A CONSTRUCTION OF THE CARTER–LUSZTIG WEYL MODULE

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1. Introduction

Let V be a d-dimensional vector space over an infinite field F. In Definition 3.2 below, we define for each partition λ of r, a polynomial representation $\Delta^\lambda(V)$ of $\mathrm{GL}(V)$ of polynomial degree r. We show that $\Delta^\lambda(V)$ is isomorphic to the module $V_{\lambda,F}$ in Green [3, Ch. 5] and so to the Carter–Lusztig 'Weyl module' $\overline{V}^\lambda$ in Carter–Lusztig [1]. In the spirit of Green's construction, we define $\Delta^\lambda(V)$ so that it is the contravariant dual of Green's module $D_{\lambda,F}$ in [3, Ch. 4]; however we avoid the relations on tensor space relations used by Carter–Lusztig and Green, and instead give in Corollary 3.6 an explicit basis for $\Delta^\lambda(V)$ as a submodule of $\bigwedge^{\lambda'} V$, where $\bigwedge^{\lambda'} V$ is the tensor product of exterior powers defined in §3.1 below.

2. Background

2.1. **Partitions and tableaux.** Fix a partition λ and the dimension d of V. Let $\ell(\lambda)$ denote the number of parts of λ. Let $[\lambda] = \{(i,j) : 1 \leq i \leq \ell(\lambda), 1 \leq j \leq \lambda_i\}$ be the set of *boxes* of λ. A λ-*tableau* is a function $t : [\lambda] \to \{1,\ldots,d\}$. We say that the image $t_{(i,j)}$ of (i,j) is the *entry* of t in box (i,j); this corresponds to the Young diagram representation of tableaux shown below in Example 2.1. The symmetric group $S_{[\lambda]}$ acts on the set of λ-tableaux by place permutation:

$$(t \cdot \sigma)_{(i,j)} = t_{(i,j)\sigma^{-1}}.$$

The inverse is correct: it ensures that the entry of t in position (i',j') appears in $t \cdot \sigma$ in position $(i',j')\sigma$. Let $\mathrm{RPP}(\lambda)$ denote the subgroup of $S_{[\lambda]}$ of permutations that permute within themselves the boxes in each row of $[\lambda]$. Let $\mathrm{CPP}(\lambda)$ be the analogous group of column place permutations. A λ-tableau is *row semistandard* its rows are weakly increasing read from left to right and *column standard* if its columns are strictly increasing read from top to bottom. It is *semistandard* if both properties hold. Let $\mathrm{RSYT}(\lambda)$, $\mathrm{CSYT}(\lambda)$ and $\mathrm{SSYT}(\lambda)$ denote the corresponding sets of tableaux.

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2.2. **The costandard module $\nabla^\lambda(V)$.** We use the construction of $\nabla^\lambda(V)$ in [2, §2]. Fix a basis $v_1, \ldots, v_d$ of V. Let $\mathrm{Sym}^\lambda V = \bigotimes_{i=1}^{\ell(\lambda)} \mathrm{Sym}^{\lambda_i} V$. In [2, Definition 2.3] we defined the GL-*tabloid* $f(t)$ corresponding to a λ-tableau t with entries from $\{1, \ldots, d\}$ by

$$f(t) = \bigotimes_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda_i} v_{t_{(i,j)}} \in \mathrm{Sym}^\lambda V$$

and the GL-*polytabloid* corresponding to t by

$$F(t) = \sum_{\sigma \in \mathrm{CPP}(\lambda)} f(t \cdot \sigma) \,\mathrm{sgn}(\sigma) \in \mathrm{Sym}^\lambda V.$$

Finally, $\nabla^\lambda(V)$ is defined to be the subspace of $\mathrm{Sym}^\lambda V$ spanned by the GL-polytabloids $F(t)$ for t a λ-tableau. By Proposition 2.11 and §2.5 in [2], $\nabla^\lambda(V)$ is a $\mathrm{GL}(V)$-module with basis all $F(t)$ for $t \in \mathrm{SSYT}(\lambda)$.

