equal or greater than the number of $(k+1)$s, for every k. If w is not latticed and the rightmost j-unpaired $j+1$ in w is a $k+1$, say that w is *k-unlatticed*. (This term is not standard but is temporarily useful.) We extend these definitions to tableaux using the word map.

Definition 4.6. Let λ be a partition of n with $\ell(\lambda) \leq N$. Let μ be a partition of n. Let

$$\mathcal{T} = \bigcup_{\sigma \in \mathrm{Sym}_N} \mathrm{SSYT}(\mu, \lambda \cdot \sigma).$$

(If $\lambda \cdot \sigma$ has a negative part then take $\mathrm{SSYT}(\mu, \lambda \cdot \sigma) = \varnothing$.) Define $J : \mathcal{T} \to \mathcal{T}$ by $J(t) = t$ if t is latticed and $J(t) = S_k E_k(t)$ if t is k-unlatticed.

Lemma 4.7. *Take notation as in Definition 4.6. The map J is an involution on $\mathcal{T}$. Let $t \in \mathrm{SSYT}(\mu, \lambda \cdot \sigma)$.* **Lecture 9**

(i) *If t is k-unlatticed then $J(t) \in \mathrm{SSYT}\big(\mu, \lambda \cdot (\sigma(k, k+1))\big)$.*

(ii) *If t is latticed then $\sigma = \mathrm{id}_{\mathrm{Sym}_N}$, $\lambda = \mu$ and t is the unique element of* $\mathrm{SSYT}(\lambda, \lambda)$.

Proof. Let $t \in \mathrm{SSYT}(\mu, \lambda \cdot \sigma)$ be k-unlatticed. The words $\mathrm{w}(t)$ and $\mathrm{w}(J(t))$ differ only in their entries lying in the positions of the k-unpaired entries of $\mathrm{w}(t)$. If the k-unpaired subword of $\mathrm{w}(t)$ is $k^a(k+1)^b$ then the k-unpaired subword of $\mathrm{w}(J(t))$ is $k^{b-1}(k+1)^{a+1}$. Let the rightmost k-unpaired $k+1$ in $\mathrm{w}(t)$ be in position i. Recall that, by Definition 4.1, a $j+1$ is j-unpaired in a word v if and only if it sets a new record for the excess of $(j+1)$s over js when v is read right to left. Since positions $i+1, \ldots, n$ of $\mathrm{w}(t)$ and $\mathrm{w}(J(t))$ are equal, it follows that the subword in these positions is latticed in both $\mathrm{w}(t)$ and $\mathrm{w}(J(t))$. Since $\mathrm{w}(J(t))_i = k+1$, we see that $\mathrm{w}(J(t))$ is k-unlatticed. Hence $J(J(t)) = S_k E_k(J(t))$. It now follows from Lemma 4.4 that $J(J(t)) = t$. Hence J is an involution.

Part (i) follows from (4.3). For (ii), if t is latticed then the content of t is a partition, so $\sigma = \mathrm{id}_{\mathrm{Sym}_N}$ by Question 22 and so $t \in \mathrm{SSYT}(\mu, \lambda)$.

Considering each row of t in turn, starting with the first row, we see that the lattice condition implies that $\mu = \lambda$ and all the entries in row i of t are equal to i, for each $i \in \{1, \ldots, \ell(\lambda)\}$. $\square$

Note that (ii) shows that J has a unique fixed point in $\mathcal{T}$ if $\mu = \lambda$ and otherwise none. When the definition of J is extended to skew-tableaux there may be many fixed points: see Question 30.

Example 4.8. Take $\lambda = (5, 3, 1)$. The orbit of $(5, 3, 1)$ under the dot action can be computed by repeatedly applying the transpositions $(1, 2)$ and $(2, 3)$ using (4.1).

$$\begin{tikzcd}
& (2,6,1) \arrow[r, "(2,3)"] & (2,0,7) \arrow[dr, "(1,2)"] & \\
(5,3,1) \arrow[ur, "(1,2)"] \arrow[dr, "(2,3)"'] & & & (-1,3,7) \\
& (5,0,4) \arrow[r, "(1,2)"'] & (-1,6,4) \arrow[ur, "(2,3)"'] &
\end{tikzcd}$$

In fact one can show using the Bruhat order that there is no need to carry on once a negative entry appears, but it is maybe interesting to see the braid relation $(1, 2)(2, 3)(1, 2) = (2, 3)(1, 2)(2, 3)$ appearing.

The four compositions $(5, 3, 1)$, $(2, 6, 1)$, $(5, 0, 4)$ and $(2, 0, 7)$ correspond to the four non-zero summands in the Jacobi–Trudi Identity for $s_{(5,3,1)}$. We have

$$\begin{aligned} a_\delta \mathrm{ev}_3 s_{(5,3,1)} &= a_\delta \mathrm{ev}_3 h_{(5,3,1)} - a_\delta \mathrm{ev}_3 h_{(2,6,1)} - a_\delta \mathrm{ev}_3 h_{(5,0,4)} + a_\delta \mathrm{ev}_3 h_{(2,0,7)} \\ &= \sum_\mu \big(|\mathrm{SSYT}(\mu, (5, 3, 1))| - |\mathrm{SSYT}(\mu, (2, 6, 1))| \\ &\qquad\quad - |\mathrm{SSYT}(\mu, (5, 0, 4))| + |\mathrm{SSYT}(\mu, (2, 0, 7))| \big) a_{\delta+\mu} \end{aligned}$$

where the second line follows from Young's Rule (Corollary 3.9) for antisymmetric polynomials. Applying the J involution we see that the coefficient of $a_{\delta+\mu}$ is the number of latticed semistandard Young tableaux of shape μ and content $(5, 3, 1)$. As remarked above, if $\mu = (5, 3, 1)$ there is a unique such tableau, namely

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & \multicolumn{2}{c}{} \\ \cline{1-3} 3 & \multicolumn{4}{c}{} \\ \cline{1-1} \end{array}$$

and otherwise there are none. Hence $a_\delta s_{(5,3,1)}(x_1, x_2, x_3) = a_{\delta+(5,3,1)}$ and $s_{(5,3,1)}(x_1, x_2, x_3) = a_{\delta+(5,3,1)}/a_\delta$.

Two explicit examples of the J involution are given below.

(i) Take $\mu = (5, 3, 1)$. Since μ does not dominate $P(2, 6, 1) = (6, 2, 1)$, $P(5, 0, 4) = (5, 4)$ or $P(2, 0, 7) = (7, 2)$, the set $\mathcal{T}$ in Definition 4.6 is simply $\mathrm{SSYT}((5, 3, 1), (5, 3, 1))$, and its unique element is the fixed point of J shown above.

(ii) Take $\mu = (6,3)$. Then

$$\mathcal{T} = \mathrm{SSYT}((6,3),(5,3,1)) \cup \mathrm{SSYT}((6,3),(2,6,1)) \cup \mathrm{SSYT}((6,3),(5,0,4))$$

$$= \left\{ \begin{array}{|c|c|c|c|c|c|}\hline 1&1&1&1&1&2\\ \hline 2&2&3 \\ \cline{1-3}\end{array}, \begin{array}{|c|c|c|c|c|c|}\hline 1&1&1&1&1&3\\ \hline 2&2&2 \\ \cline{1-3}\end{array} \right\}$$

$$\cup \left\{ \begin{array}{|c|c|c|c|c|c|}\hline 1&1&2&2&2&2\\ \hline 2&2&3 \\ \cline{1-3}\end{array} \right\} \cup \left\{ \begin{array}{|c|c|c|c|c|c|}\hline 1&1&1&1&1&3\\ \hline 3&3&3 \\ \cline{1-3}\end{array} \right\}$$

Label the tableaux t_1, t_2, t_3, t_4 from left to right. We have $J(t_1) = S_1 E_1(t_1) = t_3$ and $J(t_2) = S_2 E_2(t_2) = t_4$. Of course this implies that $J(t_3) = t_1$ and $J(t_4) = t_2$.

4.5. **Equivalence of definitions of Schur functions.** The previous example shows the main ideas required. We need the following version of the Jacobi–Trudi Identity (Theorem 2.1).

Lemma 4.9. *Let λ be a partition of n with $\ell(\lambda) \leq N$. Then*

$$s_\lambda = \sum_{\sigma \in \mathrm{Sym}_N} h_{\lambda \cdot \sigma} \,\mathrm{sgn}(\sigma).$$

Proof. The Jacobi–Trudi Identity states that if $\ell(\lambda) \leq M$ then

$$s_\lambda = \det(h_{\lambda_i - i + j})_{1 \leq i,j \leq M}.$$

Take $M = N$. For each $\tau \in \mathrm{Sym}_N$, the summand of the determinant given by taking $h_{\lambda_i - i + i\tau}$ from row i of the matrix, for each $i \in \{1, \ldots, N\}$, is $\mathrm{sgn}(\tau) \prod_{i=1}^N h_{\lambda_i - i + i\tau}$. Therefore

$$\begin{aligned} s_\lambda &= \sum_{\tau \in \mathrm{Sym}_N} \mathrm{sgn}(\tau) \prod_{i=1}^N h_{\lambda_i - i + i\tau} \\ &= \sum_{\sigma \in \mathrm{Sym}_N} \mathrm{sgn}(\sigma) \prod_{i=1}^N h_{\lambda_{i\sigma^{-1}} - i\sigma^{-1} + i} \\ &= \sum_{\sigma \in \mathrm{Sym}_N} \mathrm{sgn}(\sigma) h_{\lambda \cdot \sigma} \end{aligned}$$

where the final equality follows immediately from (4.2). □

In the path model used to prove Theorem 2.1, path tuples for which the final destination of the path starting at $(M-i, 1)$ is $(\lambda_{i\tau} + M - i\tau, N)$ correspond to the summand $\mathrm{sgn}(\tau) \prod_{i=1}^M h_{\lambda_{i\tau} - i\tau + i} = \mathrm{sgn}(\tau) h_{\lambda \cdot \tau^{-1}}$ in the expansion of the determinant. (Since the path starting at $(M-i, 1)$ makes $\lambda_{i\tau} - i\tau + i$ right steps.) Swapping the final destinations of the paths starting at $(M-i, 1)$ and $(M-j, 1)$ replaces the terms $h_{\lambda_{i\tau} + i - i\tau}$ and $h_{\lambda_{j\tau} + j - j\tau}$ in this product with $h_{\lambda_{j\tau} + i - j\tau}$ and $h_{\lambda_{i\tau} + j - i\tau}$, respectively. Thus τ is replaced with $(i,j)\tau$.

Theorem 4.10. *Let λ be a partition of n with $\ell(\lambda) \leq N$. Then $s_\lambda(x_1, \ldots, x_N) = a_{\lambda+\delta}/a_\delta$.*

Proof. By Lemma 4.9 we have

$$a_\delta s_\lambda(x_1,\ldots,x_N) = \sum_{\sigma\in\mathrm{Sym}_N} a_\delta h_{\lambda\cdot\sigma}(x_1,\ldots,x_N)\,\mathrm{sgn}(\sigma).$$

By Young's rule (as stated in Corollary 3.9) we have

$$h_{\lambda\cdot\sigma}(x_1,\ldots,x_N) = \sum_{\substack{\mu\vdash n\\ \ell(\mu)\le N}} |\mathrm{SSYT}(\mu,\lambda\cdot\sigma)|\frac{a_{\mu+\delta}}{a_\delta}.$$

Therefore

$$a_\delta s_\lambda(x_1,\ldots,x_N) = \sum_{\substack{\mu\vdash n\\ \ell(\mu)\le N}} c_\mu a_{\delta+\mu}$$

where

$$c_\mu = \sum_{\sigma\in\mathrm{Sym}_N} |\mathrm{SSYT}(\mu,\lambda\cdot\sigma)|\,\mathrm{sgn}(\sigma).$$

Fix μ. Let $\mathcal{T} = \bigcup_{\sigma\in\mathrm{Sym}_N}\mathrm{SSYT}(\mu,\lambda\cdot\sigma)$ and let $J : \mathcal{T}\to\mathcal{T}$ be as defined in Definition 4.6. By Lemma 4.7, if $t\in\mathrm{SSYT}(\mu,\lambda\cdot\sigma)$ and $J(t)\neq t$ then $J(t)\in\mathrm{SSYT}\big(\mu,\lambda\cdot(\sigma(k,k+1))\big)$ for some $k\in\{1,\ldots,N-1\}$. Using the disjointness of the union defining $\mathcal{T}$ (see Question 22) it follows that applying J therefore cancels all contributions to c_μ except those coming from tableaux $t\in\mathcal{T}$ such that $J(t)=t$. By Lemma 4.7, the unique fixed point of J on $\mathcal{T}$ is the unique element of $\mathrm{SSYT}(\lambda,\lambda)$ when $\mu=\lambda$, and otherwise there are no fixed points. Therefore $c_\lambda = 1$ and $c_\mu = 0$ unless $\mu=\lambda$. Hence $a_\delta s_\lambda(x_1,\ldots,x_N) = a_{\delta+\lambda}$, as required. □

4.6. **Results for Schur functions.** Using (1.5) in §1.6 it is now routine to translate all the results in §3 to results on Schur functions. Let λ be a partition of n and let $r\in\mathbf{N}_0$. Dividing through by a_δ in Pieri's Rule (Theorem 3.7) we get

$$s_\lambda(x_1,\ldots,x_N)e_r(x_1,\ldots,x_N) = \sum_{\substack{\mu\in P_r(\lambda)\\ \ell(\mu)\le N}} s_\mu(x_1,\ldots,x_N)$$

for all $N\in\mathbf{N}$ such that $\ell(\lambda)\le N$. Provided $N\ge r+n$ the condition $\ell(\mu)\le N$ always holds, so we have

$$\mathrm{ev}_N(s_\lambda e_r) = \sum_{\mu\in P_r(\lambda)} \mathrm{ev}_N(s_\mu)$$

for all N sufficiently large. Hence, by (1.5), $s_\lambda e_r = \sum_{\mu\in P_r(\lambda)} s_\mu$. Young's Rule and the Murnaghan–Nakayama Rule for Schur functions follow in the same way. (See Questions 25 and 28.)

