Modular plethysms for $\mathrm{SL}_2(F)$

Mark Wildon

MFO Miniworkshop 2020

Organizers: Christine Bessenrodt, Chris Bowman, Eugenio Giannelli

Outline

§1 Plethysms for $\mathrm{SL}_2(\mathbb{C})$

§2 A modular Wronskian isomorphism

§3 Modular plethystic isomorphisms for complements

§4 Obstructions to modular plethysms

Section 1 is with Rowena Paget, based on

- *Plethysms of symmetric functions and representations of* $\mathrm{SL}_2(\mathbb{C})$, arXiv:1907.07616, July 2019
  To appear in Journal of Algebraic Combinatorics.

Sections 2, 3 and 4 are with my Ph.D student Eoghan McDowell.

§1 Plethysms for $\mathrm{SL}_2(\mathbb{C})$

Are there nice isomorphisms $\mathrm{S}^2(k^n) \cong \Lambda^2(k^{n+1})$?

Asked 1 year, 1 month ago Active 1 year, 1 month ago Viewed 349 times

12

This might be forced to migrate to math.SE but let me still risk it.

The spaces $\mathrm{S}^2(k^n)$ and $\Lambda^2(k^{n+1})$ from the title have equal dimensions. Is there a *natural* isomorphism between them?

⋮

share cite edit close flag edited Jan 15 '19 at 10:52

asked Jan 15 '19 at 9:45
მამუკა ჯიბლაძე
13.9k ● 3 ● 50 ● 125

19

Let E be a 2-dimensional k-vector space. The Wronksian isomorphism is an isomorphism of $\mathrm{SL}(E)$-modules $\bigwedge^m \mathrm{S}^{m+r-1}(E) \cong \mathrm{S}^m\mathrm{S}^r(E)$. It is easiest to deduce it from the corresponding identity in symmetric functions (specialized to 1 and q), but it can also be defined explicitly: see for example Section 2.5 of this paper of Abdesselam and Chipalkatti.

In particular, identifying $\mathrm{S}^n(E)$ with the homogeneous polynomial functions on E of degree n, their definition becomes the map $\wedge^2\mathrm{S}^n(E) \to \mathrm{S}^2\mathrm{S}^{n-1}(E)$ defined by

$$f \wedge g \mapsto \frac{\partial f}{\partial X}\frac{\partial g}{\partial Y} - \frac{\partial f}{\partial Y}\frac{\partial g}{\partial X}.$$

Now $\mathrm{S}^n(E) \cong k^{n+1}$ and $\mathrm{S}^{n-1}(E) \cong k^n$, so we have the required isomorphism $\mathrm{S}^2 k^n \cong \wedge^2 k^{n+1}$.

share cite edit delete flag edited Jan 15 '19 at 11:49

answered Jan 15 '19 at 11:09
Mark Wildon
8,018 ● 1 ● 32 ● 51

Equivalent conditions for $\mathrm{SL}_2(\mathbb{C})$ plethystic isomorphisms

Recall that E is the natural 2-dimensional representation of $\mathrm{SL}_2(\mathbb{C})$.

Theorem

Let λ and μ be partitions and let $\ell, m \in \mathbb{N}$. The following are eqv:

(i) $\nabla^\lambda \mathrm{Sym}^\ell E \cong_{\mathrm{SL}(E)} \nabla^\mu \mathrm{Sym}^m E$;

(ii) $(s_\lambda \circ s_{(\ell)})(x^{-1}, x) = (s_\mu \circ s_{(m)})(x^{-1}, x)$;

(iii) $s_\lambda(1, q, \ldots, q^\ell) = s_\mu(1, q, \ldots, q^m)$ *up to a (fixed) power of q;*

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(iv) $C(\lambda) + \ell + 1 / H(\lambda) = C(\mu) + m + 1 / H(\mu)$

where $/$ is difference of multisets (negative multiplicities okay) and

- $C(\lambda) = \{j - i : (i, j) \in [\lambda]\}$ *is the multiset of contents of λ;*
- $H(\lambda) = \{h_{(i,j)} : (i, j) \in [\lambda]\}$ *is the multiset of hook lengths of λ.*