Example 2.1. For example, if $d = 3$ so V is 3-dimensional then $\nabla^{(3,1)}(V)$ contains

$$F\left(\begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \right) = e_1^2 e_2 \otimes e_3 - e_1 e_2 e_3 \otimes e_1.$$

2.3. **Highest weight vectors.** Let $s(\lambda)$ be the λ-tableau whose ith row has all entries equal to i. Like any GL-tabloid, $F(s(\lambda))$ is a weight vector for the action of the diagonal subgroup of $\mathrm{GL}(V)$. As motivation for the construction below, we observe that it is highest weight with respect to the Borel subgroup of upper triangular matrices. This is easily seen over the complex field using the Lie algebra action of $\mathsf{gl}(V)$. For instance, if $X \in \mathsf{gl}(V)$ is defined on the basis $v_1, \ldots, v_d$ by $X(v_i) = v_{i-1}$ and $X(v_j) = 0$ for $j \neq i$, then the ith column of X has a 1 in row $i - 1$ making X upper triangular and $X \cdot F(s)$ is the sum of all $F(s')$ where s' is obtained from s by changing a single i to $i - 1$; since any such change creates a repeat in a column, we have $X \cdot F\bigl(s(\lambda)\bigr) = 0$. In particular, v_1 is a highest weight vector in the representation V.

Example 2.2. If $\lambda = (r)$ then $F\bigl(s(\lambda)\bigr) = v_1^r$ is highest weight in $\mathrm{Sym}^r V$. In characteristic zero, $\mathrm{Sym}^r V$ is generated by v_1^r, but typically this fails in prime characteristic p. To see the obstruction, take $r = p^e$. Then since $\bigl(\sum_{i=1}^d \alpha_i v_i\bigr)^{p^e} = \sum_{i=1}^d \alpha_i^{p^e} v_i^{p^e}$, the submodule generated by v_1 is equal to $\langle v_1^{p^e}, \ldots, v_d^{p^e} \rangle_F$; this is isomorphic to the e^{th} Frobenius twist of the natural module, and so far smaller than the $\binom{d+r-1}{r}$-dimensional $\mathrm{Sym}^r V$.

2.4. **Contravariant duality.** The dual of a polynomial representation is typically not polynomial, because of the division by the determinant when taking the inverse of a matrix. Instead we use the contravariant duality from Green [3, §2.7]. By definition, if W is a representation of $\mathrm{GL}(V)$ then the *contravariant dual* of W, denoted W°, is the dual space $W^\star$ with the $\mathrm{GL}(V)$ action

$$(g\vartheta)(w) = \vartheta(g^{\mathrm{t}} w)$$

for $g \in \mathrm{GL}(V)$, $\vartheta \in W^\star$ and $w \in W$. If $T : U \to W$ is a homomorphism of representations of $\mathrm{GL}(V)$ then $T^\circ : W^\circ \to U^\circ$ defined by $(T^\circ\vartheta)v = \vartheta(Tv)$ satisfies

$$\big(T^\circ(g\vartheta)\big)v = (g\vartheta)(Tv) = \vartheta(g^{\mathrm{t}}(Tv)) = \vartheta(T(g^{\mathrm{t}}v)) = (T^\circ\vartheta)(g^{\mathrm{t}}v) = \big(g\big(T^\circ\vartheta\big)\big)(v)$$

and so T° is again a homomorphism of representations of $\mathrm{GL}(V)$. Moreover, if $\rho : \mathrm{GL}(V) \to \mathrm{GL}(W)$ is the representation corresponding to W with respect to a basis $w_1, \ldots, w_m$ of W then, with respect to the dual basis $w_1^\star, \ldots, w_m^\star$ of W°, the representation $\rho^\circ : \mathrm{GL}(V) \to \mathrm{GL}(W^\circ)$ is defined by

$$\rho^\circ(g) = \rho(g^{\mathrm{t}})^{\mathrm{t}} \tag{1}$$

for all $g \in \mathrm{GL}(V)$. In particular, W° is polynomial of the same degree as W. Therefore contravariant duality is a degree-preserving contravariant function on polynomial representations of $\mathrm{GL}(V)$.

3. Construction of $\Delta^\lambda(V)$

3.1. **Exterior powers and quotients.** We define the exterior power $\bigwedge^r V$ in the usual way as the quotient of $V^{\otimes r}$ by the span of all tensors $u^{(1)} \otimes \cdots \otimes u^{(r)}$ such that $u^{(i)} = u^{(j)}$ for distinct i and j. Let $u^{(1)} \wedge \cdots \wedge u^{(r)}$ denote the image in this quotient of $u^{(1)} \otimes \cdots \otimes u^{(r)}$. By construction $\bigwedge^r V$ has as a canonical basis all $v_{i_1} \wedge \cdots \wedge v_{i_r}$ with $1 \leq i_1 < \cdots < i_r \leq d$.

