Representations of Symmetric Groups 1

Question 7 and 8 are taken from problem sheet 4 for the 2009 Oxford course on the symmetric group: see `http://people.maths.ox.ac.uk/erdmann/`. Question 4 is a generalization of Exercise 1 in §10.7 of Peter J. Cameron, *Combinatorics*, CUP 1994.

1. Let G be a finite group, let F be a field and let Ω be a finite G-set. Let $V = \langle e_\omega : \omega \in \Omega \rangle_F$ be an F-vector space with a basis indexed by the elements of Ω.

(a) Show that V is an FG-module with (right) action defined by $e_\omega g = e_{\omega g}$.

(b) Suppose that $\Omega = \Delta_1 \cup \cdots \cup \Delta_d$ is the decomposition of Ω into distinct G-orbits. Show that if $v_i = \sum_{\omega \in \Delta_i} e_\omega$ then $v_1, \ldots, v_d$ is an F-basis for

$$V^G = \{v \in V : vg = v \text{ for all } g \in G\}.$$

(c) Now suppose that $F = \mathbf{C}$. Let π be the ordinary character of V.

(i) Show that $\pi(g) = |\mathrm{Fix}\, g|$ for each $g \in G$.

(ii) Explain why $\dim V^G = \langle \chi, 1_G \rangle$, where 1_G is the trivial character of G.

(iii) Using (b), prove the orbit counting formula (credited variously to Frobenius and Burnside):

$$d = \frac{1}{|G|} \sum_{g \in G} |\mathrm{Fix}\, g|.$$

(d) How many ways are there to colour the faces of a cube with 3 colours, if two cubes are regarded as the same if one can be rotated into the other?

2. Let G be a transitive permutation group acting on the set Ω. Let H be the point stabiliser of $\omega \in \Omega$. Let $\pi(g) = |\mathrm{Fix}\, g|$ be the permutation character of G.

(a) Prove that $\pi = 1_H\big\uparrow^G$. (Here the up-arrow denotes induction and 1_H is the trivial character of H.)

(b) Prove that $\langle \pi, \pi \rangle_G$ is the number of orbits of H on Ω.

(c) Show that $\pi = 1 + \chi$ where χ is irreducible if and only if G acts 2-transitively on Ω.

[A group G acts 2-transitively on a set Ω if given any two elements $(\alpha, \beta), (\alpha', \beta') \in \Omega \times \Omega$ such that $\alpha \neq \beta$ and $\alpha' \neq \beta'$, there exists $g \in G$ such that $\alpha g = \alpha'$ and $\beta g = \beta'$.]

3. Let $p(n)$ denote the number of partitions of $n \in \mathbf{N}_0$. Prove that $p(n) \leq p(n-1) + p(n-2)$ for all $n \geq 2$. Hence show that there exists a constant $c < 17/10$ such that $p(n) < c^n$ for all $n \in \mathbf{N}_0$.

4. Let λ be a partition and let s be a λ-tableau. Show that if we sort the columns of s into increasing order, to obtain a column-standard tableau t, and then sort the rows of t into increasing order, to obtain a row-standard tableau u, then u is standard.

5. Let $n \geq 2$ and let V denote the Specht module $S^{(n-1,1)}$, defined over a field F of characteristic p.

(a) Show that if $p = 0$ or $p \not| n$ then V is irreducible.

(b) Show that if p divides n then V has a unique non-trivial proper submodule.

(c) Describe the character of V in the case when $F = \mathbf{C}$.

6. Let $n \geq 2$ and let U denote the Specht module $S^{(2,1^{n-2})}$, defined over $\mathbf{Z}$. For $i \in \{1, 2, \ldots, n\}$, let t_i denote the unique $(2, 1^{n-2})$-tableau which has i in the rightmost box of its first row, and whose first column increases when read from top to bottom. Let $v_i = e(t_i)$ denote the polytabloid corresponding to t_i.

(a) Show that any element of U is a $\mathbf{Z}$-linear combination of $v_1, \ldots, v_n$.

(b) Show that $v_2, \ldots, v_n$ is a $\mathbf{Z}$-basis for U.

7. Let G be a finite group. Let Ω be a G-set and let $\Omega^{(2)}$ denote the G-set of all 2-element subsets of Ω. Let $M = F\Omega$ and $N = F\Omega^{(2)}$ be the corresponding permutation modules (defined using the same construction as Q1). Define linear maps $S : M \to N$ and $T : N \to M$ by

$$e_\alpha S = \sum_{\beta \neq \alpha} e_{\{\alpha,\beta\}} \quad \text{and} \quad e_{\{\alpha,\beta\}} T = e_\alpha + e_\beta$$

where $\alpha, \beta \in \Omega$.

(a) Check that these maps are FG-module homomorphisms.

(b) Write down a formula for $e_\alpha(S \circ T)$. Hence find the matrix of the linear map $S \circ T$ with respect to the basis $\{e_\omega : \omega \in \Omega\}$.

(c) Find the eigenvalues of this matrix. When if $S \circ T$ invertible?

8. Let F be a field, and $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition of $n \in \mathbf{N}$. Let M^λ be the FS_n-permutation module with basis all tabloids of shape λ.

(a) Show that M^λ is isomorphic to the permutation module $F\Omega$, where Ω is the set of (right) cosets of the Young subgroup S_λ in S_n.

(b) Show that $\dim M^\lambda = n!/\prod_{i=1}^{k} \lambda_i!$.