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Theorem (Stanley's Hook Content Formula, 1971)

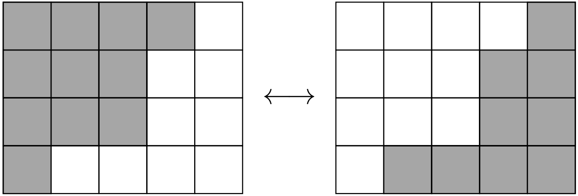
Let $[m]_q = 1 + q + \cdots + q^{m-1}$. For a suitable $B \in \mathbb{N}_0$,

$$s_\lambda(1, q, \ldots, q^\ell) = q^B \frac{\prod_{(i,j)\in[\lambda]} [j - i + \ell + 1]_q}{\prod_{(i,j)\in[\lambda]} [h_{(i,j)}]_q}$$

Plethystic complement isomorphism for $\mathrm{SL}_2(\mathbb{C})$

Let λ be a partition contained in a box with d rows and a columns. Let $\lambda^\bullet$ be its complement. For example if $a = 5$, $d = 4$ then

$$(4,3,3,1)^\bullet = (4,2,2,1).$$



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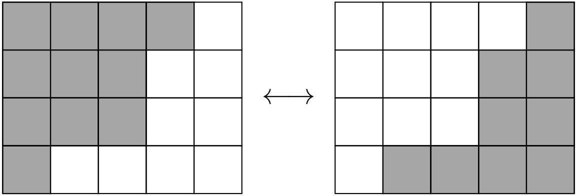

Theorem (King 1985)

Let λ have at most $\ell + 1$ parts. Let $\lambda^\bullet$ be the complement of λ in a box with $\ell + 1$ rows. Then $\nabla^\lambda \mathrm{Sym}^\ell E \cong \nabla^{\lambda^\bullet} \mathrm{Sym}^\ell E$.

In Paget–W 2019 we showed that King's Theorem gives all plethystic isomorphisms relating $\nabla^\lambda \mathrm{Sym}^\ell E$ and $\nabla^\mu \mathrm{Sym}^m E$, when λ and μ are complements in *any* box.

Stanley's HCF for the complement isomorphism

For example, using a rectangle with 4 rows and 5 columns,

$$\nabla^{(4,3,3,1)}\mathrm{Sym}^3 E \cong \nabla^{(4,2,2,1)}\mathrm{Sym}^3 E.$$

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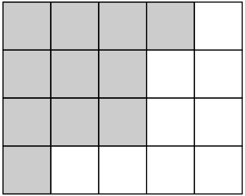

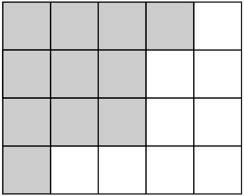

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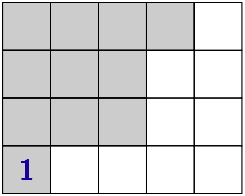
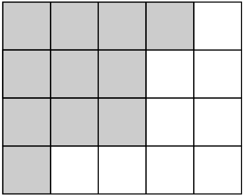
$C(\lambda) + 4$





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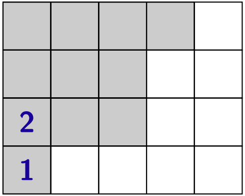

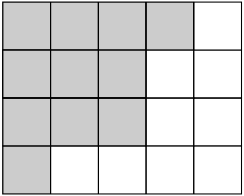

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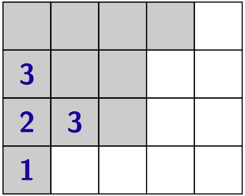

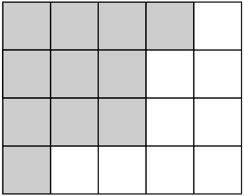

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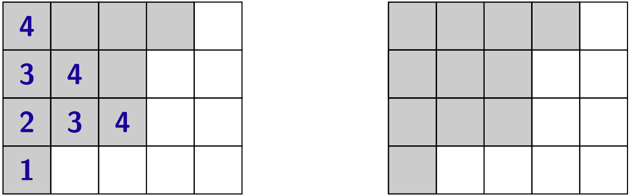