Lemma 3.1. *Let $\vartheta_{i_1 \ldots i_r} \in (\bigwedge^r V)^\circ$ for $1 \leq i_1 < \cdots < i_r \leq d$ be the dual basis of $(\bigwedge^r V)^\circ$. The linear map $v_{i_1} \wedge \cdots \wedge v_{i_r} \mapsto \vartheta_{i_1 \ldots i_r}$ is an isomorphism of representations of* $\mathrm{GL}(V)$.

Proof. Using our canonical basis, we calculate that $g \in \mathrm{GL}(V)$ acts by

$$g(v_{j_1} \wedge \cdots v_{j_r}) = \sum_{k_1} g_{k_1 j_1} v_{k_1} \wedge \cdots \wedge \sum_{k_r} g_{k_r j_r} v_{k_r}.$$

The coefficient of $v_{i_1} \wedge \cdots \wedge v_{i_r}$ in the right-hand side is obtained by taking each wedge product $v_{k_1} \wedge \cdots \wedge v_{k_r}$ such that $\{i_1, \ldots, i_r\} = \{k_1, \ldots, k_r\}$ and reordering it; therefore

$$\rho(g)_{ij} = \sum_{\sigma \in S_r} \mathrm{sgn}(\sigma) g_{i_{\sigma(1)} j_1} \cdots g_{i_{\sigma(r)} j_r}.$$

Hence, replacing g with its transpose,

$$\begin{aligned}\rho(g^{\mathrm{t}})_{ij} &= \sum_{\sigma\in S_r} \mathrm{sgn}(\sigma) g_{j_1 i_{\sigma(1)}} \ldots g_{j_r i_{\sigma(r)} j_r} \\ &= \sum_{\tau\in S_r} \mathrm{sgn}(\tau) g_{j_{\tau(1)} i_1} \ldots g_{j_{\tau(r)} i_r} \\ &= \rho(g)_{ji}\end{aligned}$$

and so $\rho(g^{\mathrm{t}})^{\mathrm{t}} = \rho(g)$. By (1), $\rho(g^{\mathrm{t}})^{\mathrm{t}}$ is the matrix representing g in its action on the dual basis. Therefore the linear map specified in the lemma is an isomorphism $\bigwedge^r V \to (\bigwedge^r V)^\circ$. $\square$

Let $\bigwedge^{\lambda'} V = \bigwedge^{\lambda'_1} V \otimes \cdots \otimes \bigwedge^{\lambda'_a} V$, where a is the first part of λ. Given a λ-tableau t, we define

$$|t| = \bigotimes_{j=1}^{a} \bigwedge_{i=1}^{\lambda'_j} v_{t_{(i,j)}} \in \bigwedge^{\lambda'} V. \tag{2}$$

The vector space $\bigwedge^{\lambda'} V$ has a canonical basis $\{|t| : t \in \mathrm{CSYT}(\lambda)\}$. It is easily seen that $F(t\cdot\tau) = F(t)\,\mathrm{sgn}(\tau)$ for $t \in \mathrm{CPP}(\lambda)$, and so there is a canonical surjection $q : \bigwedge^{\lambda'} V \to \nabla^\lambda(V)$ defined by $q(|t|) = F(t)$.

3.2. **Definition of $\Delta^\lambda(V)$.** Let $\{|t|^\circ : t \in \mathrm{CSYT}(\lambda)\}$ be the basis of $(\bigwedge^{\lambda'} V)^\circ$ dual to $\{|t| : t \in \mathrm{CSYT}(\lambda)\}$. Consider the dual map

$$q^\circ : \nabla^\lambda(V)^\circ \to \bigl(\bigwedge^{\lambda'} V\bigr)^\circ.$$