5. Inner product on Λ

Define an inner product $\langle\ ,\ \rangle : \Lambda \times \Lambda \to \mathbf{C}$ by extension of

$$\langle s_\lambda, s_\mu \rangle = \begin{cases} 1 & \text{if } \lambda = \mu \\ 0 & \text{otherwise.} \end{cases}$$

We use the convention that $\langle\ ,\ \rangle$ is linear in its first component and conjugate linear in its second, so the form as a whole is sesquilinear.

5.1. **Dual bases.** If $\{f_\mu : \mu \vdash n\}$ and $\{g_\nu : \nu \vdash n\}$ are bases of Λ_n such that

$$\langle f_\mu, g_\nu \rangle = \begin{cases} 1 & \text{if } \mu = \nu \\ 0 & \text{otherwise,} \end{cases}$$

then we say $\{f_\mu : \mu \vdash n\}$ and $\{g_\nu : \nu \vdash n\}$ are *dual bases*. Thus $\{s_\mu : \mu \vdash n\}$ is a self-dual orthonormal basis of Λ_n.

The following lemma gives an efficient way to prove duality relationships between the symmetric functions we have studied.

Lecture 10

Lemma 5.1. *Suppose that $\{f_\mu : \mu \vdash n\}$ and $\{g_\nu : \nu \vdash n\}$ are bases of Λ_n such that*

$$f = \sum_{\nu \vdash n} \langle f, g_\nu \rangle f_\nu$$

for all $f \in \Lambda_n$. Then $\{f_\mu : \mu \vdash n\}$ and $\{g_\nu : \nu \vdash n\}$ are dual bases.

Proof. Taking $f = f_\mu$ we get $f_\mu = \sum_{\nu \vdash n} \langle f_\mu, g_\nu \rangle f_\nu$. The result follows by comparing coefficients. $\square$

Theorem 5.2. *The bases $\{\mathrm{mon}_\nu : \nu \vdash n\}$ and $\{h_\mu : \mu \vdash n\}$ of Λ_n are dual.*

Proof. By the expansion of s_λ in the monomial basis in (1.11), we have

$$s_\lambda = \sum_{\nu \vdash n} |\mathrm{SSYT}(\lambda, \nu)| \mathrm{mon}_\nu$$

for all partitions λ of n. By Young's rule (as stated in Corollary 3.9 in the antisymmetric setting), we have

$$h_\nu = \sum_{\lambda \vdash n} |\mathrm{SSYT}(\lambda, \nu)| s_\lambda$$

for all partitions ν of n. Hence $\langle s_\lambda, h_\nu \rangle = |\mathrm{SSYT}(\lambda, \nu)|$. Substituting in the first equation we get $s_\lambda = \sum_{\nu \vdash n} \langle s_\lambda, h_\nu \rangle \mathrm{mon}_\nu$. Hence $f = \sum_{\nu \vdash n} \langle f, h_\nu \rangle \mathrm{mon}_\nu$ for all $f \in \Lambda_n$. Now apply Lemma 5.1. $\square$

The analogous result for power sum symmetric functions can also be obtained by this method, but the proof is a little technical. See Proposition 7.9.3 in [14] for a shorter, and more usual, alternative proof, using Cauchy's Identity

$$\prod \frac{1}{1 - x_i y_j} = \exp \sum_{n=1}^{\infty} \frac{p_n(x) p_n(y)}{n}.$$

It is helpful to use a definition from the character theory of the symmetric group. Given a partition μ of n with $\ell(\mu) = k$, and α a composition of n, let $\sigma_\alpha \in \mathrm{Sym}_n$ be a permutation of cycle-type α and let $\pi^\mu(\alpha)$ be the number of ordered set partitions $(P_1, \ldots, P_k)$ of $\{1, \ldots, n\}$ such that each P_i is a union of orbits of σ_α and $|P_i| = \mu_i$ for each i. Thus π^μ is the permutation character of Sym_n acting on the cosets of the *Young subgroup* $\mathrm{Sym}_\mu = \mathrm{Sym}_{\mu_1} \times \ldots \times \mathrm{Sym}_{\mu_{\ell(\mu)}}$.

For example, take $r \leq n/2$. Then $\pi^{(n-r,r)}$ is the permutation character of Sym_n acting on r-subsets of $\{1, \ldots, n\}$ and so $\pi^{(n-r,r)}(\sigma)$ is the number of r-subsets of $\{1, \ldots, n\}$ fixed by σ.

Theorem 5.3. *The bases $\{p_\mu : \mu \vdash n\}$ and $\{\frac{p_\nu}{z_\nu} : \nu \vdash n\}$ of Λ_n are dual.*

Proof. Fix a partition μ of n with $\ell(\mu) = k$. Given a composition α of n, having exactly a_j parts of size j for each $j \in \{1, \ldots, n\}$, define an *α-packing matrix* to be a $k \times n$ matrix C such that $\sum_{j=1}^n jC_{ij} = \mu_i$ for each $i \in \{1, \ldots, k\}$ and $\sum_{i=1}^k C_{ij} = a_j$ for each $j \in \{1, \ldots, n\}$.

Claim 1:

$$\pi^\mu(\alpha) = \sum \prod_{j=1}^{n} \frac{a_j!}{C_{1j}! \ldots C_{kj}!}$$

where the sum is over all α-packing matrices C. *Proof:* put C_{ij} of the a_j orbits of σ_α of size j in P_i. There are $\binom{a_j}{C_{1j}, \ldots, C_{kj}}$ ways to choose which orbits of size j go into each of $P_1, \ldots, P_k$. $||$

Claim 2:

$$p_\alpha = \sum_{\mu \vdash n} \pi^\mu(\alpha) \mathrm{mon}_\mu.$$

Proof: suppose that when we multiply out

$$p_1^{a_1} p_2^{a_2} \ldots p_n^{a_n} = (x_1 + \cdots + x_N)^{a_1} (x_1^2 + \cdots + x_N^2)^{a_2} \ldots (x_1^n + \cdots + x_N^n)^{a_n}$$

we choose to take x_i^j from exactly C_{ij} of the a_j terms in the product

$$(x_1^j + \cdots + x_N^j)^{a_j}.$$

We can do this in $\binom{a_j}{C_{1j}, \ldots, C_{Nj}}$ ways, for each $j \in \{1, \ldots, n\}$. Moreover, the product is x^μ if and only if C is an α-packing matrix. Claim 1 now implies that the coefficient of mon_μ is $\pi^\mu(\alpha)$. $||$

Claim 3:

$$h_r = \sum_{\beta \vdash r} \frac{p_\beta}{z_\beta}$$

Proof: This follows from the Cycle Index Formula (Theorem 1.12) and the following remark.

Claim 4:

$$h_\mu = \sum_{\alpha \vdash n} \frac{p_\alpha}{z_\alpha} \pi^\mu(\alpha).$$

Proof: This follows by multiplying out the product $h_{\mu_1} \dots h_{\mu_k}$, using Claim 3. The argument is similar to Claim 2, and is left as an exercise: see Question 26. $||$

By Claim 2 and Theorem 5.2, we have

$$\langle h_\mu, p_\alpha \rangle = \pi^\mu(\alpha). \tag{5.1}$$

Substituting in Claim 4 we get

$$h_\mu = \sum_{\alpha \vdash n} \langle h_\mu, p_\alpha \rangle \frac{p_\alpha}{z_\alpha}.$$

Therefore $f = \sum_{\alpha \vdash n} \langle f, p_\alpha \rangle \frac{p_\alpha}{z_\alpha}$ for all $f \in \Lambda_n$ and the result follows from Lemma 5.1. $\square$

5.2. **Gale–Ryser revisited.** By Question 25 we have

$$h_\mu = \sum_{\substack{\lambda \vdash n \\ \lambda \trianglerighteq \mu}} |\mathrm{SSYT}(\lambda, \mu)| s_\lambda \tag{5.2}$$

$$e_\nu = \sum_{\substack{\lambda \vdash n \\ \lambda' \trianglerighteq \nu}} |\mathrm{SSYT}(\lambda', \nu)| s_\lambda \tag{5.3}$$

Hence

$$\langle h_\mu, e_\nu \rangle = \sum_{\substack{\lambda \vdash n \\ \nu' \trianglerighteq \lambda \trianglerighteq \mu}} |\mathrm{SSYT}(\lambda, \mu)| |\mathrm{SSYT}(\lambda', \nu)|.$$

We saw in Lemma 1.3 that $e_\nu = \sum_{\mu \vdash n} N_{\mu\nu} \mathrm{mon}_\mu$. Taking the inner product with h_μ we obtain $\langle h_\mu, e_\nu \rangle = N_{\mu\nu}$. Therefore, to show that if $\nu' \trianglerighteq \mu$ then $N_{\mu\nu} > 0$ it is sufficient to prove the following (related) result on Kostka Numbers. The proof is the symmetric functions version of Theorem 2.2.20 in [5].

Lemma 5.4. *If $\lambda \trianglerighteq \mu$ then $|\mathrm{SSYT}(\lambda, \mu)| \geq 1$.*

Proof. Obviously $K_{\lambda\lambda} = 1$. Let μ and $\mu^\star$ be neighbours in the dominance order, with $\mu \triangleright \mu^\star$, so by Question 2, $\mu^\star_i = \mu_i - 1$, $\mu^\star_j = \mu_j + 1$ for some $i < j$, and $\mu^\star_k = \mu_k$ if $k \neq i, j$. Since $h_{(a,b)} = h_{(a+1,b-1)} + s_{(a,b)}$ whenever $a \geq b$, by Corollary 2.4, we have

$$h_{\mu^\star} = \Big(\prod_{k \neq i,j} h_{\mu_k}\Big) h_{\mu_i - 1} h_{\mu_j + 1} = \Big(\prod_{k \neq i,j} h_{\mu_k}\Big)\big(h_{\mu_i} h_{\mu_j} + s_{(\mu_i - 1, \mu_j + 1)}\big)$$

and so if $f = \prod_{k \neq i,j} h_{\mu_k}$ then

$$\begin{aligned} K_{\lambda\mu^\star} &= \langle s_\lambda, h_{\mu^\star} \rangle \\ &= \langle s_\lambda, h_\mu \rangle + \langle s_\lambda, f s_{(\mu_i - 1, \mu_j + 1)} \rangle \\ &= K_{\lambda\mu} + \langle s_\lambda, f s_{(\mu_i - 1, \mu_j + 1)} \rangle \\ &\geq K_{\lambda\mu}. \end{aligned}$$

The lemma follows. $\square$

This gives the direction of the Gale–Ryser Theorem not proved in §1.2. The notes for Question 11 give references for combinatorial proofs of Lemma 5.4.

5.3. **The ω involution.** The following definition is maybe best motivated by the connection with characters of the symmetric group, where it corresponds to multiplication by the sign character: see §5.5.

Definition 5.5. Let $\omega : \Lambda \to \Lambda$ be the $\mathbf{C}$-algebra homomorphism defined by $\omega(h_n) = e_n$.

It follows fairly easily from Newton's Identity (1.3) that $\omega(e_n) = h_n$: see Question 10. Thus ω is an involution. The related identities in (1.7) or Question 3(f) can be used to show that $\omega(p_n) = (-1)^{n-1}p_n$ for all $n \in \mathbf{N}$. (Again see Question 10.) Hence

$$\omega(p_\lambda) = \operatorname{sgn}(\sigma_\lambda)p_\lambda \tag{5.4}$$

where σ_λ is a permutation of cycle-type λ.

Lemma 5.6. *Let λ be a partition. Then $\omega(s_\lambda) = s_{\lambda'}$.*

Proof. Comparing (5.2) and (5.3) we get

$$\sum_{\substack{\lambda \vdash n \\ \lambda \trianglerighteq \mu}} |\mathrm{SSYT}(\lambda,\mu)|\omega(s_\lambda) = \sum_{\substack{\lambda \vdash n \\ \lambda' \trianglerighteq \mu}} |\mathrm{SSYT}(\lambda',\mu)|s_\lambda.$$

Suppose inductively that $\omega(s_\lambda) = s_{\lambda'}$ for all $\lambda \rhd \mu$. The previous equation implies that

$$\omega(s_\mu) + \sum_{\substack{\lambda \vdash n \\ \lambda \rhd \mu}} |\mathrm{SSYT}(\lambda,\mu)|\omega(s_\lambda) = s_{\mu'} + \sum_{\substack{\lambda \vdash n \\ \lambda' \rhd \mu}} |\mathrm{SSYT}(\lambda',\mu)|s_\lambda.$$

By induction, the left-hand side is

$$\omega(s_\mu) + \sum_{\substack{\lambda \vdash n \\ \lambda \rhd \mu}} |\mathrm{SSYT}(\lambda,\mu)|s_{\lambda'}.$$

If $\lambda \rhd \mu$ the coefficient of $s_{\lambda'}$ on either side is $|\mathrm{SSYT}(\lambda,\mu)|$. Therefore cancelling these equal terms we get $\omega(s_\mu) = s_{\mu'}$, as required. $\square$

Alternative proof. By the Murnaghan–Nakayama rule (rewriting Corollary 3.13 using Theorem 4.10) we have

$$p_\mu = \sum_{\lambda \vdash n} \mathrm{c}^\lambda(\mu)s_\lambda$$

where $\mathrm{c}^\lambda(\mu)$ is the sum of the signs of border-strip tableaux of shape λ and type μ. Taking the inner product with s_λ we get

$$\langle s_\lambda, p_\mu\rangle = \mathrm{c}^\lambda(\mu).$$

Hence if λ is a partition of n then, by Theorem 5.3,

$$s_\lambda = \sum_{\mu \vdash n} \langle s_\lambda, p_\mu \rangle \frac{p_\mu}{z_\mu} = \sum_{\mu \vdash n} \mathrm{c}^\lambda(\mu) \frac{p_\mu}{z_\mu}.$$

Applying ω we obtain

$$\omega(s_\lambda) = \sum_{\mu \vdash n} \mathrm{c}^\lambda(\mu) \operatorname{sgn}(\sigma_\mu) \frac{p_\mu}{z_\mu} = \sum_{\mu \vdash n} \mathrm{c}^{\lambda'}(\mu) \frac{p_\mu}{z_\mu} = s_{\lambda'}$$

where $\sigma_\mu \in \mathrm{Sym}_n$ is a permutation of cycle-type μ. Here the middle equality is proved by transposing the border-strip tableaux counted (with signs) in $\mathrm{c}^\lambda(\mu)$. $\square$

5.4. **Littlewood–Richardson Rule.** In the final lecture I sketched a proof of the Littlewood–Richardson Rule. This needs some extensions to the results proved above, specifically the Jacobi–Trudi formula for Schur functions labelled by skew-partitions, and the Lascoux–Schützenberger involution for skew-tableaux. Essentially the proof is the specialization of the proof of the SXP rule in [18] (in turn inspired by [12]), replacing multitableaux with skew-tableaux. See Question 27, 30, and 31 and the outline answers.

