A MODEL FOR THE DOUBLE COSETS OF YOUNG SUBGROUPS

Acknowlegement. I am very grateful to Darij Grinberg for many corrections to errors in these note and helpful suggestions for clarifying remarks. Of course I have full responsibility for any remaining errors.

1. Introduction

There are several results concerning double cosets of Young subgroups of symmetric groups which are often proved as corollaries of more general results about Coxeter groups. In this note we give a selection of these results and give them short combinatorial proofs.

The replacement for the machinery of Coxeter groups consists of some simple combinatorial arguments involving tableaux. Inevitably we must repeat a few definitions from and standard lemmas from Coxeter theory, but I have tried to keep this to a minimum.

The idea behind this approach is well known to experts (see for instance [1] and [2, Ch. 13]), but I have not yet seen a really explicit account of it. (The introduction of 'labelled' tableaux in Remark 3.2 may be new.)

2. Preliminaries

Permutations. Let S_n be the symmetric group of degree n. If $g \in S_n$ we write jg for the image of $j \in \{1, 2, \ldots, n\}$ under g. This convention means that permutations should be composed from left to right. For example, the composition of the transposition (12) and the 3-cycle (123) is $(12)(123) = (23)$.

Given a permutation $g \in S_n$ and numbers $i, j \in [n]$ we say that (i, j) is an *inversion* of g if $i < j$ and $ig > jg$. If $(i, i+1)$ is an inversion of g then we say that g has a *descent* in position i. The *length* of g, denoted $\ell(g)$ is defined as the total number of inversions of g. The reason for this name is given by the following lemma.

Lemma 2.1. *The permutation $g \in S_n$ has length ℓ if and only if the shortest expression of g as a product of basic transpositions $(12), (23), \ldots$ has length ℓ. Moreover, if $s_i = (i\, i{+}1)$ for $1 \leq i < n$ then*

$$\ell(s_i g) = \begin{cases} \ell(g) + 1 & \text{if } i \text{ is not a descent of } g, \\ \ell(g) - 1 & \text{if } i \text{ is a descent of } g. \end{cases}$$

and

$$\ell(g s_i) = \begin{cases} \ell(g) + 1 & \text{if } i \text{ is not a descent of } g^{-1}, \\ \ell(g) - 1 & \text{if } i \text{ is a descent of } g^{-1}. \end{cases}$$

Date: 2006, revised January 2014.

Proof. The first part follows by an easy induction on $\ell(g)$. The formula for $\ell(s_i g)$ is immediate from the definition. Since $\ell(h) = \ell(h^{-1})$ for any permutation h, the second formula follows from the first, applied to $(gs_i)^{-1} = s_i g^{-1}$. Alternatively, one may note that $(ig^{-1}, (i+1)g^{-1})$ is an inversion of gs_i if and only if $ig^{-1} < (i+1)g^{-1}$, and this holds if and only if $(i\,i+1)$ is *not* an inversion of g^{-1}. □

Compositions. A *composition* of n is a tuple of strictly positive integers whose sum is n. We write $\nu \models n$ to indicate that ν is a composition of n. We say that a composition $(\nu_1, \ldots, \nu_k)$ is a *refinement* of the composition $(\mu_1, \ldots, \mu_\ell)$ if there exist $a_0 < a_1 < \ldots < a_\ell$ with $a_0 = 0$, $a_\ell = k$ and

$$\sum_{j=a_{i-1}+1}^{a_i} \nu_j = \mu_i$$

for each $i \in \{1, \ldots, \ell\}$. The *diagram* of a composition $(\nu_1, \ldots, \nu_k)$ is, by the formal definition, the set

$$\{(i,j) : 1 \leq i \leq k, 1 \leq j \leq \nu_i\},$$

which we represent by a Young diagram in the usual way. For example, the composition $(5, 2, 3)$ is represented by

$$\ydiagram{5,2,3}\ .$$

We refer to the elements of a Young diagram as *nodes*. Finally we define the *Young subgroup* S_ν corresponding to $(\nu_1, \ldots, \nu_k)$ by

$$S_\nu = \prod_{i=1}^{k} S_{\{\nu_1+\cdots+\nu_{i-1}+1, \ldots, \nu_1+\cdots+\nu_i\}}.$$

3. The Model

Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ and $\mu = (\mu_1, \ldots, \mu_s)$ be two fixed compositions of n, fixed throughout this note.

