Introduction to plethysm

Polynomial reps of $GL_d(\mathbb{C})$ degree r ⟶ Reps of Sym_r

$E = \langle e_1 \ldots e_d \rangle_{\mathbb{C}}$ natural rep.

↘ Symmetric functions of degree r in d variables ↙

Poly reps	Sym funcs	Sym_r reps
$\mathrm{Sym}^r E$	$s_{(r)}$ complete	$\mathbb{C}$ (trivial)
$\Lambda^r E$	$s_{(1^r)}$ elementary	sign
$\nabla^\lambda E$	s_λ Schur func	S^λ (Specht module
$U \oplus V$	$\mathrm{tr}_U + \mathrm{tr}_V$	$\mathcal{Y}U \oplus \mathcal{Y}V$
$U \otimes V$ (deg r, deg s)	$\mathrm{tr}_U \times \mathrm{tr}_V$	$\mathcal{Y}U \otimes \mathcal{Y}V \uparrow_{S_r \times S_s}^{r+s}$
Composition $\nabla^\nu \nabla^\mu E$	Plethysm $s_\nu \circ s_\mu$	$S^{\mu \otimes n} \otimes \mathrm{Inf}\, S^\nu \uparrow_{S_m \wr S_n}^{S_{mn}}$
weight space decomp $V = \bigoplus_{\substack{(\alpha_1 \cdots \alpha_d) \\ \alpha_1 + \cdots + \alpha_d = r}} V_\alpha$	$\mathrm{tr}_V = \sum_\alpha x_1^{\alpha_1} \ldots x_d^{\alpha_d}$	No analogue (but $\mathcal{Y}U = V_{\underbrace{(1 \ldots 1)}_{r}}$ when $d = r$)

§1 Polynomial reps of $GL_d(\mathbb{C})$ $\quad E = \langle e_1 \ldots e_d \rangle_{\mathbb{C}} \circlearrowleft GL_d(\mathbb{C})$

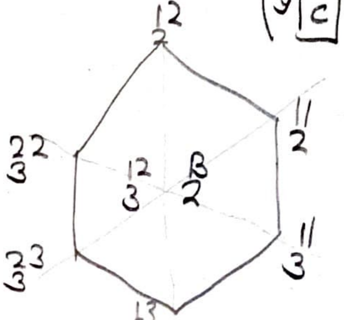
$$E \otimes E \cong \mathrm{Sym}^2 E \oplus \Lambda^2 E$$
$$= \langle e_i e_j : 1 \leq i \leq j \leq d \rangle \oplus \langle e_i \wedge e_j : 1 \leq i < j \leq d \rangle$$

$$d = 3:\ E \otimes E \otimes E \cong \underset{10D}{\mathrm{Sym}^3 E} \oplus \underset{1D}{\Lambda^3 E} \oplus\ ?$$

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \longmapsto \begin{array}{ccc} e_1^2 & e_1 e_2 & e_2^2 \\ \end{array} \begin{pmatrix} \alpha^2 & \alpha\beta & \beta^2 \\ 2\alpha\gamma & \alpha\delta + \beta\gamma & 2\beta\delta \\ \gamma^2 & \gamma\delta & \delta^2 \end{pmatrix}$$

$$\nabla^{(2,1)} E = \left\langle F\left(\begin{array}{|c|c|} \hline a \leq & b \\ \hline c & \\ \hline \end{array} \right) : a, b, c \in \{1 \ldots d\} \right\rangle \subseteq \mathrm{Sym}^2 E \otimes E$$

$$\text{where } F\left(\begin{array}{|c|c|} \hline a & b \\ \hline c & \\ \hline \end{array} \right) = e_a e_b \otimes e_c - e_c e_b \otimes e_a$$



$\leftrightarrow$ adjoint rep of $\mathfrak{sl}_3$ (or $\mathfrak{su}_3$)

Irreducibles $\nabla^\lambda E = \langle F(t) : t \in SSYT(\lambda') \rangle$
(examples for $\nabla^{(n)}, \nabla^{(1^n)}$, see fact 6)