5.5. **Connection with the symmetric group.** Let $n \in \mathbf{N}_0$. Let $\mathrm{Cl}(\mathrm{Sym}_n)$ be the set of functions $\mathrm{Sym}_n \to \mathbf{C}$ that are constant on conjugacy classes. Thus $\mathrm{Cl}(\mathrm{Sym}_n)$ has as a basis the indicator functions $\mathbb{1}_\alpha$ for α a partition of n defined by

$$\mathbb{1}_\alpha(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ has cycle-type } \alpha \\ 0 & \text{otherwise.} \end{cases}$$

The elements of $\mathrm{Cl}(\mathrm{Sym}_n)$ are called *class functions*. There is an inner product on $\mathrm{Cl}(\mathrm{Sym}_n)$ defined by

$$\langle \phi, \psi \rangle = \frac{1}{n!} \sum_{\sigma \in \mathrm{Sym}_n} \phi(\sigma)\overline{\psi(\sigma)}.$$

Proposition 5.7. *There is a linear isometry $\Lambda_n \to \mathrm{Cl}(\mathrm{Sym}_n)$ defined by $\frac{p_\alpha}{z_\alpha} \mapsto \mathbb{1}_\alpha$ for each $\alpha \vdash n$. Under this isometry the ω-involution corresponds to multiplication by the sign character and the image of s_λ is an irreducible character of* Sym_n.

Proof. By Claim 4 in the proof of Theorem 5.3 we have $h_\mu = \sum_{\alpha \vdash n} \frac{p_\alpha}{z_\alpha} \pi^\mu(\alpha)$. Therefore the image of h_μ is the permutation character π^μ. Since

$$\langle h_\mu, \frac{p_\alpha}{z_\alpha} \rangle = \frac{\pi^\mu(\alpha)}{z_\alpha} = \langle \pi^\mu, \mathbb{1}_\alpha \rangle,$$

the map is an isometric embedding. Since Λ_n and $\mathrm{Cl}(\mathrm{Sym}_n)$ both have dimension equal to the number of partitions of n, it is an isometry.

Applying the ω-involution to Claim 4 and using (5.4) we get $e_\mu = \sum_{\alpha\vdash n} \frac{p_\alpha}{z_\alpha}\pi^\mu(\alpha)\,\mathrm{sgn}(\sigma_\alpha)$, where σ_α is an element of cycle-type α. Therefore the image of e_μ is $\pi^\mu\,\mathrm{sgn}_{\mathrm{Sym}_n}$, where $\mathrm{sgn}_{\mathrm{Sym}_n}$ is the sign character of Sym_n. This proves the second claim.

By (5.2) and (5.3) we have $h_\mu = h + s_\mu$ and $e_{\mu'} = e + s_\mu$ where h is a sum of Schur functions s_λ for $\lambda \rhd \mu$ and e is a sum of Schur functions s_λ for $\lambda \lhd \mu$. Therefore $\langle h_\mu, e_{\mu'}\rangle = 1$ and so the ordinary characters π^μ and $\pi^{\mu'}\,\mathrm{sgn}$ have a unique irreducible constituent in common, namely the image of s_μ. $\square$

Given a partition λ of n, let $\chi^\lambda \in \mathrm{Cl}(\mathrm{Sym}_n)$ denote the image of s_λ. By Corollary 3.13, taking the inner product of each side with s_λ, we have

$$\langle s_\lambda, p_\alpha\rangle = \mathrm{c}^\lambda(\alpha)$$

where $\mathrm{c}^\lambda(\alpha)$ is the sum of the signs of the border-strip tableaux of shape λ and type α defined in Definition 3.12. Hence $\mathrm{c}^\lambda(\alpha) = \langle \chi^\lambda, z_\alpha \mathbb{1}_\alpha\rangle = \chi^\lambda(\sigma_\alpha)$ where σ_α has cycle-type α. The Murnaghan–Nakayama rule therefore gives the values of χ^λ, as claimed after Corollary 3.13.

In fact it is more common to work with the inverse map, known as the *characteristic isomorphism*, defined for $\phi \in \mathrm{Cl}(\mathrm{Sym}_n)$ by

$$\phi \mapsto \frac{1}{n!}\sum_{\sigma\in\mathrm{Sym}_n} \phi(\sigma)p_{\rho(\sigma)} \tag{5.5}$$

where $\rho(\sigma)$ is the cycle-type of $\sigma \in \mathrm{Sym}_n$. Let $\mathrm{ch} : \bigoplus_{n\in\mathbf{N}_0}\mathrm{Cl}(\mathrm{Sym}_n) \to \Lambda$ be the resulting isometry. (We define the inner product of two class functions of symmetric groups of different degree to be zero.) The left-hand side $\bigoplus_{n\in\mathbf{N}_0}\mathrm{Cl}(\mathrm{Sym}_n)$ is then a graded ring with product

$$\phi\circ\psi = \big(\phi\times\psi\big)\big\uparrow_{\mathrm{Sym}_m\times\mathrm{Sym}_n}^{\mathrm{Sym}_{m+n}} \tag{5.6}$$

where $\phi \in \mathrm{Cl}(\mathrm{Sym}_m)$ and $\psi \in \mathrm{Cl}(\mathrm{Sym}_n)$ and the arrow denotes induction. We have already proved most of the following remarkable theorem. In it, recall from §5.1 that π^μ is the permutation character of Sym_n acting on the cosets of the Young subgroup Sym_μ.

Theorem 5.8. *The map* $\mathrm{ch} : \bigoplus_{n\in\mathbf{N}_0}\mathrm{Cl}(\mathrm{Sym}_n) \to \Lambda$ *is an isometric ring isomorphism. It satisfies* $\mathrm{ch}\,\chi^\lambda = s_\lambda$, $\mathrm{ch}\,\pi^\mu = h_\mu$ *and* $\mathrm{ch}\,\mathbb{1}_\alpha = \frac{p_\alpha}{z_\alpha}$ *for all partitions* λ, μ, α. *Moreover* $\mathrm{ch}(\phi\,\mathrm{sgn}_{\mathrm{Sym}_n}) = \omega(\mathrm{ch}\,\phi)$ *for all class functions* ϕ *of* Sym_n.

Proof. It only remains to show that ch is a ring homomorphism. Let λ and μ be partitions of n. We have $\pi^\lambda\circ\pi^\mu = \pi^\nu$ where ν is the partition whose multiset of parts is the union of the multisets of parts of λ and μ. Therefore

$$\mathrm{ch}\,(\pi^\lambda\pi^\mu) = \mathrm{ch}\,\pi^\nu = h_\nu = h_\lambda h_\mu = (\mathrm{ch}\,\pi^\lambda)(\mathrm{ch}\,\pi^\mu)$$

as required. $\square$

Remark 5.9. Suppose we fix $N \in \mathbf{N}$ and attempt to define

$$\mathbf{C}[x_1, \ldots, x_N]^{\mathrm{Sym}_N} \to \bigoplus \mathrm{Cl}(\mathrm{Sym}_n)$$

by $s_\lambda(x_1, \ldots, x_N) \mapsto \chi^\lambda$. There is an immediate problem: this map is not well-defined because the $s_\lambda(x_1, \ldots, x_N)$ are not linearly independent when $N < \ell(\lambda)$. (Indeed, by Lemma 1.16, we have $s_{(1^n)}(x_1, \ldots, x_N) = 0$ whenever $N < n$.) This shows that working with infinitely many variables, i.e. symmetric functions rather than symmetric polynomials, is essential to get a nice correspondence with the symmetric group.

The corollary below follows from the Littlewood–Richardson rule (in a more explicit form), but has a much simpler proof using character theory.

Corollary 5.10. *Let $\mu \vdash m$ and $\nu \vdash n$. Then $s_\mu s_\nu$ is a non-negative integral linear combination of Schur functions.*

Proof. The corresponding claim for symmetric group characters, that

$$(\chi^\mu \times \chi^\nu)\uparrow_{\mathrm{Sym}_m \times \mathrm{Sym}_n}^{\mathrm{Sym}_{m+n}}$$

is a non-negative integral linear combination of irreducible characters, is obvious. $\square$

5.6. **The cycle index revisited⋆.** The Cycle Index Formula for the symmetric group was seen in Theorem 1.12. It generalizes as follows. Given $\sigma \in \mathrm{Sym}_n$, let

$$\mathrm{cyctype}(\sigma) = (1^{\mathrm{cyc}_1(\sigma)}, \ldots, n^{\mathrm{cyc}_n(\sigma)})$$

be the cycle-type of σ. Let $G \leq \mathrm{Sym}_n$ be a permutation group. The *cycle index* of G is defined by

$$\mathrm{cyc}_G = \frac{1}{|G|} \sum_{\sigma \in G} p_{\mathrm{cyctype}(\sigma)}.$$

The general cycle index just defined appears in a beautiful theorem of Polya, stated in §5.7 below and proved using the characteristic isometry. Here we give some more immediate motivation for the cycle index using the results from §1.11.

An explicit definition of the inverse of the characteristic isometry will be useful. For each partition λ of n, let $\sigma_\lambda \in \mathrm{Sym}_n$ be a chosen element such that $\mathrm{cyctype}(\sigma_\lambda) = \lambda$. Then $\mathrm{ch}^{-1} : \Lambda_n \to \mathrm{Cl}(\mathrm{Sym}_n)$ is defined on a general element $\sum_\lambda a_\lambda \frac{p_\lambda}{z_\lambda} \in \Lambda_n$ by

$$\mathrm{ch}^{-1}\Big(\sum_\lambda a_\lambda \frac{p_\lambda}{z_\lambda}\Big) = \phi \tag{5.7}$$

where $\phi \in \mathrm{Cl}(\mathrm{Sym}_n)$ is the class function such that $\phi(\sigma_\lambda) = a_\lambda$ for each partition λ of n.

Example 5.11.

(1) Since Sym_n has $n!/z_\lambda$ elements of cycle-type λ, $\mathrm{cyc}_{\mathrm{Sym}_n} = \sum_\lambda \frac{p_\lambda}{z_\lambda}$. where the sum is over all partitions of n. Hence by (5.7), we have $\mathrm{ch}^{-1} \mathrm{cyc}_{\mathrm{Sym}_n} = 1_{\mathrm{Sym}_n}$, where 1_{Sym_n} is the trivial character of Sym_n. By the remark following Theorem 1.12, $\mathrm{cyc}_{\mathrm{Sym}_n} = h_n$. Moreover, by (1.9), $\exp \sum_{k=1}^\infty \frac{p_k}{k} t^k = \sum_{n=0}^\infty \mathrm{cyc}_{\mathrm{Sym}_n} t^n$. We saw in §1.11 some of the applications of the cycle index of Sym_n to enumerating permutations.

(2) $\mathrm{cyc}_1 = p_1^n$ and $\mathrm{ch}^{-1} p_1^n = n! \mathbb{1}_{(1^n)}$ is the regular character of Sym_n; this is the permutation character of Sym_n acting on itself by right multiplication. To illustrate that multiplication of symmetric polynomials becomes the 'multiply and induce' product on class functions in (5.6), observe that the regular character is $(1_{S_1} \times \cdots \times 1_{S_1}) \uparrow^{S_n} = 1_{S_n} \uparrow^{S_n}$ where $1_{S_1} = \mathrm{ch}\, p_1$ is the trivial character of S_1.