A *μ-tableau of type λ* is a function from the nodes of μ to $\mathbf{N}$ which takes each value $i \in \{1, \ldots, r\}$ exactly λ_i times. We represent tableaux by drawing the diagram of μ and then writing the image of each node inside the corresponding box. For example

$$\begin{ytableau} 2 & 3 & 1 & 2 \\ 1 & 1 \end{ytableau}$$

is a $(4,2)$-tableau of type $(3,2,1)$. We fix a map from the nodes of the diagram of μ to the numbers $\{1, \ldots, n\}$, as shown below for $\mu = (4,2)$.

$$\begin{ytableau} \scriptstyle 1 & \scriptstyle 2 & \scriptstyle 3 & \scriptstyle 4 \\ \scriptstyle 5 & \scriptstyle 6 \end{ytableau}$$

Note this labelling gives a total ordering on the nodes of the diagram of μ. Given a tableau t we write mt for the image of the mth node under t.

We now let S_n act on the set of all μ-tableaux by place permutation. If t is a μ-tableau of type λ and $g \in S_n$ we define a new tableau tg by

$$m(tg) = (mg^{-1})t \quad \text{for } m \in \{1, \ldots, n\}.$$

The entry in box mg of tg is the entry in box m of t, so the boxes of t have, as claimed, been permuted by g. Therefore tg is another μ-tableau of type λ. For example

$$\begin{array}{|c|c|c|c|}\hline 2&1&2&3\\\hline 1&1\\\cline{1-2}\end{array}\,(123)(45) = \begin{array}{|c|c|c|c|}\hline 2&2&1&1\\\hline 3&1\\\cline{1-2}\end{array}\,.$$

The stabiliser of t in the action of S_n is a conjugate of the Young subgroup S_λ. The set of all μ-tableaux of type λ is therefore a model for the right coset space $S_\lambda\backslash S_n$. The advantage of our approach is that it is very easy to restrict the action of S_n to S_μ, and so obtain a model for the double cosets, $S_\lambda\backslash S_n/S_\mu$.

4. Lengths of elements in double cosets

Given λ and μ compositions of n, let t^μ_λ be the unique μ-tableau of type λ whose entries are increasing when we read the boxes in the order fixed above. For instance,

$$t^{(4,2)}_{(3,2,1)} = \begin{array}{|c|c|c|c|}\hline 1&1&1&2\\\hline 2&3\\\cline{1-2}\end{array}\,.$$

Theorem 4.1. *Let λ and μ be compositions of n and let $g \in S_n$. Then $x \in S_\lambda g S_\mu$ is of minimal length in its double coset if and only if*

(i) *$t^\mu_\lambda x$ has weakly increasing rows,*
(ii) *if $i < j$ and the ith and jth positions of t^μ_λ are equal then $ix < jx$.*

Proof. 'Only if': Suppose that the number in position i of $t^\mu_\lambda x$ is greater than the number in position $i+1$. Then, by definition of the action, $ix^{-1} > (i+1)x^{-1}$. Hence x^{-1} has a descent in position i, and so by Lemma 2.1, xs_i has shorter length than x. Therefore i and $i+1$ must label positions in different rows and so the rows of $t^\mu_\lambda x$ are weakly increasing. Similarly, if (ii) fails then there is an i such that nodes i and $i+1$ of t^μ_λ have the same number and $ix > (i+1)x$; now $s_i x$ has shorter length than x.