By definition $\nabla^\lambda(V)$ is a submodule of $\mathrm{Sym}^\lambda(V)$. This iterated symmetric power has as a canonical basis the GL-tabloids $f(t)$ for $t \in \mathrm{RSYT}(\lambda)$. Let $\{f(t)^\circ : t \in \mathrm{RSYT}(\lambda)\}$ be the corresponding dual basis of $(\mathrm{Sym}^\lambda V)^\circ$. Then $\nabla^\lambda(V)^\circ$ is spanned by the restrictions of the $f(t)^\circ$. Committing a minor abuse of notation, we use the same notation for the restrictions. We now apply q° to $f\bigl(s(\lambda)\bigr)^\circ \in \nabla^\lambda(V)^\circ$. By definition of q° we have

$$\begin{aligned} q^\circ\bigl(f\bigl(s(\lambda)\bigr)^\circ\bigr)(|t|) &= f\bigl(s(\lambda)\bigr)^\circ\bigl(q(f(t)\bigr) \\ &= f\bigl(s(\lambda)\bigr)^\circ\bigl(F(t)\bigr) \\ &= f\bigl(s(\lambda)\bigr)^\circ\bigl(\sum_{\tau\in\mathrm{CPP}(\lambda)} f(t\cdot\tau)\,\mathrm{sgn}(\tau)\bigr).\end{aligned}$$

The permutation $\tau \in \mathrm{CPP}(\lambda)$ gives a non-zero contribution to the sum above if and only if $f\bigl(s(\lambda)\bigr) = f(t\cdot\tau)$; equivalently, $s(\lambda)$ and $t\cdot\tau$ have the same rows, up to order. In particular, t must have exactly λ_i entries equal to i for each $i \in \{1,\ldots,\ell(\lambda)\}$. But then, since t is column standard, the λ_1 entries of t equal to 1 must be in the row 1 of t, the λ_2 entries of t equal to 2 must be in row 2 of t, and so on. Therefore $t = s(\lambda)$ and τ is the identity permutation. It follows that

$$f\big(s(\lambda)\big)^\circ = |s(\lambda)|^\circ \in \nabla^\lambda(V)^\circ.$$

By Lemma 3.1 there is an isomorphism $\big(\bigwedge^{\lambda'} V\big)^\circ \cong \bigwedge^{\lambda'} V$ sending $|t|^\circ$ to $|t|$ for each $t \in \mathrm{CSYT}(\lambda)$. Since q is surjective, q° is injective. Therefore $\nabla^\lambda(V)^\circ$ is isomorphic to a submodule of $\bigwedge^{\lambda'} V$ containing $|s(\lambda)|$.

Definition 3.2. We define $\Delta^\lambda(V)$ to be the submodule of $\bigwedge^{\lambda'} V$ generated by $|s(\lambda)|$.

3.3. **A basis for $\Delta^\lambda(V)$.** Given a tableau t, let $\mathrm{RStab}(t)$ be the subgroup of $\mathrm{RPP}(t)$ leaving t invariant. We define

$$F^\circ(t) = \sum_{\sigma \in \mathrm{RPP}(\lambda)/\mathrm{RStab}(t)} |t \cdot \sigma| \tag{3}$$

where the sum is over a set of representatives for the right cosets $\mathrm{RStab}(t)\sigma$ of $\mathrm{RStab}(t)$ in $\mathrm{RPP}(\lambda)$. (Note that $|t \cdot \sigma| = 0$ when $t \cdot \sigma$ has two equal entries in the same column.) In this step we show that $\Delta^\lambda(V)$ contains all the $F^\circ(s)$ for s semistandard. Since this is the non-routine part of the argument, we pause to give an example.

Example 3.3. Let V be 3-dimensional and let $g \in \mathrm{GL}(V)$ satisfy $gv_1 = v_1 + \alpha v_2 + \beta v_3$ and $gv_2 = v_2 + \gamma v_3$. The vector $|s\big((3,1)\big)|$ was seen at the end of §3.1. By definition, it generates $\Delta^{(3,1)}(V)$. We have