(3) Let $n = 4$, let $H = \langle (1,2,3,4) \rangle$ and let $G = \langle (1,2,3,4), (1,3) \rangle$ be the dihedral group. Then

$$\begin{aligned}
\mathrm{cyc}_H &= \frac{1}{4}(p_{(1,1,1,1)} + p_{(2,2)} + 2p_{(4)}) \\
&= 6\mathrm{mon}_{(1,1,1,1)} + 3\mathrm{mon}_{(2,1,1)} + 2\mathrm{mon}_{(2,2)} + \mathrm{mon}_{(3,1)} + \mathrm{mon}_{(4)} \\
\mathrm{cyc}_G &= \frac{1}{8}(p_{(1,1,1,1)} + 2p_{(2,1,1)} + 3p_{(2,2)} + 2p_{(4)}) \\
&= 3\mathrm{mon}_{(1,1,1,1)} + 2\mathrm{mon}_{(2,1,1)} + 2\mathrm{mon}_{(2,2)} + \mathrm{mon}_{(3,1)} + \mathrm{mon}_{(4)}
\end{aligned}$$

and the corresponding class functions on Sym_4 are

	$(1,1,1,1)$	$(2,1,1)$	$(2,2)$	$(3,1)$	(4)
$\mathrm{ch}^{-1} \mathrm{cyc}_H$	6	0	2	0	2
$\mathrm{ch}^{-1} \mathrm{cyc}_G$	3	1	3	0	1

For instance $\mathrm{ch}^{-1} \mathrm{cyc}_H$ has value $\frac{1}{4} 2 z_{(4)} = 2$ on 4-cycles. Observe that $\mathrm{ch}^{-1} \mathrm{cyc}_H$ is the permutation character of Sym_4 acting on the cosets of H, and similarly for G. We record the monomial expansions for later use as an example of Theorem 5.14.

More generally we have the following lemma.

Lemma 5.12. *Let $G \leq \mathrm{Sym}_n$ be a permutation group. Then $\mathrm{ch}^{-1} \mathrm{cyc}_G$ is the permutation character of Sym_n acting on the cosets of G.*

Proof. Suppose that G has m_λ elements of cycle-type λ. By (5.7),

$$\mathrm{ch}^{-1} \mathrm{cyc}_G = \phi$$

where $\phi \in \mathrm{Cl}(\mathrm{Sym}_n)$ is the class function defined by

$$\phi(\sigma) = \frac{z_{\mathrm{cyctype}(g)}}{|G|} m_{\mathrm{cyctype}(\sigma)}. \tag{5.8}$$

Let π be the permutation character of S_n acting on the cosets of G. Thus $\pi(\sigma)$ is the number of cosets $G\tau$ such that $G\tau\sigma = G\tau$, or equivalently, such that $\tau\sigma\tau^{-1} \in G$. Therefore

$$\pi(\sigma_\lambda) = \frac{1}{|G|}\big|\{\tau \in \mathrm{Sym}_n : \tau\sigma_\lambda\tau^{-1} \in G\}\big| = \frac{z_\lambda}{|G|} m_\lambda. \tag{5.9}$$

where the second equality uses that z_λ is the size of the centralizer of a permutation of cycle-type λ. Comparing (5.8) and (5.9) we get the required conclusion. $\square$

5.7. **Pólya's Cycle Index Theorem.** Polya's Theorem concerns the action of Sym_n on the set W_n of words of length n with entries from $\mathbf{N}$. Given $w \in W_n$ let $\mathrm{cont}(w)$ be the composition α of n such that α_i is the number of entries of w equal to i. As seen in §4.3, the symmetric group Sym_n acts on the set of such words by place permutation: $(w\sigma)_i = w_{i\sigma^{-1}}$. Given a permutation subgroup $G \leq \mathrm{Sym}_n$ let W_n/G denote the set of orbits of G on W_n. We write $\mathrm{cont}([w])$ for the common content of words in the orbit of w. Observe that if α is a composition rearranging to the partition λ then the number of orbits with content α and λ agree. We may therefore enumerate all the orbits by the symmetric function

$$\mathrm{enum}_G = \sum_\lambda c_\lambda \mathrm{mon}_\lambda$$

where the sum is over all partitions of n and c_λ is the number of orbits of content λ.

Example 5.13. Let $H = \langle(1,2,3,4)\rangle \leq \mathrm{Sym}_4$. Then W_4/H is the set of 4-bead *necklaces*, where two necklaces are identified if they are equal up to a rotation. There are two necklaces with content $(2,2)$, namely 1122 and 1212, and six necklaces with content $(1,1,1,1)$, namely 1234, 1243, 1324, 1342, 1423, 1432. Note that, as can be seen from the monomial expansions in Example 5.11(3), these are the coefficients of $\mathrm{mon}_{(2,2)}$ and $\mathrm{mon}_{(1,1,1,1)}$ in cyc_H. We leave it as an exercise to check the other coefficients, and the corresponding result for W_4/G, thought of as the set of 4-bead necklaces up to both rotation and reflection.

Pólya's Theorem says this holds in general, i.e. enum_G is cyc_G written in the monomial basis.

Theorem 5.14. *Let $G \leq \mathrm{Sym}_n$ be a permutation group and let λ be a partition of n. The coefficient of* mon_λ *in* cyc_G *is the number of orbits of G on W_n with content λ.*

Proof. Let m_λ be the number of orbits of G on W_n with content λ. The subset W_λ of W_n of words with content λ is, as a Sym_n-set, equivalent to the coset space $\mathrm{Sym}_\lambda \setminus \mathrm{Sym}_n$, where Sym_λ is the Young subgroup for λ seen in §5.1. Therefore m_λ is the number of double cosets $\mathrm{Sym}_\lambda\, \sigma G$ for $\sigma \in \mathrm{Sym}_n$. The coefficient in the statement of the theorem is

$$\begin{aligned}
[\mathrm{mon}_\lambda]\, \mathrm{cyc}_G &= \langle h_\lambda, \mathrm{cyc}_G \rangle_{\Lambda_n} \\
&= \langle \pi^\lambda, 1\big\uparrow_G^{\mathrm{Sym}_n} \rangle_{\mathrm{Sym}_n} \\
&= \langle 1\big\uparrow_{\mathrm{Sym}_\lambda}^{\mathrm{Sym}_n}, 1\big\uparrow_G^{\mathrm{Sym}_n} \rangle_{\mathrm{Sym}_n} \\
&= \langle 1\big\uparrow_{\mathrm{Sym}_\lambda}^{\mathrm{Sym}_n}\big\downarrow_G, 1_G \rangle_G \\
&= m_\lambda
\end{aligned}$$

where the equalities hold by the duality between the monomial and complete symmetric functions in Theorem 5.2, Lemma 5.12, Theorem 5.8, the interpretation of π^λ as a permutation character, Frobenius reciprocity and finally Mackey's Theorem. □

See Exercise 32 below for the connection between Pólya's Theorem and the more basic orbit counting method using the result commonly known as Burnside's Lemma.

6. Problems

Hints, references or solutions for these problems are given in the final section.

(Lecture 1) Gale–Ryser Theorem and dominance order

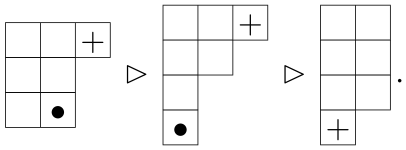
1. Let λ be a partition.

(a) Show that there is a unique 0-1 matrix with row sums λ and column sums λ'. [*Hint:* for existence, use the Young diagram of λ.]

(b) Suppose that $\mu \lhd \lambda'$. By the Gale–Ryser Theorem (the direction not proved in Lecture 1), there is a 0-1 matrix with row sums λ and column sums μ. Assuming this, prove that there are at least two such matrices.

2. (a) Let λ and ν be partitions of n. Show that if λ and ν are neighbours in the dominance order (i.e. $\lambda \rhd \nu$ and if $\lambda \trianglerighteq \mu \trianglerighteq \nu$ then either $\lambda = \mu$ or $\mu = \nu$) then $[\nu]$ can be obtained from $[\lambda]$ by moving a single box down in the Young diagram, and conversely, after any such move on $[\lambda]$, we obtain a partition ν with $\lambda \rhd \nu$.

For example, $\lambda = (3, 2, 2)$ and $\nu = (3, 2, 1, 1)$ are neighbours in the dominance order, obtained by moving the box $(3, 2) \in [\lambda]$ to the first available position $(4,1) \in [\nu]$. If instead $(1,3)$ is moved to $(4,1)$ we get $(2,2,2,1)$, which is a neighbour of ν, but not of λ:



(b) Let α and β be partitions of n. Show that $\alpha \trianglerighteq \beta$ if and only if $\alpha' \trianglelefteq \beta'$. (In particular, this implies that the condition $\lambda' \trianglerighteq \mu$ in the Gale–Ryser theorem is symmetric with respect to λ and μ.)

(Lectures 2 and 3) Newton's Identities

3. Newton's Identity $\sum_{k=0}^{n}(-1)^k h_{n-k}e_k = 0$ was proved using the generating functions $H(t) = \sum_{n=0}^{\infty} h_n t^n = \prod_{i=1}^{\infty}(1 - x_i t)^{-1}$ and $E(t) = \sum_{n=0}^{\infty}(-1)^n e_n t^n = \prod_{i=1}^{\infty}(1 - x_i t) = H(t)^{-1}$.

This exercise gives some related identities involving the power sum symmetric functions $p_k = \mathrm{mon}_{(k)} = x_1^k + x_2^k + \cdots$ that have similar short proofs using generating functions.

(a) Let $Q(t) = \sum_{k=1}^{\infty} p_k t^k / k$. Show that $Q(t) = -\sum_{i=1}^{\infty} \log(1 - x_i t)$.

(b) From (a), or directly, prove that $\sum_{k=1}^{\infty} p_k t^k = \sum_{i=1}^{\infty} x_i t/(1 - x_i t)$.

(c) Prove that $tH'(t) = tQ'(t)H(t)$.

(d) Deduce that $nh_n = \sum_{k=1}^{n} p_k h_{n-k}$.

(e) Prove an identity analogous to (d) for ne_n.

(f) Prove that $p_n = \sum_{k=0}^{n-1}(-1)^k(n-k)h_{n-k}e_k$.

(g) For $\mu \vdash n$ define $p_\mu = p_{\mu_1} \dots p_{\mu_{\ell(\mu)}}$. Show that $\{p_\mu : \mu \vdash n\}$ is a basis for Λ_n.

(h) Show that $\exp Q(t) = H(t)$.

(Lecture 2) Complete homogeneous symmetric functions and another version of the complete/elementary duality.

4. Give a combinatorial interpretation to the coefficients $M_{\lambda\mu}$ expressing the complete homogeneous symmetric functions in the monomial basis,

$$h_\mu = \sum_\lambda M_{\lambda\mu} \mathrm{mon}_\lambda$$

analogous to Lemma 1.3. In particular show that the matrix M is symmetric.

5. Let μ be a partition of n. Define a change of basis matrix R by

$$h_\mu = \sum_\lambda R_{\lambda\mu} e_\lambda.$$

(a) Show that $R_{\mu\mu} = (-1)^{n-\ell(\mu)}$.

(b) Show that if $R_{\lambda\mu} \neq 0$ then $\mu \trianglerighteq \lambda$.

(c) Show that $R_{(1^n)\mu} = 1$ for all $\mu \vdash n$. [*Hint:* consider the coefficient of x_1^n in h_μ.]

(d) Show that $R_{(2,1^{n-2})\mu} = -n + \ell(\mu)$.

(e) Show that R is self-inverse, or equivalently, $e_\mu = \sum_\lambda R_{\lambda\mu} h_\lambda$. [*Hint:* use the ω involution in Question 10.]

(Lecture 2) MacMahon's Master Theorem

6. Let a, b, $c \in \mathbf{N}_0$. Use MacMahon's Master Theorem to prove the following generalization of Dixon's Identity

$$\sum_k (-1)^k \binom{a+b}{a+k}\binom{b+c}{b+k}\binom{c+a}{c+k} = \frac{(a+b+c)!}{a!b!c!}.$$

7. Given a composition α of n let $v(\alpha) = (1,\ldots,1,\ldots,n,\ldots,n)$ where the number j appears exactly α_j times. Let d_α be the number of sequences $(u_1,\ldots,u_n)$ such that $\{u_1,\ldots,u_n\} = \{1,\ldots,n\}$ and $u_i \neq v(\alpha)_i$ for any i.

(a) Show that $d_{(1^n)}$ is the number of permutations of $\{1,\ldots,n\}$ with no fixed points. (Such permutations are called *derangements*.)

(b) Let e_n be the number of derangements of $\{1,\ldots,n\}$ with sign $+1$ and let o_n be the number of derangements of $\{1,\ldots,n\}$ with sign -1. By evaluating the determinant of the matrix

$$\begin{pmatrix} 0 & 1 & \ldots & 1 \\ 1 & 0 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 0 \end{pmatrix}$$

in two different ways, show that $o_n - e_n = (-1)^n(n-1)$. [For an alternative proof see Question 9(b).]

Now suppose that $\ell(\alpha) = m$. Work with symmetric polynomials in m variables $x_1,\ldots,x_m$.

(c) Show that $d_\alpha = [x^\alpha](p_1(x) - x_1)^{\alpha_1} \ldots (p_1(x) - x_m)^{\alpha_m}$ where

$$p_1(x) = p_1(x_1,\ldots,x_m) = x_1 + \cdots + x_m.$$

(d) Deduce from MacMahon's Master Theorem that

$$d_\alpha = [x^\alpha] \det \begin{pmatrix} 1 & -x_1 & \dots & -x_1 \\ -x_2 & 1 & \dots & -x_2 \\ \vdots & \vdots & \ddots & \vdots \\ -x_m & -x_m & \dots & 1 \end{pmatrix}^{-1}.$$

(e) Using (b) and (d) deduce that

$$d_\alpha = [x^\alpha] \frac{1}{1 - e_2(x) - 2e_3(x) - \dots - (m-1)e_m(x)}.$$

(f) Show that

$$(1 - e_2(x) - 2e_3(x) - \cdots)(1 + d_{(1)}e_1 + d_{(1^2)}e_2 + d_{(1^3)}e_3 + \cdots) = 1 + f$$

where every monomial summand in f is divisible by some x_i^2. Hence show that

$$d_{(1^n)} = \binom{n}{2} d_{(1^{n-2})} + 2\binom{n}{3} d_{(1^{n-3})} + \cdots + (n-1)\binom{n}{n} d_{(1^0)}$$

for every $n \in \mathbf{N}$.