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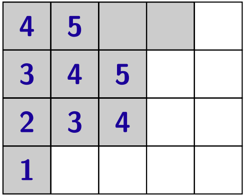

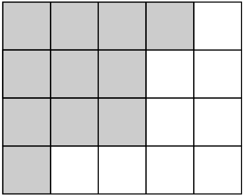

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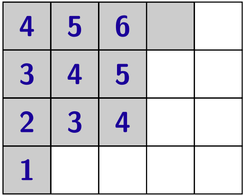

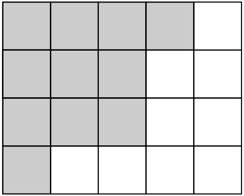

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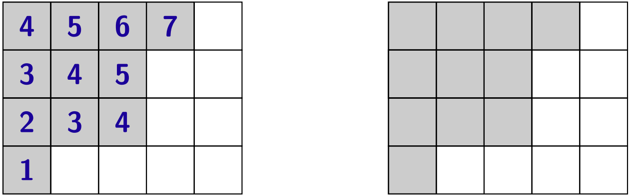

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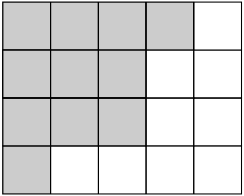
$C(\lambda) + 4$

4	**5**	**6**	**7**	
3	**4**	**5**		
2	**3**	**4**		
1				

$H(\lambda)$



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			1	
		1		
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3	**4**	**5**		
2	**3**	**4**		
1				

$H(\lambda)$

			1	
		2		
	2	**1**		
1				

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$C(\lambda) + 4$

4	5	6	7	
3	4	5		
2	3	4		
1				

$H(\lambda)$

		4	1	
	3	2		
4	2	1		
1				

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4	**5**	**6**	**7**	
3	**4**	**5**		
2	**3**	**4**		
1				

$H(\lambda)$

	5	**4**	**1**	
5	**3**	**2**		
4	**2**	**1**		
1				

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$C(\lambda) + 4$

4	5	6	7	
3	4	5		
2	3	4		
1				

$H(\lambda)$

7	5	4	1	
5	3	2		
4	2	1		
1				

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$C(\lambda) + 4$

4	**5**	**6**	**7**	**1**
3	**4**	**5**	**1**	**3**
2	**3**	**4**	**2**	**4**
1	**1**	**2**	**5**	**7**

$H(\lambda^\bullet)$

$H(\lambda)$

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5	**3**	**2**		
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3	4	5	1	3
2	3	4	2	4
1	1	2	5	7

$H(\lambda^\bullet)$

$H(\lambda)$

7	5	4	1	1
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1	7	6	5	4

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1	1	2	5	7

$H(\lambda^\bullet)$

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7	5	4	1	1
5	3	2	3	2
4	2	1	4	3
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$C(\lambda^\bullet) + 4$

Either way: $\{1^4, 2^3, 3^3, 4^4, 5^3, 6, 7^2\}$

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$C(\lambda) + 4$

4_0	5_1	6_2	7_3	1_0
3_0	4_1	5_2	1_0	3_1
2_0	3_1	4_2	2_0	4_1
1_0	1_0	2_1	5_2	7_3

$H(\lambda^\bullet)$

$H(\lambda)$

7_3	5_2	4_1	1_0	1_0
5_2	3_1	2_0	3_1	2_0
4_2	2_1	1_0	4_1	3_0
1_0	7_3	6_2	5_1	4_0

$C(\lambda^\bullet) + 4$

Either way: $\{1^4, 2^3, 3^3, 4^4, 5^3, 6, 7^2\}$

Using a theorem of Bessenrodt: stronger version with arm lengths

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$C(\lambda) + 4$

4_0	5_1	6_2	7_3	1_0
3_0	4_1	5_2	1_0	3_1
2_0	3_1	4_2	2_0	4_1
1_0	1_0	2_1	5_2	7_3

$H(\lambda^\bullet)$

$H(\lambda)$

7_3	5_2	4_1	1_0	1_0
5_2	3_1	2_0	3_1	2_0
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Either way: $\{1^4, 2^3, 3^3, 4^4, 5^3, 6, 7^2\}$

Using a theorem of Bessenrodt: stronger version with arm lengths

Problem

Interpret this using Jack symmetric functions and prove a stronger symmetric functions identity

Plane partitions

For example

$$\begin{matrix} 5 & 3 & 3 & 1 \\ 5 & 2 & 1 & \\ 1 & 1 & & \end{matrix} \quad \in \mathcal{PP}(4,3,5)$$

is a plane partition of 22 with 4 columns, 3 rows, and entries ≤ 5.