'If': We may find $h \in S_\lambda$ and $k \in S_\mu$ such that $\ell(hxk)$ is minimal. By the 'only if' part, we know that hxk satisfies conditions (i) and (ii). By (i), this can only happen if $t^\mu_\lambda hxk = t^\mu_\lambda x$. So far, all this says is that $hxk = h'x$ for some $h' \in S_\lambda$. However, by (ii), both x and hxk preserves the relative order of boxes of t^μ_λ containing the same number, so we must have $h' = 1$. Thus $x = hxk$ is the unique minimal element. □

Remark 4.2. A convenient way to work with condition (ii) uses the following definition. Say that a *labelled* μ-tableau of type λ is an injective function $\mathbf{t}$ from the nodes of μ to $\mathbf{N} \times \mathbf{N}$ such that (i, j) is in the image of $\mathbf{t}$ if and only if $j \in \{1, \ldots, \lambda_i\}$. We represent labelled tableaux by putting subscripts on numbers; for example,

$$\begin{array}{|c|c|c|c|}\hline 1_1&1_2&1_3&2_1\\\hline 2_2&3_1\\\cline{1-2}\end{array}$$

is one of 6 labelled tableaux corresponding to

$$t^{(4,2)}_{(3,2,1)} = \begin{array}{|c|c|c|c|}\hline 1&1&1&2\\\hline 2&3\\\cline{1-2}\end{array}\,.$$

More generally, let $\mathbf{t}^\mu_\lambda$ be the labelled μ-tableau of type λ obtained from t^μ_λ by allocating labels as in the example above. For ease of reference we shall say that the symbol i_j has *number* i and *index* j.

As before, S_n acts on the set of labelled μ-tableaux of type λ by place permutation. Using this action we can rewrite condition (ii) as

(ii)$'$ If $i < j$ then r_i appears in an earlier position than r_j in $\mathbf{t}^\mu_\lambda x$ (assuming both appear)

Here is an alternative proof of the 'if' part of the proof of Theorem 4.1 using labelled tableaux. Let $x \in S_\lambda g S_\mu$ satisfy (i) and (ii)$'$. As before, choose $h \in S_\lambda$ and $k \in S_\mu$ such that hxk has minimum length. By the 'only if' part, hxk satisfies (i) and (ii)$'$. But $\mathbf{t}^\mu_\lambda x$ and $\mathbf{t}^\mu_\lambda hxk$ have the same set of numbers in each row (since h permutes indices, leaving numbers unchanged, and k permutes the symbols within each row), and by (i) and (ii)$'$ the symbols in each row appear in the same order in $\mathbf{t}^\mu_\lambda x$ and $\mathbf{t}^\mu_\lambda hxk$. Hence

$$\mathbf{t}^\mu_\lambda x = \mathbf{t}^\mu_\lambda hxk.$$

Since a labelled tableau is only stabilised by the identity element, this implies that $x = hxk$.

Theorem 4.3. *Let x be a minimal length double coset representative for a $S_\lambda \backslash S_n / S_\mu$ double coset. If $g \in S_\lambda x S_\mu$ then there exist $h \in S_\lambda$ and $k \in S_\mu$ such that $g = hxk$ and $\ell(g) = \ell(h) + \ell(x) + \ell(k)$.*

Proof. We work by induction on $\ell(g)$. If $\ell(g) = \ell(x)$ then by Theorem 4.1, $g = x$ so we may take $h = k = 1$. Now suppose that $\ell(g) > \ell(x)$. By the proof of Theorem 4.1, either g does not satisfy condition (i), in which case we may find i such that $s_i \in S_\mu$ and $\ell(gs_i) < \ell(g)$, or g does not satisfy condition (ii), in which case we may find i such that $s_i \in S_\lambda$ and $\ell(s_i g) < \ell(g)$. The result now follows by induction. □

A general element of the double coset $S_\lambda x S_\mu$ is given by starting with the labelled tableau $\mathbf{t}^\mu_\lambda x$, and then applying a permutation of the indices (this corresponds to an element of S_λ^x), then a permutation of the rows (corresponding to an element of S_μ). The corresponding algebraic formulation

$$hxk = x(x^{-1}hx)k = xh^x k$$

where $h \in S_\lambda$ and $k \in S_\mu$ makes it clear that the non-uniqueness in these expressions comes from $S_\lambda^x \cap S_\mu$. This is made more precise in the next theorem.