Composition of reps Exc with $d=2$

$$\langle (e_1^2)(e_2^2) - (e_1e_2)^2 \rangle \longrightarrow Sym^2(Sym^2 E) \xrightarrow{(uv)(wz) \mapsto uvwz} Sym^4 E$$

$$= \left\langle F\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \right\rangle$$

$$\cong \nabla^{(2,2)} E$$

$$\cong \det^2$$

$$Sym^2(Sym^2 E) = \langle (e_1^2)(e_1^2), (e_1^2)(e_1e_2), (e_1^2)(e_2^2), (e_1e_2)(e_1e_2), (e_1e_2)(e_2^2), (e_2^2)(e_2^2) \rangle$$

$$(e_1^2)(e_2^2),\ (e_1e_2)(e_1e_2) \mapsto e_1^2e_2^2$$

$$Sym^2(Sym^2 E) \cong \nabla^{(2,2)}E \oplus Sym^4 E$$

Weight space decomp $V = \bigoplus_{\substack{(\alpha_1 \cdots \alpha_d) \\ \alpha_1 + \cdots + \alpha_d = r}} V_\alpha$ where $V_\alpha = \left\{ v \in V : \mathrm{diag}\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_d \end{pmatrix} v = x_1^{\alpha_1} \cdots x_d^{\alpha_d} v \right\}$

e.g. $V_{(4)} = \langle (e_1^2)(e_2^2) \rangle$ hw, $V_{(2,2)} = \langle (e_1^2)(e_2^2), (e_1e_2)(e_1e_2) \rangle$ if $V = Sym^2 Sym^2 E$

$\omega = (e_1^2)(e_2^2) - (e_1e_2)(e_1e_2)$ is hw

Highest weight vector $v \in V_\alpha$ such that $\begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} v = v$.

Wonderful fact 1: a highest weight vector in V_λ generates $\nabla^\lambda E$.

Exercise $Sym^2 Sym^n E$ has hw vectors $(e_1^2)^n, (e_1^2)^{n-2}\omega, (e_1^2)^{n-4}\omega^3, \dots$
and $Sym^2 Sym^n E \cong \nabla^{(2n)}E \oplus \nabla^{(2n-2)}E \oplus \cdots \oplus \begin{cases} \nabla^{(n,n)}E & n \text{ even} \\ \nabla^{(n+1,n-1)}E & n \text{ odd} \end{cases}$
(omitted)

§2 Symmetric functions

$$d=2 \quad tr_{Sym^2 E}\begin{pmatrix} \alpha & \\ & \delta \end{pmatrix} = tr\begin{pmatrix} \alpha^2 & & \\ & \delta^2 & \\ & & \alpha\delta \end{pmatrix} = \alpha^2 + \delta^2 + \alpha\delta = s_{(2)}(\alpha, \delta)$$

$$tr_{Sym^2 E}\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_d \end{pmatrix} = \sum_i x_i^2 + \sum_{i<j} x_i x_j = s_{(2)}(x_1 \cdots x_d)$$

$$d=3 \quad tr_{\nabla^{(2,1)}E}\begin{pmatrix} x_1 & & \\ & x_2 & \\ & & x_3 \end{pmatrix} = x^{\begin{smallmatrix} 1 & 1 \\ 2 & \end{smallmatrix}} + x^{\begin{smallmatrix} 1 & 2 \\ 2 & \end{smallmatrix}} + \cdots + x^{\begin{smallmatrix} 2 & 3 \\ 3 & \end{smallmatrix}}$$

$$= x_1^2 x_2 + x_1 x_2^2 + \cdots + x_2 x_3^2$$

Wonderful fact 2 The traces of diagonal matrices determine the representation

Motivation: clearly...

Defn $tr_{\nabla^\lambda E}\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_d \end{pmatrix} = \sum_{\substack{t \in SSYT(\lambda) \\ \text{entries} \le d}} x^t = s_\lambda(x_1 \cdots x_d)$ Schur function

Plethysm $d=2$

$$\mathrm{tr}_{\mathrm{Sym}^2(\mathrm{Sym}^2 E)}\begin{pmatrix}\alpha & \\ & \delta\end{pmatrix} = \mathrm{tr}_{\mathrm{Sym}^2 E}\begin{pmatrix}\alpha^2 & & \\ & \alpha\delta & \\ & & \delta^2\end{pmatrix} = (\alpha^2)(\alpha^2) + (\alpha^2)(\alpha\delta) + (\alpha^2)(\delta^2) + (\alpha\delta)(\ldots) + \cdots$$

$$= s_{(2)}(x_1, x_2, x_3)\Big|_{\text{evaluated at } x_1=\alpha^2,\ x_2=\alpha\delta,\ x_3=\delta^2} = (s_{(2)} \circ s_{(2)})(\alpha,\delta)$$

Rule for $f\circ g$: substitute the monomials in g for the variables of f.

