Modular plethysms of symmetric functions

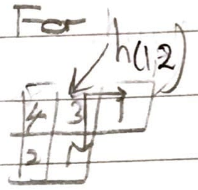
12:30 – 13:20

"Brief thanks
Cat in H"

§1 Three categorifications

Let SYT(λ) be set of standard tableaux shape $\lambda \in \mathrm{Par}(n)$. For $(i,j) \in [\lambda]$ let $h_{(i,j)}$ be hook length at (i,j):

E.g. $\mathrm{SYT}(3,2) = \left\{ \begin{matrix} 1\,2\,3 \\ 4\,5 \end{matrix}, \ldots, \begin{matrix} 1\,3\,5 \\ 2\,4 \end{matrix} \right\}$, $\dfrac{5!}{4\,3\,2\,1\,1} = 5$



① $$|\mathrm{SYT}(\lambda)| = \frac{n!}{\prod_{(i,j)\in[\lambda]} h_{(i,j)}}$$
$$\begin{tikzcd} {|\mathrm{SYT}(\lambda)|} \arrow[d, equal] & {\dfrac{n!}{\prod_{(i,j)\in[\lambda]} h_{(i,j)}}} \arrow[d, equal] \\ \dim S^\lambda \arrow[r, Rightarrow] & \mathrm{tr}_{S^\lambda}(1_{S_n}) \end{tikzcd}$$

Frobenius character formula
"matter of taste which easier"

② $$n! = \sum_{\lambda \in \mathrm{Par}(n)} |\mathrm{SYT}(\lambda)|^2$$
$$\mathbb{C}S_n \cong \bigoplus_{\lambda\in\mathrm{Par}(n)} \mathrm{End}_{\mathbb{C}} S^\lambda$$

"Spread
$\mathbb{F}S_n$ is cellular"

②′ if really have time
$n^d = \sum_{\lambda \in \mathrm{Par}(n)} |\mathrm{SSYT}_{d}(\lambda)|\,|\mathrm{SYT}(\lambda)|$

$V^{\otimes d} = \bigoplus_{\lambda\in\mathrm{Par}(n)} \nabla^\lambda V \otimes S^\lambda$

③ $$\left(\!\!\binom{d}{r}\!\!\right) = \binom{d+r-1}{r}$$

\# r-multisets of $\{1 \ldots d\}$ — \# r-subsets of $\{1\ldots d\}$

e.g. $\left(\!\!\binom{3}{2}\!\!\right) = 6 = \binom{4}{2}$
$r=2$ $d=1$

$$\mathrm{Sym}^r \langle v_1 \ldots v_d \rangle \cong_{\mathbb{C}} \Lambda^r \langle v_1 \ldots v_{d+r-1} \rangle$$

$$\mathrm{Sym}^r \mathrm{Sym}^{d-1} E \cong_{SL_2(\mathbb{C})} \Lambda^r \mathrm{Sym}^{d+r-2} E \quad \text{where } E = \langle e_1, e_2 \rangle \circlearrowleft SL_2(\mathbb{C})$$

"False in prime char"   char F = 2.

The Wronskian iso in ③ is false in prime characteristic. E.g. in char 2

$$\mathrm{Sym}^2 E \cong \begin{matrix} L_{(2)} E \\ \det \end{matrix}$$

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \xrightarrow{\rho} \begin{array}{c} \begin{matrix} e_1^2 & e_2^2 & e_1 e_2 \end{matrix} \\ \begin{pmatrix} \alpha^2 & \beta^2 & 2\alpha\beta \\ \gamma^2 & \delta^2 & 2\gamma\delta \\ \alpha\gamma & \beta\delta & \alpha\delta+\beta\gamma \end{pmatrix} \end{array}$$

$$\langle e_1 e_2 \rangle \cong \det$$

$$\Big\downarrow \mathrm{Sym}'$$

$$\mathrm{Sym}_2 E \cong \begin{matrix} \det \\ L_{(2)} E \end{matrix}$$

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \xrightarrow{\rho'} \begin{array}{c} \begin{matrix} e_1\otimes e_1 & e_2 \otimes e_2 & e_1\otimes e_2 + e_2 \otimes e_1 \end{matrix} \\ \begin{pmatrix} \alpha^2 & \beta^2 & \alpha\beta \\ \gamma^2 & \delta^2 & \gamma\delta \\ 2\alpha\beta & 2\gamma\delta & \alpha\delta+\beta\gamma \end{pmatrix} \end{array}$$