Plane partitions

For example

$$\begin{matrix} 5 & 3 & 3 & 1 \\ 5 & 2 & 1 & \\ 1 & 1 & & \end{matrix} \quad \in \mathcal{PP}(4,3,5)$$

is a plane partition of 22 with 4 columns, 3 rows, and entries ≤ 5. By rotating a half-turn and adding $i-1$ to all entries in row i get a bijection between $\mathcal{PP}(a,b,c)$ and $\mathrm{SSYT}_{\{0,1,\ldots,b+c-1\}}(a^b)$. Hence

$$q^{-a\binom{b}{2}} s_{(a^b)}(1,q,\ldots,q^{b+c-1}) = \sum_{\pi \in \mathcal{PP}(a,b,c)} q^{|\pi|}.$$

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Theorem (MacMahon 1896)

$$\sum_{\pi \in \mathcal{PP}(a,b,c)} q^{|\pi|} = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{q^{i+j+k-1}-1}{q^{i+j+k-2}-1}.$$

Since right-hand side is invariant under permuting a, b, c get

$$\nabla^{(a^b)} \mathrm{Sym}^{b+c-1} E \cong_{\mathrm{SL}(E)} \nabla^{(b^a)} \mathrm{Sym}^{a+c-1} E \cong_{\mathrm{SL}(E)} \nabla^{(b^c)} \mathrm{Sym}^{c+a-1} E \cong_{\mathrm{SL}(E)} \cdots.$$

§2: A modular Wronskian isomorphism

Let F be an infinite field and let V be a polynomial representation of $\mathrm{SL}_2(F)$. Recall from Schur–Weyl duality that S_r acts on $V^{\otimes r}$ by permuting tensor factors.

- $\mathrm{Sym}_r V = (V^{\otimes r})^{S_r}$
- $\mathrm{Sym}^r V = V^{\otimes r} / \langle (v^{(1)} \otimes \cdots \otimes v^{(r)}) \cdot \sigma - v^{(1)} \otimes \cdots \otimes v^{(r)} : v^{(i)} \in V, \sigma \in S_r \rangle$

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For example, if E is the natural representation of $\mathrm{SL}_2(F)$ then $\mathrm{Sym}^2 E = \langle e_1^2, e_2^2, e_1 e_2 \rangle$ and

$$\begin{array}{cc} e_1 & e_2 \end{array} \qquad\qquad \begin{array}{ccc} e_1^2 & e_2^2 & e_1 e_2 \end{array}$$

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \longmapsto \begin{pmatrix} \alpha^2 & \beta^2 & \alpha\beta \\ \gamma^2 & \delta^2 & \gamma\delta \\ 2\alpha\gamma & 2\beta\delta & \alpha\delta + \beta\gamma \end{pmatrix}$$

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Hence for $\bigwedge^2 \mathrm{Sym}^2 E$,

$$\begin{array}{cc} e_1 & e_2 \end{array} \qquad\qquad \begin{array}{ccc} e_1^2 \wedge e_1 e_2 & e_2^2 \wedge e_1 e_2 & e_1^2 \wedge e_2^2 \end{array}$$
$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \longmapsto \begin{pmatrix} \alpha^3\delta - \alpha^2\beta\gamma & \alpha\beta^2\delta - \alpha\beta^2\gamma & 2\alpha^2\beta\delta - 2\alpha\gamma\beta^2 \\ \alpha\gamma^2\delta - \alpha\gamma^2\delta & \alpha\delta^3 - \beta\gamma\delta^2 & 2\beta\gamma^2\delta - 2\alpha\gamma\delta^2 \\ \alpha^2\gamma\delta - \gamma^2\alpha\beta & \beta^2\gamma\delta - \alpha\beta\delta^2 & \alpha^2\delta^2 - \beta^2\gamma^2 \end{pmatrix}$$