(g) Prove more generally that if $\ell \in \mathbf{N}$ and $n \geq \ell$ then

$$d_{(1^n)} = \frac{n!}{\ell!} d_{(1^\ell)} + (-1)^{\ell+1} \sum_{m=\ell+1}^{n} \binom{m-1}{\ell} \binom{n}{m} d_{(1^{n-m})}.$$

[This question is based on §71 of MacMahon, *Combinatory Analysis Vol. I.* According to MacMahon, the identity in (f) was not known before the Master Theorem. It has a short proof using generating functions, which is the only way I know to prove (g); this identity is maybe not so easy to discover without the Master Theorem.]

(Lecture 3) Power sum symmetric functions and cycle indices.

8. Recall that if λ is a partition of n with exactly a_i parts of size i, for each $i \in \{1, \dots, n\}$, then

$$z_\lambda = 1^{a_1} 2^{a_2} \dots n^{a_n} a_1! a_2! \dots a_n!.$$

(a) Prove that z_λ is the size of the centralizer in Sym_n of an element of cycle-type λ. (Or, more combinatorially, you might prefer to prove that $n!/z_\lambda$ is the size of the conjugacy class, and deduce the result for centralizers.)

(b) Prove that $\displaystyle\sum_{\lambda \vdash n} \frac{p_\lambda}{z_\lambda} = h_n$.

(c) What is the analogous identity for elementary symmetric functions?

9. (a) Consider the specialization $p_k \mapsto (-1)^{k-1}$ for $k \in \mathbf{N}$. Let λ be a partition of n. Show that $p_\lambda \mapsto \operatorname{sgn} \sigma_\lambda$ where σ_λ is a permutation of cycle-type λ.

(b) Let o_n be the number of derangements in Sym_n that are odd permutations, and let e_n be the number of derangements in Sym_n that are even permutations. By specializing the Cycle Index Formula, prove that $o_n - e_n = (-1)^n(n-1)$ for all $n \in \mathbf{N}_0$.

(c) [American Mathematical Monthly Problem 11668] Let O_n be the number of derangements of $\{1, 2, \ldots, n\}$ with an odd number of cycles in their disjoint cycle decomposition, and let E_n be the number of derangements of $\{1, 2, \ldots, n\}$ with an even number of cycles in their disjoint cycle decomposition. Prove that $O_n - E_n = n - 1$ for all $n \in \mathbf{N}_0$.

10. Let $\omega : \Lambda \to \Lambda$ be the $\mathbf{C}$-algebra homomorphism defined by $\omega(h_n) = e_n$. Use Newton's Identity to show that $\omega(e_n) = h_n$. Show also that $\{p_\lambda : \lambda \vdash n\}$ is a basis of Λ_n consisting of eigenvectors for ω and determine the eigenvalues.

(Lecture 4) Schur functions

11. Let λ and μ be partitions of n.

(a) Show that if $\mathrm{SSYT}(\lambda, \mu) \neq \varnothing$ then $\lambda \trianglerighteq \mu$.

(b) Show, conversely, that if $\lambda \trianglerighteq \mu$ then $|\mathrm{SSYT}(\lambda, \mu)| \geq 1$.

(c) It follows from (a) that

$$s_\lambda = \sum_{\mu \trianglelefteq \lambda} K_{\lambda\mu} \mathrm{mon}_\mu$$

where $K_{\lambda\mu} = |\mathrm{SSYT}(\lambda, \mu)|$. (These are the *Kostka numbers*.) Show that $K_{\lambda\lambda} = 1$ and deduce that $\{s_\lambda : \lambda \vdash n\}$ is a basis for Λ_n.

(d) Show that $s_{(n-a,a)}(x_1, x_2) = \sum_{a \leq b \leq n/2} \mathrm{mon}_{(n-b,b)}(x_1, x_2)$.

(Lecture 4) Jacobi–Trudi Identity

12. Give a proof of the general Jacobi–Trudi Identity by generalizing Example 2.2. In particular, show that the symmetric polynomial in $x_1, \ldots, x_n$ obtained by expanding the determinant by choosing $h_{\lambda_{j\sigma} - j\sigma + j}$ from column j for each $j \in \{1, \ldots, M\}$ is the sum of the signed weights of the path tuples $(P_M, \ldots, P_1)$ such that P_j, the path starting at $(M - j, 1)$, finishes at $(M - j\sigma + \lambda_{j\sigma}, N)$ for each j.

13. Another version of the 'proof-by-example' of Theorem 2.1 had an error, pointed out to me by Darij Grinberg. Suppose, in the setting of Example 2.2, we define a map $K : \mathcal{A}\backslash\mathcal{S} \to \mathcal{A}\backslash\mathcal{S}$ on path triples (P_3, P_2, P_1) with at least one intersection as follows:

(a) choose i minimal such that P_i meets another path;
(b) choose j minimal such that P_i meets P_j;

then (as before) swap P_i and P_j after their first intersection. Show that K is not an involution.

14. Let λ be a partition with $M \geq \ell(\lambda)$. Applying the ω involution (jump ahead to Definition 5.5) to the Jacobi–Trudi Identity and using Lemma 5.6 gives

$$s_{\lambda'} = \det \begin{pmatrix} e_{\lambda_1} & e_{\lambda_1+1} & \cdots & e_{\lambda_1+(M-1)} \\ e_{\lambda_2-1} & e_{\lambda_2} & \cdots & e_{\lambda_2+(M-2)} \\ \vdots & \vdots & \ddots & \vdots \\ e_{\lambda_M-(M-1)} & e_{\lambda_M-(M-2)} & \cdots & e_{\lambda_M} \end{pmatrix}$$

Give an involutive proof of this identity.

A polarization identity: see MathOverflow 61884

15. (a) Let $N, n \in \mathbf{N}$. Let

$$P(x_1, \ldots, x_N) = \sum_{I \subseteq \{1,\ldots,N\}} (-1)^{n-|I|} \Big(\sum_{i \in I} x_i\Big)^n.$$

Show that

$$P(x_1, \ldots, x_N) = \begin{cases} 0 & \text{if } n < N \\ N! x_1 \ldots x_N & \text{if } n = N. \end{cases}$$

(b) Hence show that if V is a $\mathbf{C}$-vector space and $f : V \times \cdots \times V \to \mathbf{C}$ is an n-multilinear form symmetric in its variables then

$$f(v_1, \ldots, v_n) = \frac{1}{n!} \sum_{I \subseteq \{1,\ldots,n\}} (-1)^{n-|I|} g\Big(\sum_{i \in I} v_i\Big)$$

where $g(u) = f(u, \ldots, u)$ for $u \in V$. (A special case is the well known polarization identity $2f(v, v') = f(v + v', v + v') - f(v, v) - f(v', v')$.)

(c) Express P in the monomial basis of symmetric polynomials in N variables when $n > N$.

(Lecture 6) Young's Rule

16. Let λ be a partition of n and let $r \in \mathbf{N}_0$. Let $Y_r(\lambda)$ be the set of partitions μ such that $[\mu]$ is obtained from $[\lambda]$ by adding r boxes, no two in the same column. Prove Young's Rule that

$$a_{\lambda+\delta} h_r(x_1, \ldots, x_N) = \sum_{\substack{\mu \in Y_r(\lambda) \\ \ell(\mu) \leq N}} a_{\mu+\delta}$$

using a sign-reversing involution on the set

$$\mathcal{P} = \{(A, S) : A \in \mathrm{Abc}(\lambda),\ S \text{ a multisubset of } \{1, \ldots, N\},\ |S| = r\}.$$

Related to Lecture 7 and Murnaghan–Nakayama rule: cores and quotients of partitions. See [5, §2.7] *for background definitions. Given $\lambda \vdash n$, let $f^\lambda = \chi^\lambda(1^n)$ be the number of standard tableaux of shape λ.*

17. Let $\ell \in \mathbf{N}$ and let λ be a partition of $n \in \mathbf{N}_0$ with ℓ-core γ. Let $(\lambda(0), \ldots, \lambda(\ell-1))$ be the ℓ-quotient of λ. Define $w \in \mathbf{N}_0$ so that $n = |\gamma| + w\ell$ and let $\alpha = (1^{|\gamma|}, \ell^w)$. Show that

$$\chi^\lambda(\alpha) = \pm f^{\lambda(0)} f^{\lambda(1)} \ldots f^{\lambda(\ell-1)}.$$

18. Use 2-quotients of partitions to prove the following identity of Gauss:

$$\sum_{n=0}^{\infty} x^{\frac{n(n+1)}{2}} = \frac{(1-x^2)(1-x^4)(1-x^6)\ldots}{(1-x)(1-x^3)(1-x^5)\ldots}.$$

19. Let $\ell, m \in \mathbf{N}$ be coprime.

(a) Show that $\ell m - \ell - m$ is the largest number that cannot be expressed in the form $a\ell + bm$ where $a, b \in \mathbf{N}_0$. [*Hint:* show that if N is the largest such number then $N \leq \ell m$, $N + \ell$ is a multiple of m and $N + m$ is a multiple of ℓ.]

(b) Using (a) and the abacus, show that there are only finitely many partitions that are simultaneous ℓ-cores and m-cores.

(c) Prove that the number of simultaneous ℓ-core and m-core partitions is

$$\frac{1}{\ell+m}\binom{\ell+m}{\ell}.$$

An alternative proof of the Murnaghan–Nakayama rule

20. Observe that $p_2 = 2h_2 - h_1^2$. Using Young's rule, show that if λ is any partition then

$$s_\lambda(2h_2 - h_1^2) = \sum_{\mu = \lambda + \square\square} s_\mu - \sum_{\nu = \lambda + \begin{smallmatrix}\square\\ \square\end{smallmatrix}} s_\nu,$$

where the notation indicates that μ is obtained from λ by adding a 2-rim-hook consisting of two boxes in the same row, and ν is obtained from λ by adding a 2-rim-hook consisting of two boxes in the same column. Deduce the Murnaghan–Nakayama rule for the product $s_\lambda p_2$.

21. Generalizing the previous question, use Question 3(f) and the Young and Pieri rules to prove the Murnaghan–Nakayama rule by a cancelling involution.

(Lecture 8) Lascoux–Schützenberger involution

22. Let λ be a partition such that $\ell(\lambda) \leq N$.

(a) Let $\sigma \in \mathrm{Sym}_N$ and suppose that $j\sigma > (j+1)\sigma$. Show that if $\beta = \lambda \cdot \sigma$ then $\beta_{j\sigma} > \beta_{(j+1)\sigma}$.

(b) Deduce that if $\tau \in \mathrm{Sym}_N$ and $\lambda \cdot \tau$ is a partition then $\tau = \mathrm{id}_{\mathrm{Sym}_N}$.

(c) Hence show that the orbit $\{\lambda \cdot \sigma : \sigma \in \mathrm{Sym}_N\}$ has size $N!$. Thus the union in Definition 4.6 is disjoint.

23. Fix $\ell \in \mathbf{N}_0$. Let $\mathcal{S}$ be the set of words of length ℓ with entries $+$ and $-$. Read from left to right, say that a $-$ in w is *bad* if it sets a new record for the excess of $-$ over $+$. A $-$ is *good* if it is not bad. For example, in $+--++---$ the $-$ in position 3 and the $-$ in position 8 are bad and the others are good.

For $m, n \in \mathbf{N}_0$, let $\mathcal{S}(m,n)$ be the set of words in $\mathcal{S}$ having $-$ entries in exactly n positions, such that at least m of these entries are good. (Note that all words have length ℓ.)

(a) Let $m \in \mathbf{N}_0$ and let $n \in \mathbf{N}$. By adapting the coplactic maps define a bijection

$$\mathcal{S}(m,n)\backslash\mathcal{S}(m+1,n) \to \mathcal{S}(m,n-1)\backslash\mathcal{S}(m+1,n-1).$$

(b) Hence define a bijection $\mathcal{S}(m,n)\backslash\mathcal{S}(m+1,n) \to \mathcal{S}(m,m)$.

(c) Hence prove Theorem 15.14 in [6].

24. Recall that $\delta = (N-1, N-2, \ldots, 2, 1)$. Let λ be a partition with $\ell(\lambda) \leq N$. Let $\mathrm{SSYT}_{\leq N}(\lambda)$ denote the set of semistandard λ-tableaux with entries from $\{1, \ldots, N\}$. Use a sign-reversing involution and the coplactic maps to prove that

$$\sum_{A \in \mathrm{Abc}(\varnothing)} \sum_{t \in \mathrm{SSYT}_{\leq N}(\lambda)} x^A x^T \operatorname{sgn}(A) = \sum_{A' \in \mathrm{Abc}(\lambda)} x^{A'} \operatorname{sgn}(A')$$

and deduce that $s_\lambda(x_1, \ldots, x_N) = a_{\delta+\lambda}/a_\delta$.

(Lecture 9) Corollaries of $a_{\lambda+\delta}/a_\delta = s_\lambda(x_1, \ldots, x_N)$

25. (a) Deduce the equivalent forms of Young's Rule and Pieri's Rule, Theorems 3.8 and 3.7 respectively:

$$h_\mu = \sum_{\lambda \vdash n} |\mathrm{SSYT}(\lambda, \mu)| s_\lambda$$

$$e_\mu = \sum_{\lambda \vdash n} |\mathrm{SSYT}(\lambda', \mu)| s_\lambda.$$

(b) The ω-involution was defined by $\omega(h_n) = e_n$ for $n \in \mathbf{N}_0$. Show that $\omega(s_\lambda) = s_{\lambda'}$ for all partitions λ.