Theorem 4.4. *Let x be a minimal length double coset representative for an $S_\lambda \backslash S_n / S_\mu$ double coset. Let $H = S_\lambda^x \cap S_\mu$. Let $g \in S_\lambda x S_\mu$.*

(i) *There are exactly $|H|$ ways to express g in the form hxk with $h \in S_\lambda$ and $k \in S_\mu$.*

(ii) *There is a unique element $h_{\min} \in S_\lambda$ of minimal length such that $h_{\min} x S_\mu = g S_\mu$, or equivalently, such that the rows of the labelled tableaux $\mathbf{t}^\mu_\lambda h_{\min} x$ and $\mathbf{t}^\mu_\lambda g$ agree setwise. If $g = h_{\min} x k$ where $k \in S_\mu$ then $\ell(g) = \ell(h_{\min}) + \ell(x) + \ell(k)$.*

(iii) *There is a unique element $k_{\min} \in S_\mu$ of minimal length such that $S_\lambda x k_{\min} = S_\lambda g$, or equivalently, such that the labelled tableaux $\mathbf{t}^\mu_\lambda x k_{\min}$*

and $\mathbf{t}^{\mu}_{\lambda} g$ agree up to a permutation of indices. If $g = hxk_{\min}$ where $h \in S_\lambda$ then $\ell(g) = \ell(h) + \ell(x) + \ell(k_{\min})$.

Proof. Part (i) is clear from the displayed equation immediately before the theorem.

For (ii), first note that if $g, g' \in S_n$ then, by the action of S_μ on labelled tableaux, the rows of $\mathbf{t}^{\mu}_{\lambda} g$ and $\mathbf{t}^{\mu}_{\lambda} g'$ agree setwise if and only if $gS_\mu = g'S_\mu$. Let $h_{\min} \in S_\lambda$ be a permutation of minimum length such that the rows of $\mathbf{t}^{\mu}_{\lambda} hx$ and $\mathbf{t}^{\mu}_{\lambda} g$ agree setwise, and whenever $i < j$ and r_i and r_j both appear *in different rows* of $\mathbf{t}^{\mu}_{\lambda} h_{\min} x$, r_i appears in a higher row than r_j. It is then clear that $h_{\min} x \in gS_\mu$ and that any other permutation $h \in S_\lambda$ such that $hx \in gS_\mu$ has strictly more inversions than $h_{\min}$. Finally if $h_{\min} xk = g$, then applying k to $\mathbf{t}^{\mu}_{\lambda} h_{\min} x$ increases the number of inversions by $\ell(k)$.

Part (iii) is analogous to (ii). $\square$

Here is an example to illustrate all the results in this section.

Example 4.5. Let $\lambda = (3, 3, 2)$ and let $\mu = (5, 3)$. We work with position permutations of labelled $(5, 3)$-tableaux of type $(3, 3, 2)$, with nodes numbered from 1 up to 8 as shown below.

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 5 \\ \hline 6 & 7 & 8 \\ \cline{1-3}\end{array}$$

We have

$$\mathbf{t}^{(5,3)}_{(3,3,2)} = \begin{array}{|c|c|c|c|c|}\hline 1_1 & 1_2 & 1_3 & 2_1 & 2_2 \\ \hline 2_3 & 3_1 & 3_2 \\ \cline{1-3}\end{array}$$

Let $g = (2, 3, 6)(4, 5, 7, 8)$. We will find a minimal length double coset representative x and elements $h \in S_\lambda$, $k \in S_\mu$ such that $hxk = g$.