$$\begin{aligned}
g|s\big((3,1)\big)| &= \left|\begin{array}{c|c|c} v_1+\alpha v_2+\beta v_3 & v_1+\alpha v_2+\beta v_3 & v_1+\alpha v_2+\beta v_3 \\ v_2+\gamma v_3 & v_2+\gamma v_3 & \end{array}\right| \\
&= \left|\begin{array}{c|c|c} 1 & 1 & 1 \\ 2 & 2 & \end{array}\right| + \alpha \left|\begin{array}{c|c|c} 1 & 1 & 2 \\ 2 & 2 & \end{array}\right| \\
&\qquad + \beta\left(\left|\begin{array}{c|c|c} 1 & 1 & 3 \\ 2 & 2 & \end{array}\right| + \left|\begin{array}{c|c|c} 3 & 1 & 1 \\ 2 & 2 & \end{array}\right| + \left|\begin{array}{c|c|c} 3 & 1 & 1 \\ 2 & 2 & \end{array}\right| \right) \\
&\qquad + \gamma\left(\left|\begin{array}{c|c|c} 1 & 1 & 2 \\ 2 & 3 & \end{array}\right| + \left|\begin{array}{c|c|c} 1 & 1 & 2 \\ 3 & 2 & \end{array}\right| \right) \\
&\qquad + \alpha\gamma\left(\left|\begin{array}{c|c|c} 1 & 1 & 2 \\ 2 & 3 & \end{array}\right| + \left|\begin{array}{c|c|c} 1 & 1 & 2 \\ 3 & 2 & \end{array}\right| + \left|\begin{array}{c|c|c} 1 & 2 & 1 \\ 2 & 3 & \end{array}\right| + \left|\begin{array}{c|c|c} 2 & 1 & 1 \\ 3 & 2 & \end{array}\right| \right) \\
&\qquad + \beta\gamma\left(\left|\begin{array}{c|c|c} 1 & 1 & 3 \\ 2 & 3 & \end{array}\right| + \left|\begin{array}{c|c|c} 1 & 3 & 1 \\ 3 & 2 & \end{array}\right| + \left|\begin{array}{c|c|c} 3 & 1 & 1 \\ 2 & 3 & \end{array}\right| \right) + \cdots
\end{aligned}$$

where the first line should be interpreted formally, as a shorthand for

$$(v_1+\alpha v_2+\beta v_3)\wedge(v_2+\gamma v_3)\otimes(v_1+\alpha v_2+\beta v_3)\wedge(v_2+\gamma v_3)\otimes(v_1+\alpha v_2+\beta v_3).$$

This product is expanded in the second step above. To illustrate this expansion and one of the summands omitted above, $\alpha\beta\gamma$ is the coefficient of

$$\left|\begin{array}{c|c|c} 1 & 2 & 3 \\ 2 & 3 & \end{array}\right| + \left|\begin{array}{c|c|c} 1 & 3 & 2 \\ 3 & 2 & \end{array}\right| + \left|\begin{array}{c|c|c} 2 & 1 & 3 \\ 3 & 2 & \end{array}\right| + \left|\begin{array}{c|c|c} 3 & 1 & 2 \\ 2 & 3 & \end{array}\right| + \left|\begin{array}{c|c|c} 2 & 3 & 1 \\ 3 & 2 & \end{array}\right| + \left|\begin{array}{c|c|c} 3 & 2 & 1 \\ 2 & 3 & \end{array}\right|$$

Since α, β, γ are arbitrary elements of the infinite field F, it follows (see Lemma 3.4) that $\Delta^{(3,2)}(V)$ contains all the vectors

$$\begin{array}{|c|c|c|}1&1&2\\2&2&\end{array}\,,\quad \begin{array}{|c|c|c|}1&1&3\\2&2&\end{array}+\begin{array}{|c|c|c|}3&1&1\\2&2&\end{array}+\begin{array}{|c|c|c|}3&1&1\\2&2&\end{array}\,,\quad \begin{array}{|c|c|c|}1&1&2\\2&3&\end{array}+\begin{array}{|c|c|c|}1&1&2\\3&2&\end{array}\,,\ldots$$

and so on. These are

$$F^\circ\Bigl(\begin{ytableau}1&1&2\\2&2\end{ytableau}\Bigr),\quad F^\circ\Bigl(\begin{ytableau}1&1&3\\2&3\end{ytableau}\Bigr),\quad F^\circ\Bigl(\begin{ytableau}1&1&3\\2&3\end{ytableau}\Bigr),\ldots$$

and so on.

The following lemma justifies the final claim above more carefully.

Lemma 3.4. *Let V be an F-vector space and let $w_1,\ldots,w_M \in V$. Let*

$$f_1,\ldots,f_M \in F[x_1,\ldots,x_N]$$

be distinct monomials. If

$$f_1(\alpha_1,\ldots,\alpha_N)w_1+\cdots+f_M(\alpha_1,\ldots,\alpha_N)w_M \in V$$

for all $(\alpha_1,\ldots,\alpha_N)$ in F^N then $w_1,\ldots,w_M\in V$.