(Lecture 10) Dual bases

26. Prove Claim 4 in the proof of Theorem 5.3, that

$$h_\mu = \sum_{\alpha \vdash n} \frac{p_\alpha}{z_\alpha} \pi^\mu(\alpha).$$

(Lecture 10) Skew-Schur functions. A skew-tableau is semistandard *if its rows are weakly increasing from left to right and the columns are strictly increasing from top to bottom. For example, the skew-tableau shown below is semistandard.*

$$\begin{array}{cccc} & & 1 & 1 \\ & 1 & 2 & \\ 1 & 3 & & \\ 4 & & & \end{array}$$

27. Let λ/ν be a skew-partition of n. Generalizing the combinatorial definition of Schur functions, define

$$s_{\lambda/\nu} = \sum_{T \in \mathrm{SSYT}(\lambda/\nu)} x^T.$$

For α a composition of n, let $K_{\lambda/\nu,\alpha} = |\mathrm{SSYT}(\lambda/\nu, \alpha)|$ be the number of semistandard Young tableaux of shape λ/ν and content α.

(a) Generalize the Bender–Knuth involution to show that $s_{\lambda/\nu}$ is a symmetric function.

(b) Use Young's Rule to show that $\langle s_\lambda, s_\nu h_\alpha \rangle = K_{\lambda/\nu,\alpha}$.

(c) Use the orthogonality of the complete symmetric and monomial symmetric functions to show that $\langle s_{\lambda/\nu}, h_\alpha \rangle = K_{\lambda/\nu,\alpha}$.

(d) Hence show that $\langle s_\lambda, s_\nu f \rangle = \langle s_{\lambda/\nu}, f \rangle$ for any $f \in \Lambda$.

Murnaghan–Nakayama Rule for skew-Schur functions

28. Prove the Murnaghan–Nakayama rule for $s_\lambda p_\alpha$ using the antisymmetric version of the rule given in Theorem 3.11.

Jacobi–Trudi Rule for skew-partitions

29. Let λ/ν be a skew-partition. Let $M \geq \ell(\lambda)$. Prove that

$$s_{\lambda/\nu} = \det \begin{pmatrix} h_{\lambda_1-\nu_1} & h_{\lambda_1-\nu_2+1} & \cdots & h_{\lambda_1-\nu_M+(M-1)} \\ h_{\lambda_2-\nu_1-1} & h_{\lambda_2-\nu_2} & \cdots & h_{\lambda_2-\nu_M+(M-2)} \\ \vdots & \vdots & \ddots & \vdots \\ h_{\lambda_M-\nu_1-(M-1)} & h_{\lambda_M-\nu_2-(M-2)} & \cdots & h_{\lambda_M-\nu_M} \end{pmatrix}$$

Deduce that

$$s_{\lambda/\nu} = \sum_{\sigma \in \mathrm{Sym}_M} h_{\lambda\cdot\sigma-\nu}\,\mathrm{sgn}(\sigma).$$

Lascoux–Schützenberger involution for skew-tableaux

30. Extend the word map w, the coplactic maps and the Lascoux–Schützenberger involution to skew-tableaux of a fixed shape and prove a generalization of Lemma 4.7.

Littlewood–Richardson Rule

31. Let λ/ν be a skew-partition of n and let μ be a partition of m with $\ell(\mu) \leq N$. Let $c^{\lambda}_{\mu\nu}$ be the number of semistandard λ/ν skew-tableaux t of content μ such that $\mathrm{w}(t)$ is latticed. The Littlewood–Richardson Rule states that

$$\langle s_\mu s_\nu, s_\lambda\rangle = c^{\lambda}_{\mu\nu}.$$

Prove the Littlewood–Richardson Rule using this outline.

(a) Show, using Question 26(d), that the Littlewood–Richardson rule is equivalent to $s_{\lambda/\nu} = \sum_\mu c^{\lambda}_{\mu\nu} s_\mu$.

(b) Using Theorem 2.1, show that

$$\langle s_{\lambda/\nu}, s_\mu\rangle = \sum_{\sigma\in \mathrm{Sym}_N} \langle s_{\lambda/\nu}, h_{\mu\cdot\sigma}\,\mathrm{sgn}(\sigma)\rangle.$$

(c) Deduce from Question 27 that

$$\langle s_{\lambda/\nu}, s_\mu\rangle = \sum_{\sigma\in \mathrm{Sym}_N} |\mathrm{SSYT}(\lambda/\nu, \mu\cdot\sigma)|\,\mathrm{sgn}(\sigma).$$

(d) Using Question 30, show that the right-hand side above is $c^{\lambda}_{\mu\nu}$.

Pólya's Cycle Index Theorem

32. (a) Let $G \leq \mathrm{Sym}_n$ be a permutation group and consider the orbits of G on words in W_n with entries from $\{1, \ldots, c\}$. Deduce from Pólya's Theorem that the number of such orbits is

$$\frac{1}{|G|}\sum_{g\in G} c^{\ell(\mathrm{cyctype}(g))}.$$

Remark: this result can also be proved using that the number of orbits of a group G on a set Ω is $\frac{1}{|G|}\sum_{g\in G}|\operatorname{Fix}_\Omega(g)|$, i.e. the result commonly known as Burnside's Lemma.

(b) Deduce from (a) and Examples 5.11 and 5.13 that the number of 4-bead necklaces using beads of c colours is $\frac{1}{4}(c^4+c^2+2c)$ up to rotation, and $\frac{1}{8}(c^4+2c^3+3c^2+c)$ up to both rotation and reflection.

7. Hints, references or solutions for problems

1. For (a), observe that the $\ell(\lambda)\times\ell(\lambda')$-matrix A with $A_{ij}=1$ if $(i,j)\in[\lambda]$ and $A_{ij}=0$ otherwise has row sums λ and column sums λ'. For (b), let A be a 0-1 matrix with row sums λ and column sums μ. The positions (i,j) with $A_{ij}\neq 1$ do not form a partition diagram (since then we are in the case of (a), with $\mu=\lambda'$). Therefore there exists (i,j) with $A_{ij}=0$ and either $A_{(i+1)j}=1$ or $A_{i(j+1)}=1$. In the former case, since the row sums of A are non-increasing, there exists j' with $A_{ij'}=1$ and $A_{(i+1)j'}=0$. Define A' by swapping 0s and 1s in the positions $(i,j),(i+1,j),(i,j'),(i+1,j')$ as indicated below:

$$\begin{matrix}0 & \ldots & 1\\ 1 & \ldots & 0\end{matrix}\longrightarrow\begin{matrix}1 & \ldots & 0\\ 0 & \ldots & 1\end{matrix}.$$

Then A' has the same row and column sums as A. The latter case is similar.

2. For (a), see [5, Theorem 1.4.10] (for a stronger result characterizing neighbours in the dominance order). By (a), $\alpha\trianglerighteq\beta$ if and only if $[\beta]$ can be obtained from $[\alpha]$ by repeatedly moving boxes from higher rows to lower rows, so if and only if $[\beta']$ can be obtained from $[\alpha']$ by repeatedly moving boxes from lower rows to higher rows. Hence $\alpha\trianglerighteq\beta$ if and only if $\beta'\trianglerighteq\alpha'$.

3. We have

$$Q(t)=\sum_{k=1}^{\infty}p_k\frac{t^k}{k}=\sum_{k=1}^{\infty}\sum_{i=1}^{\infty}x_i^k\frac{t^k}{k}=\sum_{i=1}^{\infty}-\log(1-x_it).$$

Differentiating with respect to t we get

$$Q'(t)=\sum_{k=1}^{\infty}p_kt^{k-1}=\sum_{i=1}^{\infty}\frac{x_i}{1-x_it}.$$

On the other hand, differentiating $H(t)=\prod_{i=1}^{\infty}1/(1-x_it)$ using the product rule gives

$$H'(t)=\sum_{i=1}^{\infty}\frac{x_i}{1-x_it}H(t) \tag{7.1}$$

so we have $H'(t) = H(t)Q'(t)$. Multiply through by t to get $tH'(t) = H(t)(tQ'(t))$; now take coefficients of t^n to get $nh_n = \sum_{k=1}^{n} h_{n-k}p_k$. This proves (a), (b), (c), (d).

For (e), argue in a similar way using $E(t) = \prod_{i=1}^{\infty}(1 - x_i t)$; (f) then follows from the identity

$$tH'(t)E(t) = \sum_{i=1}^{\infty} \frac{x_i t}{1 - x_i t} = tQ'(t).$$

The standard argument, used for example in Proposition 1.6, now shows that $\{p_\lambda : \lambda \vdash n\}$ is a basis for Λ_n, as stated in (g). See (1.6) for (h).

Remark. See Claim 4 in the proof of Theorem 5.3 for a combinatorial interpretation of the coefficients in the transition matrix expressing each h_μ as a linear combination of p_λ. As observed in Remark (2) in §1.9, these coefficients are rational but not necessarily integral.

4. $M_{\lambda\mu}$ is the number of $\ell(\lambda) \times \ell(\mu)$-matrices with row sums λ, column sums μ and entries in $\mathbf{N}$. This can be proved in the same way as Lemma 1.3; use the jth column of such a matrix Z to record the monomial chosen from the jth bracket in

$$h_\mu = \prod_{j=1}^{\ell(\mu)} h_{\mu_j} = \prod_{j=1}^{\ell(\mu)} (x_1^{\mu_j} + \cdots + x_1^{\mu_j - 1} x_2 + \cdots + \cdots)$$

in a sequence of choices whose product is the monomial x^λ.

5. By Newton's Identity (1.3), we have

$$h_k = (-1)^{k-1}e_k + (-1)^{k-2}e_{k-1}h_1 + \cdots + (-1)^{k-j-1}e_{k-j}h_j + \cdots + h_{k-1}e_1$$

for each $k \in \mathbf{N}$. It follows inductively that if $\mu \vdash n$ then

$$h_\mu = (-1)^{n-\ell(\mu)}e_\mu + f$$

where f is an integral linear combination of elementary symmetric functions labelled by partitions λ such that $\mu \rhd \lambda$. This proves (a) and (b).

Let $\mu \vdash n$. Since $[x_1^n]h_\mu = 1$ whereas $[x_1^n]e_\mu = 0$ unless $\mu = (1^n)$, we have $R_{(1^n)\mu} = 1$ for any μ. Similarly $[x_1^{n-1}x_2]h_\mu = \ell(\mu)$, since we must choose a unique h_{μ_j} in the product $h_\mu = h_{\mu_1} \ldots h_{\mu_{\ell(\mu)}}$ from which to take $x_1^{\mu_j - 1}x_2$, whereas

$$[x_1^{n-1}x_2]e_\mu = \begin{cases} n & \text{if } \mu = (1^n) \\ 1 & \text{if } \mu = (2, 1^{n-2}) \\ 0 & \text{otherwise.} \end{cases}.$$

Since $h_\mu = \cdots + R_{(2,1^{n-2})\mu}e_{(2,1^{n-2})} + e_{(1^n)}$, it follows that $\ell(\mu) = R_{(2,1^{n-2})\mu} + n$, hence $R_{(2,1^{n-2})\mu} = -n + \ell(\mu)$. This proves (c) and (d).

Finally (e) follows immediately from the ω involution, which swaps e_μ and h_μ for each μ. (Is there a combinatorial interpretation of this symmetry?)

Remark. Computer algebra can be useful for exploring the transition matrices between the different bases of Λ. In MAGMA [2], the matrix R above is the transpose of `HomogeneousToElementaryMatrix(n)`. In SAGE [15] one can create Λ by `Sym = SymmetricFunctions(QQ)`; then `Sym.inject_shorthands()` sets up conversions between the different bases. For example `h(e[[3]])` evaluates to `h[1,1,1]-2*h[2,1]+h[3]`, giving the coefficients in the first column of R when $n = 3$.

6. See [4].

7. For (a) to (f), see MacMahon's book, cited in the question. For (g), which generalizes (f), recall that the exponential generating function for the derangement numbers $d_{(1^n)}$ is $\exp(-t)/(1-t)$. Therefore

$$\frac{1}{n!}\sum_m \binom{m-1}{\ell}\binom{n}{m} d_{(1^{n-m})}$$

is the coefficient of t^n in

$$\left(\sum_m \binom{m-1}{\ell}\frac{t^m}{m!}\right)\frac{\exp(-t)}{1-t}.$$

(Note that the sum is now over all m, so there is a non-zero contribution when $m = 0$.) By the claim below, this simplifies to

$$\frac{(-1)^\ell}{1-t}\big(1 - t + \frac{t^2}{2!} - \cdots + (-1)^\ell \frac{t^\ell}{\ell!}\big) = A_\ell(t) + (-1)^\ell \frac{d_{(1^\ell)}}{\ell!}\frac{t^\ell}{1-t}$$

where $A_\ell(t)$ is a polynomial of degree at most $\ell - 1$. Taking out the contribution from $m = 0$ we get

$$(-1)^\ell \frac{d_{(1^n)}}{n!} + \frac{1}{n!}\sum_{m=\ell+1}^{n}\binom{m-1}{\ell}\binom{n}{m} d_{(1^{n-m})} = (-1)^\ell \frac{d_{(1^\ell)}}{\ell!}$$

for all $n \geq \ell$. This rearranges to give the claimed identity.