First note that

$$\mathbf{t}^{(5,3)}_{(3,3,2)} g = \begin{array}{|c|c|c|c|c|}\hline 1_1 & 2_3 & 1_2 & 3_2 & 2_1 \\ \hline 1_3 & 2_2 & 3_1 \\ \cline{1-3}\end{array}.$$

The permutation of minimum length such that condition (ii)$'$ holds permutes the elements $2_1, 2_2, 2_3$ as the 3-cycle $(2_1 2_2 2_3)$ and the elements $3_1, 3_2$ as the transposition $(3_1 3_2)$. We have $xk = h^{-1} g$, we will must take $h^{-1} = (465)(78)$. Hence $h^{-1} g = (23674)$ and

$$\mathbf{t}^{(5,3)}_{(3,3,2)} h^{-1} g = \begin{array}{|c|c|c|c|c|}\hline 1_1 & 2_1 & 1_2 & 3_1 & 2_2 \\ \hline 1_3 & 2_3 & 3_2 \\ \cline{1-3}\end{array}.$$

The unique permutation k^{-1} such that the rows of $\mathbf{t}^{(5,3)}_{(3,3,2)} h^{-1} g k^{-1}$ satisfy (i) and (ii)$'$ is $k^{-1} = (23)(45)$. Set

$$x = h^{-1} g k^{-1} = (36754).$$

We check that

$$\mathbf{t}^{(5,3)}_{(3,3,2)} x = \begin{array}{|c|c|c|c|c|}\hline 1_1 & 1_2 & 2_1 & 2_2 & 3_1 \\ \hline 1_3 & 2_3 & 3_2 \\ \cline{1-3}\end{array}$$

satisfies (i) and (ii)$'$. Therefore $g = hxk$ where where $h = (456)(78), x = (36754)$ and $k = (23)(45)$ and $g \in S_\lambda x S_\mu$ where x is the unique minimal length element in its double coset.

However, this h does not have minimum possible length. It was unnecessary to permute the indices so that (ii)$'$ holds: it is sufficient if $2_1, 2_2$ and 3_1 are in the first row. In fact the minimal length element of $S_{(3,3,2)}$ acts as the product of transpositions $(2_2 2_3)(3_1 3_2)$. So we take $h_{\min}^{-1} = (56)(78)$. Now $h_{\min}^{-1} g = (236745)$ and

$$\mathbf{t}^{(5,3)}_{(3,3,2)} h_{\min}^{-1} g \; = \; \begin{array}{|c|c|c|c|c|}\hline 1_1 & 2_2 & 1_2 & 3_1 & 2_1 \\ \hline 1_3 & 2_3 & 3_1 \\ \cline{1-3} \end{array}.$$

We now take $k'^{-1} = (2453)$ to straighten out the first row. As expected we have $g = h_{\min} x k'$.

Since $\ell(x) = 4$ we have $\ell(g) = \ell(h) + \ell(x) + \ell(k) = 3 + 4 + 2 = 9$. This agrees with $\ell(g) = \ell(h_{\min}) + \ell(x) + \ell(k') = 2 + 4 + 3$.

5. Permutation actions

We say that a tableau is *row semistandard* if its rows are non-decreasing. As an corollary of the proof of Theorem 4.1 we have the following useful result.

Corollary 5.1. *Let X be the set of minimal length representatives for the double cosets $S_\lambda \backslash S_n / S_\mu$. The map sending $x \in X$ to $t^\mu_\lambda x$ gives a bijection between X and the set of row semistandard μ-tableaux of type λ.*

Hence the orbits of S_μ on $S_\lambda \backslash S_n$ are canonically labelled by row semistandard μ-tableaux of type λ.