Proof. We work by induction on N. If $N=1$ then $f_1=x_1^{r_1},\ldots,f_M=x_1^{r_M}$ for distinct $r_1,\ldots,r_M\in\mathbf{N}_0$ and the hypothesis is that

$$\alpha^{r_1}w_1+\cdots+\alpha^{r_M}w_M\in V$$

for all $\alpha\in F$. Define an $M\times M$ matrix A by $A_{ij}=y_i^{r_j}$ where $y_1,\ldots,y_M$ are indeterminates. Since the r_i are distinct, the leading term in $\det A$ is $y_1^{r_1}y_2^{r_2}\ldots y_M^{r_M}$, in the obvious order on the indeterminates. Hence $\det A$ is a non-zero multivariable polynomial and so there is a specialization $y_i=\beta_i$ for $1\le i\le M$ such that $\det A\neq 0$. Fix a basis for V and write $w_1,\ldots,w_M$ as row vectors. Using this specialization we have

$$\begin{pmatrix}\beta_1^{r_1}&\beta_1^{r_2}&\ldots&\beta_1^{r_M}\\ \beta_2^{r_1}&\beta_2^{r_2}&\ldots&\beta_2^{r_M}\\ \vdots&\vdots&\ddots&\vdots\\ \beta_M^{r_1}&\beta_M^{r_2}&\ldots&\beta_M^{r_M}\end{pmatrix}\begin{pmatrix}w_1\\w_2\\ \vdots\\ w_M\end{pmatrix}=\begin{pmatrix}\beta_1^{r_1}w_1+\beta_1^{r_2}w_2+\cdots+\beta_1^{r_M}w_M\\ \beta_2^{r_1}w_1+\beta_2^{r_2}w_2+\cdots+\beta_2^{r_M}w_M\\ \vdots\\ \beta_M^{r_1}w_1+\beta_M^{r_2}w_2+\cdots+\beta_M^{r_M}w_M\end{pmatrix}.$$

By hypothesis, each row of the right-hand $M\times(\dim V)$ matrix is in V. By choice of the β_i, the $M\times M$ matrix on the left is invertible. Therefore $w_1,\ldots,w_M\in V$ as required.

For the inductive step, let $f_i=x_N^{r_i}g_i$ where $g_i\in F[x_1,\ldots,x_{N-1}]$. For each r let $\mathcal{D}_r=\{i\in\{1,\ldots,M\}:r_i=i\}$. By hypothesis

$$\sum_r \alpha_N^r\sum_{i\in\mathcal{D}_r} g_i(\alpha_1,\ldots,\alpha_{N-1})w_i\in V$$

for all $(\alpha_1,\ldots,\alpha_{N-1},\alpha_N)\in F^N$. By the $M=1$ case already proved, we have $\sum_{i\in D_r} g_i(\alpha_1,\ldots,\alpha_{N-1})w_i\in V$ for all $(\alpha_1,\ldots,\alpha_{N-1})\in F^{N-1}$. Since the monomials f_i are distinct, each g_i can be written as a sum of distinct monomials. Therefore, taking each r separately, it follows by induction that $w_i \in V$ for each $i \in \mathcal{D}_r$. $\square$

We now generalize Example 3.3. Define $g \in \mathrm{GL}(V)$ on the chosen basis $v_1, \ldots, v_d$ of V by

$$g(v_i) = v_i + \sum_{j>i} \alpha_{ij} v_j$$

where $\alpha_{ij} \in F$ is an arbitrary field element. Given $c_{ij} \in \mathbf{N}$ such that $c_{ii} + \cdots + c_{id} = \lambda_i$ for each i, let $s(\lambda, c)$ denote the unique row semistandard λ-tableau t obtained from $s(\lambda)$ by changing exactly c_{ij} entries of i in the i^{th} row of $s(\lambda)$ to j. Regarding the α_{ij} as indeterminates, it is clear that $|s(\lambda, c)|$ appears with coefficient $\prod_{ij} \alpha_{ij}^{c_{ij}}$ in $g|s|$. The same holds for any $|u|$ where u is obtained from $s(\lambda, c)$ by permuting the entries within each row; the sum of all such $|u|$, by definition, $F^\circ\bigl(s(\lambda, c)\bigr)$. Moreover, all summands of $g|s|$ are obtained in this way. Therefore

$$g|s| = \sum_c \Bigl(\prod_{ij} \alpha_{ij}^{c_{ij}}\Bigr) F^\circ\bigl(s(\lambda, c)\bigr)$$

where the sum is over all c_{ij} satisfying the condition above. By Lemma 3.4, it follows that $\Delta^\lambda(V)$ contains all $F^\circ\bigl(s(\lambda, c)\bigr)$. In particular, $\Delta^\lambda(V)$ contains all $F^\circ(s)$ for s a semistandard λ-tableau.