Claim. If $\ell \in \mathbf{N}$ then

$$\sum_m \binom{m-1}{\ell}\frac{t^m}{m!} = (-1)^\ell \exp(t)\big(1 - t + \frac{t^2}{2!} - \cdots + (-1)^\ell \frac{t^\ell}{\ell!}\big).$$

Proof. Suppose inductively that the left-hand side is $\exp(t)P_\ell(t)$ for some polynomial $P_\ell(t)$. Then for $\ell + 1$ the left-hand side is

$$\big(t^{-(\ell+1)}\exp(t)P_\ell(t)\big)'\,\frac{t^{\ell+2}}{\ell+1}.$$

Hence the right-hand side for $\ell+1$ is $\exp(t)P_{\ell+1}(t)$ where
$$P_{\ell+1}(t) = \frac{(t-\ell-1)P_\ell(t) + tP'_\ell(t)}{\ell+1}.$$
It is routine to check that the polynomials
$$Q_\ell(t) = (-1)^\ell\bigl(1 - t + \frac{t^2}{2!} - \cdots + (-1)^\ell \frac{t^\ell}{\ell!}\bigr)$$
satisfy this recurrence, hence $P_\ell(t) = Q_\ell(t)$ for all $\ell \in \mathbf{N}_0$. $\square$

Remark: Darij Grinberg has sent me an easier proof of (g) by induction. He also observed that the result also holds when $\ell = 0$.

8. It is worth noting the group-theoretic refinement of (a): the centralizer of an element of cycle-type $\lambda \vdash n$ where λ has exactly a_i parts of size i is the direct product of wreath products $\prod_i C_i \wr \mathrm{Sym}_{a_i}$, of order z_λ. For (b) use Question 3(h). The analogous identity for e_n, obtained most simply by applying the ω involution from Question 10, is
$$\sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} \frac{p_\lambda}{z_\lambda} = e_n.$$

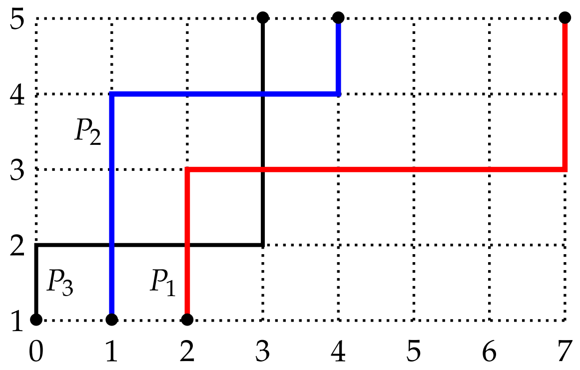
9. For (b), specialize the version of the Cycle Index Formula in (1.9) by $p_1 \mapsto 0$ and $p_k \mapsto (-1)^{k-1}$. For (c), specialize by $p_1 \mapsto 0$, $p_k \mapsto -1$; this gives
$$\prod_{k=2}^\infty \exp\bigl(-\frac{1}{k}t^k\bigr) = \sum_{n=0}^\infty \frac{1}{n!} \sum_{\sigma \in \mathrm{Sym}_n} (-1)^{\mathrm{cyc}(\sigma)}$$
where $\mathrm{cyc}(\sigma) = \mathrm{cyc}_1(\sigma) + \mathrm{cyc}_2(\sigma) + \cdots$ is the number of disjoint cycles in σ. The left-hand side is $(1-t)\exp(t)$, so taking coefficients of t^n we get
$$\frac{E_n - O_n}{n!} = \frac{1}{n!} - \frac{1}{(n-1)!} = \frac{1-n}{n!}$$
hence $O_n - E_n = n - 1$, as claimed.

10. Using Question 3(f), one gets $\omega(p_n) = (-1)^{n-1}p_n$. Therefore $\omega(p_\lambda) = (-1)^{n-\ell(\lambda)}p_\lambda$ if $\lambda \vdash n$. Note that, as seen in 9(a), $(-1)^{n-\ell(\lambda)} = \mathrm{sgn}\,\sigma_\lambda$, where $\sigma_\lambda \in \mathrm{Sym}_n$ is a permutation of cycle-type λ.

11. Part (b) is not nearly as obvious as it might seem: one proof uses the coplactic maps in §4 of the notes. An algebraic proof using the Jacobi–Trudi Identity is given in [5, page 44] and Lemma 5.4 above. For a combinatorial proof of the lemma see Matthew Fayers' answer to MathOverflow question 226537. A small simplification to this proof is here: `wildonblog.wordpress.com/2016/01/05/non-zero-kostka-numbers/`.

12. See [13, §4.5], or for the more general version of the Jacobi–Trudi Identity for skew-tableaux, [14, §7.16], where the proof is a short application of a more general involution due to Lindström and Gessel–Viennot.

13. The problem occurs when we define $\overline{P}_1$ and $\overline{P}_3$ by swapping P_1 and P_3 after their first intersection, and P_3 meets P_2 after this first intersection, so $\overline{P}_1$ now meets $\overline{P}_2 = P_2$. A specific example is shown below.



14. *Outline.* We define an involution on M-tuples of paths, as in Example 2.2, although the start and end points are now different, and we also change the definition of weight.

Number the steps of paths from 1. Give a path whose right-steps are step numbers $i_1 < i_2 < \ldots < i_k$ weight $x_{i_1} \ldots x_{i_k}$. For an involution defined by swapping paths after a first intersection to be weight preserving we need this key property: *if two paths intersect then when they reach this intersection they have both made the same number of steps* This is achieved by making the starting points $(M-i, i)$ for $1 \leq i \leq M$; the weights obtained by the path P_i from $(M-i, i)$ should give $e_{\lambda_i}(x_1, \ldots, x_N)$, so the path must have exactly N steps, of which exactly λ_i are right. Therefore P_i ends at $(M-i+\lambda_i, i+N-\lambda_i)$. (Note that if $N < \lambda_1$ then there are no paths at all from $(M-1, 1)$ to $(M-1+\lambda_1, 1+N-\lambda_1)$; correspondingly $e_{\lambda_1}(x_1, \ldots, x_N) = 0$ and $s_{\lambda'}(x_1, \ldots, x_N) = 0$.)

Given a non-intersecting tuple of paths $(P_M, \ldots, P_1)$, let t be the λ' tableau with entries in column i recording the step numbers of the right-steps made by P_i. Suppose that $t(a, i) > t(a, i+1)$. Then P_i has made strictly more steps before its ath step right than P_{i+1}; therefore P_i and P_{i+1} meet on or before their ath step right. The converse also holds. Hence non-intersecting paths are in a weight-preserving bijection with semistandard tableaux of shape λ'.

15. (a), (c) Since $P(x_1, \ldots, x_N)$ is symmetric, it is determined by the coefficient of x^λ for each $\lambda \vdash n$ with $\ell(\lambda) \leq N$. Let $\ell(\lambda) = M$. The monomial x^λ appears in $(\sum_{i\in I} x_i)^n$ if and only if $I \supseteq \{1,\ldots,M\}$; the coefficient is

$$\binom{n}{\lambda_1,\ldots,\lambda_N}$$

since we must take x_i from exactly λ_i of the brackets, for each $i \in \{1,\ldots,M\}$. There are exactly $\binom{N-M}{k-M}$ k-subsets of $\{1,\ldots,N\}$ that contain $\{1,\ldots,M\}$, hence

$$P(x_1,\ldots,x_N) = \sum_{\substack{\lambda\vdash n\\ \ell(\lambda)\le N}} \binom{n}{\lambda_1,\ldots,\lambda_N}\Big(\sum_{k=M}^{N}(-1)^{n-k}\binom{N-M}{k-M}\Big)\mathrm{mon}_\lambda(x_1,\ldots,x_N).$$

The inner sum vanishes unless $N = M$. Hence

$$P(x_1,\ldots,x_N) = (-1)^{n-N}\sum_{\substack{\lambda\vdash n\\ \ell(\lambda)=N}} \binom{n}{\lambda_1,\ldots,\lambda_N}\mathrm{mon}_\lambda(x_1,\ldots,x_N).$$

This answers (c) and shows that if $n < N$ then $P(x_1,\ldots,x_N) = 0$ and if $n = N$ then

$$P(x_1,\ldots,x_n) = \binom{n}{1,\ldots,1}\mathrm{mon}_{(1,\ldots,1)}(x_1,\ldots,x_N) = N!x_1\ldots x_N.$$

(b) Symmetric n-multilinear maps $V\times\cdots\times V \to \mathbf{C}$ correspond to elements of $(\mathrm{Sym}^n V)^\star$. There is a map $V^\star \to (\mathrm{Sym}^n V)^\star$ defined by $\theta \mapsto \theta^n \in (\mathrm{Sym}^n V)^\star$, where θ^n is defined by

$$\theta^n(v_1\ldots v_n) = \theta(v_1)\ldots\theta(v_n).$$

The set $\{\theta^n : \theta \in V^\star\}$ is dense in $(\mathrm{Sym}^n V)^\star$. Taking $f(v_1,\ldots,v_n) = \theta^n(v_1,\ldots,v_n) = \theta(v_1)\ldots\theta(v_n)$ we see that $g(u) = f(u,\ldots,u) = \theta(u)^n$. So it is sufficient to prove that if $\theta \in V^\star$ then

$$\theta(v_1)\ldots\theta(v_n) = \frac{1}{n!}\sum_{I\subseteq\{1,\ldots,n\}}(-1)^{n-|I|}\theta(\sum_{i\in I} v_i)^n.$$

Setting $\theta(v_i) = x_i$, this is precisely the identity in (a).

Remark: for an alternative proof see [16]. Darij Grinberg has shown me an elegant proof of the strong result that if $f(x_1,\ldots,x_n)$ is any n-multilinear map then

$$\sum_{I\subseteq\{1,\ldots,n\}}(-1)^{n-|I|}g(\sum_{i\in I} v_i) = \sum_{\sigma\in\mathrm{Sym}_n} f(v_{1\sigma^{-1}},\ldots,v_{n\sigma^{-1}})$$

where $g(u) = f(u,\ldots,u)$. This immediately implies (b).

16. See [8, §3.2].

17. An equivalent result is stated after Theorem 1.1 in [3]. The key point is that the signs of any two border-strip tableaux of shape λ and content

$(1^{|\gamma|}, \ell^w)$ agree. This can most easily be seen using the abacus: if A is an ℓ-runner abacus representing λ and b and b' are beads on A then in any sequence of single-step upward bead moves taking A to an abacus representing γ, the parity of the number of times bead b moves past bead b' is the same.

18. See [17, §2.3].

19. (a) If $n \geq \ell m$ then the numbers $n, n-\ell, \ldots, n-(m-1)\ell$ cover all residue classes modulo m, and so $n = a\ell + bm$ for some $a, b \in \mathbf{N}_0$. Let N be the largest number not of this form. If $N + \ell = a\ell + bm$ then $a = 0$; similarly $N + m$ is a multiple of ℓ. Hence $N = \ell m - \ell - m$.

(b) Take an abacus A representing a simultaneous ℓ-core and m-core partition with beads in all negative positions and a gap in position 0. If there is a bead in position k of A then there are beads in all positions $k - (a\ell + bm)$ for $a, b \in \mathbf{N}_0$. Hence k is not of the form $a\ell + bm$, so by (a), $k \leq \ell m - \ell - m$. Therefore A has only finitely many beads in positive positions.

(c) See [1]. There are some simplifications in the case $m = \ell + 1$, when the number of simultaneous cores is the Catalan number $\frac{1}{2\ell+1}\binom{2\ell+1}{\ell} = \frac{1}{\ell+1}\binom{2\ell}{\ell}$.

I am grateful to Darij Grinberg for drawing my attention to Drew Armstrong's slides from FPSAC 2017 `www.math.miami.edu/~armstrong/Talks/RCC_FPSAC_17.pdf` which, in his words, 'give gorgeous picture proofs of all parts of Question 19 (as well as mentioning further results)'.

20. By two applications of Young's Rule, $s_\lambda h_1^2 = \sum_\mu c_\mu s_\mu$ where $[\mu]$ is obtained from $[\lambda]$ by adding two boxes in any way. If these additions make a rim-hook shape, so the added boxes are either in the same row, or in the same column, then $c_\mu = 1$, otherwise $c_\mu = 2$. Another application of Young's Rule to $s_\lambda h_2$ now shows that $s_\lambda p_2 = s_\lambda(2h_2 - h_1^2) = \sum_\mu \mathrm{sgn}(\mu/\lambda) s_\mu$. This proves the Murnaghan–Nakayama rule for addition of 2-rim-hooks.

21. By Question 3(f) we have $p_r = \sum_{k=0}^{r-1} (-1)^k (r-k) h_{r-k} e_k$. Fix a skew-partition μ/λ of r and let $\mathcal{T}$ be the set of all μ/λ-tableaux with entries from $\{1,2\}$ such that

(a) No two 1s appear in the same column;
(b) No two 2s appear in the same row;
(c) Exactly one 1 is distinguished;
(d) Each row and each column is weakly increasing.

Given $t \in \mathcal{T}$, define $\operatorname{sgn}(t) = (-1)^k$ where k is the number of 2s. By the Young and Pieri rules, and the identity above, $\sum_{t \in \mathcal{T}} \operatorname{sgn}(t)$ is the coefficient of s_μ in $s_\lambda p_r$. Note that $\mathcal{T}$ is empty unless μ/λ is a disjoint union of rim-hooks (equivalently, $[\mu/\lambda]$ contains no 2×2 box), so we may assume that this is the case.

If ν/λ is a rim-hook in μ/λ, define its *terminal box* to be the box $(i, j) \in [\nu/\lambda]$ such that $(i-1, j) \notin [\nu/\lambda]$ and $(i, j+1) \notin [\nu/\lambda]$. For example, the 7-rim-hook shown in Figure 1 has terminal box in position $(2, 5)$.