Proposition 5.2. *Let λ and μ be compositions of n. If $x \in S_n$ is of minimal length in its coset xS_μ then $(S_\lambda)^x \cap S_\mu$ is a Young subgroup of S_n.*

Proof. First note that if x has minimum length in xS_μ then the (unlabelled) tableau $t^\mu_\lambda x$ is row semistandard. Since $(S_\lambda)^x$ is the stabiliser of $t^\mu_\lambda x$, we see that $(S_\lambda)^x \cap S_\mu$ is the subgroup of S_μ that permutes the equal entries within each row. Since $t^\mu_\lambda x$ is row semistandard, these entries form contiguous blocks and so $(S_\lambda)^x \cap S_\mu$ is a Young subgroup. $\square$

In particular, the hypotheses of the lemma are satisfied whenever x has minimum length in the double coset $S_\lambda x S_\mu$. For example, $x = (456)$ is a minimal length coset representative for $S_{(3,2,1)} \backslash S_6 / S_{(4,2)}$, corresponding to the tableau

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 3 \\ \hline 2 & 2 \\ \cline{1-2} \end{array}.$$

We can read off from this tableau that

$$(S_{(3,2,1)})^{(456)} \cap S_{(4,2)} = S_{\{1,2,3\}} \times S_{\{5,6\}}.$$

Remark 5.3. The converse of Proposition 5.2 does not hold. For example, if $\lambda = (2,1)$ and $\mu = (3)$ then $(S_{(2,1)})^{(123)} = S_{\{2,3\}}$ is a Young subgroup, but (123) is not of minimal length in the coset $(123)S_3 = S_3$.

The definition of a μ-tableau of type λ is asymmetric between λ and μ. For row semistandard tableaux there is an alternative description which removes this asymmetry.

Lemma 5.4. *Let t be a row semistandard μ-tableau of type λ. Define a matrix T by letting T_{ij} be the number of entries of row i of t equal to j. This gives a bijection between row semistandard μ-tableaux of type λ and non-negative integral matrices with row sums μ and column sums λ.* $\square$

An immediate consequence is that the number of row semistandard μ-tableaux of type λ is equal to the number of row semistandard λ-tableaux of type μ. (See Corollary 5.7 below for another remark along these lines.) The next definition is most easily given in terms of matrices.

Definition 5.5. *Say that a non-negative integral matrix T has* content ν *if when we read the entries row-by-row, reading rows from left to right, starting at the top row and finishing at the bottom row, the non-negative entries form the composition ν. Let $m^{\mu}_{\lambda}(\nu)$ be the number of non-negative integral matrices with row sums μ, column sums λ and content ν.*

Using our bijection, we may also refer to the *content* of a row semistandard tableau, which is obtained by counting numbers of equal entries in each row. For example, the $(3,1)$-tableaux of type $(2,2)$

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 \\ \cline{1-1} \end{array} \quad \text{and} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 1 \\ \cline{1-1} \end{array}$$

correspond to the matrices

$$\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$$

and have contents $(2,1,1)$ and $(1,2,1)$ respectively.

Proposition 5.6. *Let Ω^λ denote the coset space $S_\lambda \backslash S_n$. If we restrict the action of S_n to S_μ then the following Mackey formula holds:*

$$\Omega^\lambda = \bigcup_{\nu} m^{\mu}_{\lambda}(\nu)\, S_\nu \backslash S_\mu.$$

where the union is over all compositions ν refining μ.

Proof. By Corollary 5.1, if we identify $S_\lambda \backslash S_n$ with the set of μ-tableaux of type λ then each orbit of S_μ contains a unique row semistandard tableau. Let t be such a tableau, with content ν say. It follows immediately from the definition of content that ν is a refinement of μ and that S_ν is the stabiliser of t in the action of S_μ. The result follows. □

Corollary 5.7. *The number of orbits of S_λ on Ω^μ is equal to the number of orbits of S_μ on Ω^λ.* □

This corollary can also be proved using character theory. I am grateful to Darij Grinberg for pointing out a short proof: the orbits of S_μ on Ω^λ are in bijection with the double cosets $S_\lambda \backslash S_n / S_\mu$ (via the map that sends an orbit to the union of the cosets in it), and similarly, the orbits of S^λ on Ω^μ are in bijection with the double cosets $S_\mu \backslash S_n / S_\lambda$. The two sets of double cosets are in bijection by the map defined by $S_\lambda g S_\mu \mapsto S_\mu g^{-1} S_\lambda$.