Lemma 3.5. *The $F^\circ(s)$ for $s \in \mathrm{SSYT}(\lambda)$ are linearly independent.*

Proof. We order the canonical basis of $\bigwedge^{\lambda'} V$ by $|u| \prec |v|$ if and only if in the right-most column where the distinct column standard tableaux u and v differ, the greater entry is in v. If t is a semistandard tableau then, for all $\sigma \in \mathrm{RPP}(\lambda)$, either $t \cdot \sigma = t$ or $|t| \prec |t \cdot \sigma|$. Therefore t is the unique least basis element appearing in $F^\circ(t)$. Moreover, if s and u are distinct semistandard tableaux then $|s| \neq |u|$. Therefore the coefficients of the $F^\circ(s)$ for $s \in \mathrm{SSYT}(\lambda)$ expressed in the canonical basis $\{|t| : t \in \mathrm{CSYT}(\lambda)\}$ of $\bigwedge^{\lambda'} V$ form a triangular matrix. $\square$

Corollary 3.6. *The module $\Delta^\lambda(V)$ has as a basis $\{F^\circ(s) : s \in \mathrm{SSYT}(\lambda)\}$ and is isomorphic to $\nabla^\lambda(V)^\circ$.*

Proof. By definition, $\Delta^\lambda(V)$ is the submodule of $\bigwedge^{\lambda'} V$ generated by $|s(\lambda)|$. We have seen that it contains all $F^\circ(s)$ for $s \in \mathrm{SSYT}(\lambda)$. By Lemma 3.5, these elements are linearly independent. Since $\Delta^\lambda(V)$ is contained in the image of the composition

$$\nabla^\lambda(V)^\circ \xrightarrow{q^\circ} \Bigl(\bigwedge^{\lambda'} V\Bigr)^\circ \cong \bigwedge^{\lambda'} V,$$

the dimension of $\Delta^\lambda(V)$ is at most $\dim \nabla^\lambda(V)$. We remarked in §2.2 that $\nabla^\lambda(V)$ has a basis indexed by semistandard λ-tableaux. Therefore, by dimension counting, the $F^\circ(s)$ for $s \in \mathrm{SSYT}(\lambda)$ span $\Delta^\lambda(V)$ and the composition above is an isomorphism onto $\Delta^\lambda(V)$. $\square$

3.4. **Identification with the Carter–Lusztig module and a final remark.** In §5.1 of [3], Green defines for each partition λ a polynomial representation $V_{\lambda,F}$ of $\mathrm{GL}(V)$ isomorphic to the contravariant dual $D^\circ_{\lambda,K}$ of the module $D_{\lambda,K}$ he constructed earlier in Chapter 4 of [3]. By Remark 2.16 in [2] and §4.8 in [3], $\nabla^\lambda(V)$ is isomorphic to $D_{\lambda,F}$. By Corollary 3.6, $\Delta^\lambda(V) \cong \nabla^\lambda(V)^\circ$. Therefore $\Delta^\lambda(V)$ is isomorphic to $V_{\lambda,F}$, and so to the Carter–Lusztig Weyl module by the remark on page 44 of [3].

Remark 3.7. We end by remarking that, unlike normal duality, contravariant duality preserves the highest-weight property. It is therefore not a surprise that $|s(\lambda)|$ is a highest-weight vector in $\bigwedge^{\lambda'} V$. Similarly, it is not a surprise that we needed only the action of Borel subgroup of lower-triangular matrices to find the $\mathrm{GL}(V)$-submodule of $\bigwedge^{\lambda'} V$ that it generates. The more surprising feature is that this submodule is the full image of $q^\circ(\nabla^\lambda(V))$ under the identification $\bigl(\bigwedge^{\lambda'} V\bigr)^\circ \cong \bigwedge^{\lambda'} V$. In contrast, for $\nabla^\lambda(V)$, the highest-weight vector $F\bigl(s(\lambda)\bigr)$ generates its unique *minimal* submodule. This was seen in Example 2.2 above for the symmetric power.

References

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