We define an involution $J : \mathcal{T} \to \mathcal{T}$ as follows: let ν/λ be the highest rim-hook in μ/λ not containing the distinguished entry $1^\star$, or the rim-hook containing $1^\star$ if this is the only rim-hook. If $1^\star$ is not in the terminal box of ν/λ then flip the entry in this box; this gives a new element of $\mathcal{T}$ with opposite sign. Otherwise leave t fixed. The involution J is sign-reversing on its non-fixed-points. Moreover there is a fixed tableau t if and only if μ/λ is an r-rim-hook, and in this case, t is unique and the number of 2s in t is $\operatorname{ht}(\mu/\lambda)$. Hence

$$\sum_{t \in \mathcal{T}} \operatorname{sgn}(t) = \begin{cases} \operatorname{sgn}(\mu/\lambda) & \text{if } \mu/\lambda \text{ is a rim-hook} \\ 0 & \text{otherwise.} \end{cases}$$

as required.

Example: if $\lambda = (1)$ and $\mu = (3, 2)$ then $|\mathcal{T}| = 5$; the unique fixed point and the two pairs swapped by J are shown below:

$$\begin{ytableau} \none & 1 & 1^\star \\ 1 & 2 \end{ytableau}\,, \quad \begin{ytableau} \none & 1^\star & 1 \\ 1 & 2 \end{ytableau} \longleftrightarrow \begin{ytableau} \none & 1^\star & 2 \\ 1 & 2 \end{ytableau}\,, \quad \begin{ytableau} \none & 1 & 1 \\ 1^\star & 2 \end{ytableau} \longleftrightarrow \begin{ytableau} \none & 1 & 2 \\ 1^\star & 2 \end{ytableau}\,.$$

If $\lambda = (2, 1)$ and $\mu = (3, 2, 2)$ then J has no fixed points, $|\mathcal{T}| = 8$ and J is defined by

$$\begin{ytableau} \none & \none & 1^\star \\ \none & 1 \\ 1 & 2 \end{ytableau} \longleftrightarrow \begin{ytableau} \none & \none & 1^\star \\ \none & 2 \\ 1 & 2 \end{ytableau}\,, \quad \begin{ytableau} \none & \none & 1 \\ \none & 1^\star \\ 1 & 2 \end{ytableau} \longleftrightarrow \begin{ytableau} \none & \none & 2 \\ \none & 1^\star \\ 1 & 2 \end{ytableau}\,,$$

$$\begin{ytableau} \none & \none & 1 \\ \none & 1 \\ 1^\star & 2 \end{ytableau} \longleftrightarrow \begin{ytableau} \none & \none & 2 \\ \none & 1 \\ 1^\star & 2 \end{ytableau}\,, \quad \begin{ytableau} \none & \none & 1 \\ \none & 2 \\ 1^\star & 2 \end{ytableau} \longleftrightarrow \begin{ytableau} \none & \none & 2 \\ \none & 2 \\ 1^\star & 2 \end{ytableau}\,.$$

Reference: I am grateful to Darij Grinberg for observing that a similar argument is used to prove Theorem 6.3 in [11].

22. By definition $\beta = \lambda \cdot \sigma$. By (4.2) we have

$$(\lambda \cdot \sigma)_i = \lambda_{i\sigma^{-1}} + i - i\sigma^{-1}.$$

Hence if $j\sigma > (j+1)\sigma$ then $(\lambda \cdot \sigma)_{j\sigma} = \lambda_j + j\sigma - j > \lambda_{j+1} + (j+1)\sigma - (j+1) = (\lambda \cdot \sigma)_{(j+1)\sigma}$, so $\beta_{j\sigma} > \beta_{(j+1)\sigma}$, proving (a). Since $j\sigma > (j+1)\sigma$, β is not a partition. Therefore if $\lambda \cdot \sigma$ is a partition then σ has no inversions, and so $\sigma = \mathrm{id}_{\mathrm{Sym}_N}$, proving (b). Hence the stabiliser under the dot action of λ is trivial, so the orbit has size $N!$, proving (c).

23. (a) Note that $\mathcal{S}(m,n)\backslash\mathcal{S}(m+1,n)$, respectively $\mathcal{S}(m,n-1)\backslash\mathcal{S}(m+1,n-1)$, is the set of $+/-$ words with exactly m good $-$ entries out of n, respectively $n-1$.

Pair up $+$ and $-$ in the expected way, so in the example word $+ - - + + - - - \in \mathcal{S}(3,5)$ the unpaired $-$ entries are in red. The analogue of the F_1 coplactic map is defined by replacing the rightmost unpaired $-$ with (an unpaired) $+$. This does not change the number of good $-$ entries; by Lemma 4.4 it gives a bijection between the two sets.

(b) Repeatedly apply the bijection in (a) until all $-$ entries are good. The resulting word is in $\mathcal{S}(m,m)$. Note that since each $-$ left of the rightmost unpaired $-$ has the same paired/unpaired status after applying F_1, the map is simply: *change all unpaired $-$ entries to $+$.*

(c) In James' notation,

$$\begin{aligned} \mathcal{S}(m,n) &= s\big((\ell-n,m),(\ell-n,n)\big), \\ \mathcal{S}(m+1,n) &= s\big((\ell-n,m+1),(\ell-n,n)\big) = s\big((\ell-n,m)A_2,(\ell-n,n)\big) \\ \mathcal{S}(m,m) &= s\big((\ell-m,m),(\ell-m,m)\big) = s\big((\ell-n,m),(\ell-n,n)R_2\big). \end{aligned}$$

Thus the bijection in (b) is precisely the bijection in James' Theorem 15.14 for two row partitions. The general case of Theorem 15.14 follows easily.

Remark 1: to apply the coplactic map, as originally defined, in (a) or (b), we rewrite $+$ as 2 and $-$ as 1. This goes the opposite way to the notation in [6], where $+$ corresponds to 1 and $-$ to 2.

Remark 2: This result is used in [6, Chapter 16] to prove the Littlewood–Richardson rule (for a different proof see Question 31) and in [6, Chapter 17] to prove a very elegant characteristic-free version of Young's Rule for the symmetric group.

24. See [8, Theorem 4.2]. (Our proof of Theorem 4.10 is intended to be a more motivated version of Loehr's argument; it is somewhat longer since we go via the Jacobi–Trudi identity.)

25. (a) The equivalent identity for Young's Rule for symmetric polynomials, with $a_{\lambda+\delta}/a_\delta$ replacing s_λ, was proved in Corollary 3.9; now apply Theorem 4.10. The required form of Pieri's Rule can be proved similarly.

(b) *Outline:* suppose that $\omega(s_\lambda) = \sum_\nu A_{\nu\lambda} s_\lambda$. Show that $AK = \widetilde{K}$ where $K_{\lambda\mu} = |\mathrm{SSYT}(\lambda,\mu)|$ and $\widetilde{K}_{\lambda\mu} = |\mathrm{SSYT}(\lambda',\mu)|$. Now use that K is unitriangular, and so invertible, to determine A.

26. By Claim 3 in the proof of Theorem 5.3 we have

$$h_{\mu_1} \dots h_{\mu_k} = \prod_{i=1}^{k} \sum_{\beta(i) \vdash \mu_i} \frac{p_{\beta(i)}}{z_{\beta(i)}} = \prod_{i=1}^{k} \sum_{\beta(i) \vdash \mu_i} p_{\beta(i)} \prod_{j=1}^{n} \frac{1}{j^{B_{ij}} B_{ij}!}$$

where B_{ij} is the number of parts of $\beta(i)$ equal to j for each $j \in \{1, \dots, n\}$. Suppose that $\beta \vdash n$ has exactly a_j parts equal to j for each $j \in \{1, \dots, n\}$. We get a contribution to the coefficient of p_β from every k-tuple of partitions $\beta(1), \dots, \beta(k)$ such that $\sum_i B_{ij} = a_j$ for each $j \in \{1, \dots, n\}$ and $\sum_j jB_{ij} = |\beta(i)| = \mu_i$. Thus the $k \times n$ matrix B is an β-packing matrix, and the coefficient of p_β is

$$\begin{aligned} \sum_B \prod_{i=1}^{k} \prod_{j=1}^{n} \frac{1}{j^{B_{ij}} B_{ij}!} &= \sum_B \prod_{j=1}^{n} \binom{a_j}{B_{1j}, \dots, B_{kj}} \frac{1}{a_j! j^{a_j}} \\ &= \frac{1}{z_\beta} \sum_B \prod_{j=1}^{n} \binom{a_j}{B_{1j}, \dots, B_{kj}} = \frac{1}{z_\beta} \pi^\mu(\beta) \end{aligned}$$

where the final equality uses Claim 1 in the proof of Theorem 5.3.

27. For (a) see [14, 7.10.2]: the proof given above of Proposition 1.17 generalizes routinely. The identity $s_\nu h_\alpha = \sum_{\lambda \vdash n} K_{\lambda/\nu,\alpha} s_\lambda$ can be proved by repeated applications of Young's Rule, as in Corollary 3.9 (which it generalizes). By (a), $s_{\lambda/\nu}$ is a symmetric function, so

$$s_{\lambda/\nu} = \sum_{\alpha \vdash n} K_{\lambda/\nu,\alpha} \mathrm{mon}_\alpha.$$

Using the orthogonality of the monomial and complete homogeneous symmetric functions (see Theorem 5.2) we get

$$\langle s_{\lambda/\nu}, h_\alpha \rangle = K_{\lambda/\nu,\alpha} = \langle s_\nu h_\alpha, s_\lambda \rangle.$$

This proves (c). Since the h_α for α form a basis of Λ, this implies that $\langle s_{\lambda/\nu}, f \rangle = \langle s_\lambda, f s_\nu \rangle$ for all $f \in \Lambda$, proving (d).

28. See §7.17 in [14].

29. See [14, Theorem 7.16.2]. *Outline.* Theorem 2.1 is the special case $\nu = \varnothing$. The path model used in the proof generalizes as follows: take starting points $(\nu_i + M - i, 1)$ and final destinations $(\lambda_i + M - i, N)$ for $i \in \{1, \dots, M\}$. The sum of the weights of non-intersecting paths is $s_{\lambda/\nu}(x_1, \dots, x_N)$. Suppose that the final destination of the path starting at $(\nu_i + M - i, 1)$ is $(\lambda_{i\tau} + M - i\tau, N)$ for each i. By (4.2), these paths are counted in the summand $\prod_{i=1}^{N} h_{\lambda_{i\tau} - \nu_i - i\tau + i} = h_{\lambda \cdot \tau^{-1} - \nu}$ of the right-hand side of

$$s_{\lambda/\nu} = \sum_{\sigma \in \mathrm{Sym}_M} h_{\lambda \cdot \sigma - \nu} \operatorname{sgn}(\sigma).$$

If we swap the paths starting at $(\nu_i + M - i, 1)$ and $(\nu_j + M - j, 1)$ after their first intersection, they now end at $(\lambda_{j\tau} + M - j\tau, N)$ and $(\lambda_{i\tau} + M - i\tau, N)$, so as expected, the new permutation is $(i, j)\tau$.

30. See [12, §2] for the definition of the J involution on skew-tableaux: Proposition 4 is the generalization of Lemma 4.4. The generalization of Lemma 4.7 is as follows: let α/β be a skew-partition and let μ be a partition with $\ell(\mu) \leq N$. Then J is an involution on

$$\bigcup_{\sigma \in \mathrm{Sym}_N} \mathrm{SSYT}(\alpha/\beta, \mu \cdot \sigma)$$

with fixed points precisely the latticed tableaux in $\mathrm{SSYT}(\alpha/\beta, \mu)$ and such that if $t \in \mathrm{SSYT}(\alpha/\beta, \mu \cdot \sigma)$ is k-unlatticed (with k chosen in the definition of the involution J as before) then $J(t) \in \mathrm{SSYT}\big(\alpha/\beta, \mu \cdot (\sigma(k, k+1))\big)$.

31. Each part is a routine application of results already proved.

32. (a) Observe that when we specialize by $x_1, \ldots, x_c \mapsto 1$, and x_{c+1}, $\ldots \to 0$, the monomial symmetric function mon_λ where $\lambda \vdash n$ is sent to 0 if $\ell(\lambda) > c$ and otherwise to the number of different contents of words in W_n that are rearrangements of λ.

For example, if $c = 4$ and $\lambda = (2, 1)$, so we specialize by $x_1, x_2, x_3, x_4 \mapsto 1$ and $x_5, \ldots \mapsto 1$ then

$$\mathrm{mon}_{(2,1)} = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3, \cdots + x_3 x_4^3 + \cdots \mapsto 12$$

corresponding to the 12 different contents $(2, 1, 0, 0)$, $(1, 2, 0, 0)$, $(2, 0, 1, 0)$, $\ldots$, $(0, 0, 1, 2)$; each is a possible content of a word in W_4 and each is a rearrangement of $(2, 1, 0, 0)$.

By Pólya's Theorem, the coefficient of mon_λ in cyc_G is the number of orbits of G on words of content λ. Therefore specializing in this way counts the number of words in W_n whose content is a rearrangement of λ, up to the action of G. Since p_λ specializes to $c^{\ell(\lambda)}$, we get that the number of orbits of G on words in W_n with entries from $\{1, \ldots, c\}$, is

$$\frac{1}{|G|} \sum_{g \in G} c^{\ell(\mathrm{cyctype}(g))}$$

as required.

(b) This is a routine application of (a). To illustrate the specialization above, if $c = 3$ then $3\mathrm{mon}_{(2,2)} = 3(x_1 x_2 + x_2 x_3 + x_1 x_3 + \cdots)$, seen in cyc_G in Example 5.11(3) corresponds to 3 orbits on words with content $(2, 2, 0)$, and so to 9 orbits of words whose content is a rearrangement of $(2, 2, 0)$.

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