$$\cong (\mathrm{Sym}^2 E)^{\circ}$$

$$\langle e_1\otimes e_1, e_2 \otimes e_2 \rangle \cong L_{(2)} E$$

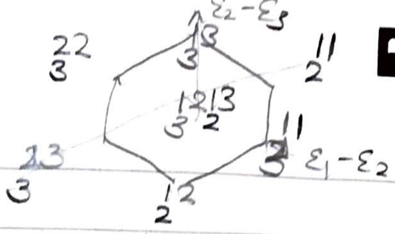
$$\cong_{SL_2(\mathbb{C})} (\mathrm{Sym}^2 E)^{*}$$

$$\rho' \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \left(\rho \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{t} \right)^{t} = \rho^{\circ} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

$$\cong \Lambda^2 \mathrm{Sym}^2 E$$

Theorem [McDowell–W 19] $\mathrm{Sym}_r \mathrm{Sym}^{d-1} E \cong_{SL_2(F)} \Lambda^r \mathrm{Sym}^{r+d-2} E$
where E is nat rep of $SL_2(F)$

"not Harry but Chris"



§2 Schur functors and Schur functions

$V \otimes V \cong \mathrm{Sym}^2 V \oplus \Lambda^2 V$

$V \otimes V \otimes V \cong \mathrm{Sym}^3 V \oplus \Lambda^3 V \overset{?}{\oplus} \nabla^{(2,1)} V \oplus \nabla^{(2,1)} V$

$3^3 \quad \left(\binom{3}{3}\right) = \binom{5}{3} = 10 \quad 1$

where $\nabla^{(2,1)} V = \langle F(t) : t \in SSYT(\lambda) \rangle$

- $SSYT(2,1) = \left\{ \begin{smallmatrix}1&1\\2\end{smallmatrix}, \begin{smallmatrix}1&2\\2\end{smallmatrix}, \begin{smallmatrix}1&2\\3\end{smallmatrix}, \begin{smallmatrix}1&3\\2\end{smallmatrix}, \dots, \begin{smallmatrix}2&2\\3\end{smallmatrix} \right\}$
- $F\left(\begin{smallmatrix}1&2\\3\end{smallmatrix}\right) = V_1V_2 \otimes V_3 - V_3V_1 \otimes V_2 \in \mathrm{Sym}^2 V \otimes V$

Generally $\nabla^\lambda E = \langle F(t) : t \in SSYT(\lambda) \rangle \circlearrowleft GL(V)$ act diagonally with $\begin{pmatrix}\alpha_1 & \\ & \alpha_d\end{pmatrix} F(t) = \prod_i \alpha_i^{\# i \text{ in } t} F(t) = \alpha^t F(t)$

<u>Defn</u> s_λ is formal character of $\nabla^\lambda V$.

For example $s_{(2)}(x_1, x_2) = x^{\boxed{1\,1}} + x^{\boxed{1\,2}} + x^{\boxed{2\,2}} = x_1^2 + x_1x_2 + x_2^2$

$s_\lambda(x_1 \dots x_d) = \sum_{t \in SSYT(\lambda)} x^t$

$\begin{pmatrix}\alpha & \\ & \delta\end{pmatrix} \xrightarrow{\mathrm{Sym}^2} \begin{pmatrix}\alpha^2 & & \\ & \delta^2 & \\ & & \alpha\delta\end{pmatrix}$ and $\alpha^2 + \alpha\delta + \delta^2 \quad s_{(2)}(\alpha, \delta)$

$\big\downarrow \mathrm{Sym}^2$

$\begin{pmatrix}(\alpha^2)^2 & & \\ & (\alpha^2)\alpha\delta & \\ & & (\delta^2)^2\end{pmatrix}$

Littlewood '36 1950

§3 πλnθυσμοσ (Plethysms)

Let $E = \langle e_1, e_2 \rangle_{\mathbb{C}}$

Example $\mathrm{Sym}^2(\mathrm{Sym}^2 E) \xrightarrow{\quad} \mathrm{Sym}^4 E$

$(uv)(wx) \mapsto uvwx$

LHS $= \langle (e_1^2)^2, (e_1^2)(e_1e_2), (e_1^2)(e_2^2)$
$(e_2^2)^2, (e_1e_2)(e_2^2), (e_1e_2)^2 \rangle \supseteq \ker\theta = \langle (e_1^2)(e_2^2) - (e_1e_2)^2 \rangle$
$= \left\langle F\left(\begin{smallmatrix}1&1\\2&2\end{smallmatrix}\right)\right\rangle \cong \nabla^{(2,2)} E$

Chris next | Hence $\mathrm{Sym}^2(\mathrm{Sym}^2 E) \cong \mathrm{Sym}^4 E \oplus \nabla^{(2,2)} E$

$s_{(2)} \circ s_{(2)} \qquad s_{(4)} + s_{(2,2)}$

where $(s_{(2)} \circ s_{(2)})(\alpha, \delta) = s_{(2)}(\alpha^2, \alpha\delta\, \delta^2) = (\alpha^2)^2 + (\alpha^2)(\alpha\delta) + \cdots$