Corollary 5.8.

$$\Omega^\lambda \times \Omega^\mu = \bigcup_{\nu \models n} m^{\mu}_{\lambda}(\nu) \Omega^\nu.$$

Sketch proof. This follows from the well known result that if G is a group acting transitively on sets Ω and Γ and $\omega \in \Omega$ then there is a bijection between the orbits of the point stabiliser G_ω on Γ and the orbits of G on $\Omega \times \Gamma$. This bijection is given by mapping γG_ω to $(\omega, \gamma)G$. □

The analogues of the two previous corollaries for permutation modules are as follows.

Corollary 5.9. *If M^λ denote the $\mathbf{Z}S_n$ permutation module obtained from the action of S_n on Ω^λ then*

$$M^\lambda \big\downarrow_{S_\mu} = \bigoplus_\nu m^\mu_\lambda(\nu)\mathbf{Z}\big\uparrow^{S_\mu}_{S_\nu},$$

$$M^\lambda \otimes M^\mu = \bigoplus_\nu m^\mu_\lambda(\nu) M^\nu$$

where the sums are over all compositions ν refining μ.

In particular, the last formula implies that if π^λ is the permutation character of Ω^λ then

$$\pi^\lambda\pi^\mu = \sum_{\nu \models n} m^\mu_\lambda(\nu)\pi^\nu.$$

This is a shadow of the multiplication formula in the Solomon Descent Algebra. For an introduction see [3].

Remark 5.10. In fact, there are already signs that the non-commutative setting is the more natural one. For example, if the parts of the compositions ν and ρ may be rearranged to give the same partition of n then $\Omega^\nu \cong \Omega^\rho$ as S_n-sets, and $M^\nu \cong M^\rho$ as $\mathbf{Z}S_n$-modules. (In each case the converse also holds.) So the right-hand-sides in the last corollary are not sums over 'independent' elements. For example, we can write

$$M^\lambda \otimes M^\mu = \bigoplus c^\mu_\lambda(\nu) M^\nu$$

where the sum is over all *partitions* ν, and the coefficients are defined by

$$c^\mu_\lambda(\nu) = \sum_{\rho\sim\nu} m^\mu_\lambda(\rho).$$

(Here $\rho \sim \nu$ means that when we arrange the parts of ρ in order, we obtain ν.) But in the non-commutative setting we can replace M^λ with the element Ξ^λ of the Solomon Descent Algebra, and obtain the coefficients $m^\mu_\lambda(\nu)$ as structure constants using the formula

$$\Xi^\lambda\Xi^\mu = \sum_{\nu\models n} m^\mu_\lambda(\nu)\Xi^\nu.$$

6. Gale–Ryser Theorem

We conclude with a remark related to the Gale–Ryser Theorem. Recall that this theorem states that there is a $0-1$ matrix with row sums λ and column sums μ if and only if $\lambda \trianglelefteq \mu'$. The following proposition gives an algebraic formulation:

Proposition 6.1. *Let λ and μ be partitions of n. There is a double coset $S_\lambda g S_\mu$ containing $|S_\lambda||S_\mu|$ elements if and only if $\lambda \trianglelefteq \mu'$.*

Proof. There is such a coset if and only if there is a row semistandard μ-tableau of type λ such that no entry is repeated in any row. The matrix corresponding to such a tableau is a $0-1$ matrix with row sums μ and column sums λ. □

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