Schur function decomposition

- $$s_{(2)} \circ s_{(2)}(x_1,x_2) = s_{(2)}(x_1^2, x_1x_2, x_2^2) = x_1^4 + x_1^3x_2 + 2x_1^2x_2^2 + x_1x_2^3 + x_2^4 = s_{(4)}(x_1,x_2) + s_{(2,2)}(x_1,x_2)$$
- $$s_{(1,1)} \circ s_{(2)}(x_1,x_2) = s_{(1,1)}(x_1^2, x_1x_2, x_2^2) = x_1^3x_2 + x_1^2x_2^2 + x_1x_2^3 = s_{(3,1)}(x_1,x_2)$$

$$\Rightarrow \Lambda^2 \mathrm{Sym}^2 E \cong \nabla^{(3,1)} E$$

(multiplicity free classical)

§3 Symmetric groups

$$d=2\quad \mathrm{Sym}^2 E = \underset{(2,0)}{\langle e_1^2\rangle} \oplus \underset{(0,2)}{\langle e_2^2\rangle} \oplus \underset{(1,1)}{\langle e_1e_2\rangle} \xrightarrow{\ \mathcal{F}\ } \langle e_1e_2\rangle \cong \mathbb{C}$$

as rep of Sym_2

$$d=4\quad \mathrm{Sym}^2E \otimes \mathrm{Sym}^2 E = \langle e_1^2, \ldots\rangle \otimes \langle e_1^2 \ldots\rangle$$

$$= \underset{(4,0,0,0)}{\langle e_1^2 \otimes e_1^2\rangle} \oplus \underset{(3,1,0,0)}{\langle e_1^2 \otimes e_1e_2\rangle} \oplus \cdots \oplus \underset{(1,1,1,1)}{\left\langle \begin{matrix} e_1e_2 \otimes e_3e_4 \\ e_1e_3 \otimes e_2e_4 \\ \vdots \\ e_3e_4 \otimes e_1e_2\end{matrix}\right\rangle}$$

$\cong S_4$ acting on $\{(\{i\,j\},\{k\,l\}) : \{i,j,k,l\} = \{1,2,3,4\}\}$

$$\mathrm{Sym}^2(\mathrm{Sym}^2 E) = \langle (e_1^2)(e_1^2)\rangle \oplus \cdots \oplus \langle (e_1e_2)(e_3e_4)\rangle$$

$\cong \mathbb{C}_{S_2} \otimes \mathbb{C}_{S_2} \big\uparrow_{S_2\times S_2}^{S_4}$

$\cong S_4$ acting on $\{\{\{i,j\},\{k,l\}\} : \{i,j,k,l\} = \{1,2,3,4\}\}$

$\cong \mathbb{C}_{S_2 \wr S_2} \big\uparrow_{S_2\wr S_2}^{S_4}$

Generally

$$(\nabla^\mu E)^{\otimes n} \xrightarrow{\ \mathcal{F}\ } S^\mu \otimes \cdots \otimes S^\mu \big\uparrow_{S_m\times\cdots\times S_m}^{S_{mn}}$$

$$\mathrm{Sym}^n \nabla^\mu E \xrightarrow{\ \mathcal{F}\ } \widetilde{S^\mu \otimes \cdots \otimes S^\mu} \big\uparrow_{S_m \wr S_n}^{S_{mn}}$$

$$\nabla^\nu \nabla^\mu E \xrightarrow{\ \mathcal{F}\ } (\widetilde{S^\mu \otimes \cdots \otimes S^\mu}) \otimes \mathrm{Inf}\, S^\nu \big\uparrow_{S_m \wr S_n}^{S_{mn}}$$

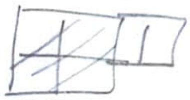
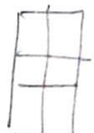
§4. $\mathrm{Sym}^n \mathrm{Sym}^2 E$

Lemma Let $w_t = \sum_{\sigma \in S_t} \prod_{i=1}^{t} e_i e_{\sigma(i)} \,\mathrm{sgn}(\sigma)$. Then w_t is highest weight in $\mathrm{Sym}^t \mathrm{Sym}^2 E$.

Ex
- $w_1 = e_1^2$
- $w_2 = (e_1^2)(e_2^2) - (e_1 e_2)(e_2 e_1)$ = w from earlier

Ex $\mathrm{Sym}^3 \mathrm{Sym}^2 E \cong \nabla^{(6)} E \oplus \nabla^{(4,2)} E \oplus \nabla^{(2,2,2)} E$

w_1^3 $\quad w_2 w_1$ $\quad w_3$

Exercise $\mathrm{Sym}^n \mathrm{Sym}^2 E \cong \bigoplus_{\lambda \in \mathrm{Par}(n)} \nabla^{2\lambda} E$ and contains $\mathrm{Sym}^2 \mathrm{Sym}^n E$.

§3 Applications (gn): minimal constituents by Specht terms (w/ Rowena Paget)
decomposition numbers (w/ Eugenio Giannelli)