<u>Generally</u> $s_\lambda \circ s_\ell(x, y) = s_\lambda(x^\ell, x^{\ell-1}y, \dots, y^\ell)$

<u>Theorem</u> Let $\lambda, \mu \in \mathrm{Par}(n)$. Let $\ell, m \in \mathbb{N}$. TFAE

(i) $\nabla^\lambda \mathrm{Sym}^\ell E \cong_{SL_2(\mathbb{C})} \nabla^\mu \mathrm{Sym}^m E$

(ii) $(s_\lambda \circ s_\ell)(q, q^{-1}) = (s_\mu \circ s_{(m)})(q, q^{-1})$

(iii) $s_\lambda(q^\ell, q^{\ell-2}, \dots, q^{-\ell}) = s_\mu(q^m, q^{m-2} \dots, q^{-m})$

(iv) $C(\lambda) + \ell + 1 / H(\lambda) = C(\mu) + m + 1 / H(\mu)$

where $H(\lambda) = \{ h_{ij} : (i,j) \in [\lambda] \}$

$C(\lambda) = \{ j - i : (i,j) \in [\lambda] \}$ content

(iii) ⟺ (iv) is Stanley's hook content formula

Cor [Woronshia] Take $\lambda = (r)$, $\mu = (1^r)$ $m = r+l-1$

$c(r) + l + 1 = \boxed{0 \mid 1 \mid \cdots \mid r-1} + l + 1 = \boxed{l+1 \mid +l \mid l+r}$

$c(1^r) + m + 1 = \boxed{\begin{array}{c} 0 \\ -1 \\ \vdots \\ -r+1 \end{array}} + r + l = \boxed{\begin{array}{c} r+l \\ r+l-1 \\ \vdots \\ l+1 \end{array}}$ wrong.

$H(r) = H(1^r) = [1, 2 \cdot, r]$. So by (iv) $\Rightarrow$ (c), $\mathrm{Sym}^r \mathrm{Sym}^l E \cong \Lambda^{r+l-1} \mathrm{Sym}^l E$

Theorem [King 84, Paget-W 20] $\nabla^\lambda \mathrm{Sym}^l E \cong \nabla^{\lambda'} \mathrm{Sym}^m E$

$\Longleftrightarrow$ conditions on λ, $l - m = \ell(\lambda) - \ell(\lambda')$.

§4 Complements

Stacey rectangular comp

Example For row length 5. Let $\lambda = (4,3,3,1)$, $\lambda^{\circ 4} = (4,2,2,1)$, $l = m = 4$

(Stacey had (4,2,2))

(4,3,3,1) →

$c(\lambda)+4$

4_0	5_1	6_2	7_3	1_0
3_0	4_1	5_2	1_0	3_1
2_0	3_1	4_2	2_0	4
1_0	1_0	2_1	5_2	7_3

← (4,2,2,1)

$2_0 2_0 2_1$

$\nLeftrightarrow$ $H(\lambda^{\circ 4})$ $H(\lambda)$ $\nLeftrightarrow$

7_3	5_2	4_1	1_0	1_0
5_2	3_1	2_0	3_1	2_0
4_2	2_1	1_0	4_1	3_0
1_0	7_3	6_2	5	4_0

$c(\lambda^\circ)+4$

$5_1 5_2 5_2$ with arms

$c(\lambda)+4 \cup H(\lambda^{\circ 4}) = \{1^4 2^3 3^2 4^4 5^3 6\, 7^2\} = c(\lambda^\circ)+4 \cup H(\lambda)$

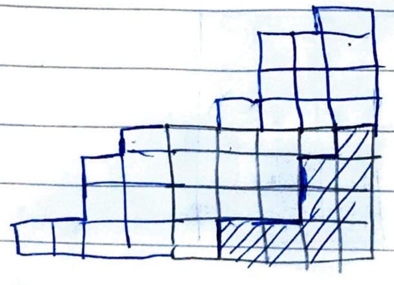
By (iv) $\Leftrightarrow$ (c) $\nabla^\lambda \mathrm{Sym}^3 E \cong \nabla^{\lambda \circ 4} \mathrm{Sym}^3 E$

Theorem [King 84, Paget-W 19, McDowell-W 20] $\nabla^\lambda \mathrm{Sym}^l E \cong \nabla^{\lambda \circ r} \mathrm{Sym}^l E$

$\Longleftrightarrow r = l + 1$.

Theorem [Bessenrodt 79, 20] Hook/content bijection holds with arm lengths

Proof Use result in 79 paper on hook lengths in



$\leftrightarrow H(4,2,2,1) \cup H(5,5,5,5)$

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