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Hence for $\bigwedge^2 \mathrm{Sym}^2 E$, writing $\Delta = \alpha\delta - \beta\gamma$,

$$\begin{array}{cc} e_1 & e_2 \end{array} \qquad \begin{array}{ccc} e_1^2 \wedge e_1 e_2 & e_2^2 \wedge e_1 e_2 & e_1^2 \wedge e_2^2 \end{array}$$
$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \longmapsto \begin{pmatrix} \alpha^2\Delta & -\beta^2\Delta & 2\alpha\beta\Delta \\ -\gamma^2\Delta & \delta^2\Delta & -2\gamma\delta\Delta \\ \alpha\gamma\Delta & -\beta\delta\Delta & (\alpha\delta + \beta\gamma)\Delta \end{pmatrix}$$

Duality and the modular Wronskian isomorphism

- $\nabla^{(r)}E = \mathrm{Sym}^r E$: costandard module
- $\Delta^{(r)}E = \mathrm{Sym}_r E$: standard module, Weyl module, Carter–Lusztig module.
- $(\mathrm{Sym}^r E)^\circ \cong \mathrm{Sym}_r E$ where $\circ$ denotes contravariant duality defined for a representation ρ by $g \mapsto \rho(g^{\mathrm{tr}})^{\mathrm{tr}}$;
- More generally, $(\nabla^\lambda V)^\circ \cong \Delta^\lambda(V^\circ)$
- For $\mathrm{SL}_2(F)$-representations, contravariant duality agrees with normal duality: $E \cong E^\star$ and $(\mathrm{Sym}^r E)^\star \cong \mathrm{Sym}_r E$.

Theorem (McDowell–W 2020)

Let F be an infinite field. There is an isomorphism

$$\mathrm{Sym}_r \mathrm{Sym}^\ell E \cong_{\mathrm{SL}_2(F)} \bigwedge^r \mathrm{Sym}^{r+\ell-1} E.$$

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Outline of proof: guess the right map. It is highly non-obvious that it is $\mathrm{SL}_2(F)$-equivariant.

When is there a subring of the complex numbers surjecting onto a given field of prime characteristic?

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To make use of the Lie algebra action of $\mathfrak{gl}_2(\mathbb{C})$ to establish a isomorphism in modular representation theory, I would like an answer to this question:

> Let K be a field of prime characteristic. When is there a subring R of the complex numbers and a maximal ideal M of R such that $R/M \cong K$?

Clearly no such ring R exists if K has strictly more than $|\mathbb{C}|$ independent transcendental elements. Is this the only obstruction? Is there a reasonably explicit way to construct a suitable R when K is the algebraic closure of $\mathbb{F}_p$?

As a follow-up (which at first I thought I needed, but I now see I can get around by working with $\mathrm{GL}_2(\mathbb{C})$ rather than $\mathrm{SL}_2(\mathbb{C})$), note that if $R/M \cong K$ then the induced map $\mathrm{GL}_d(R) \rightarrow \mathrm{GL}_d(K)$, defined on a $d \times d$ matrix with entries in R by applying $R \twoheadrightarrow R/M \cong K$ to each entry, is a surjective group homomorphism.

> Is is true in general that the restriction of the group homomorphism $\mathrm{GL}_d(R) \rightarrow \mathrm{GL}_d(K)$ to $\mathrm{SL}_d(R)$ is surjective onto $\mathrm{SL}_d(K)$, or are there further obstructions?

gr.group-theory rt.representation-theory algebraic-number-theory modular-representation-theory

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asked yesterday
Mark Wildon
8,018 ● 1 ● 32 ● 51

Using $\mathrm{SL}_2(R) \twoheadrightarrow \mathrm{SL}_2(F)$ and $\mathrm{SL}_2(R) \subseteq \mathrm{SL}_2(\mathbb{C})$, this technical trick lets us prove the isomorphism using the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$.

§3: Modular plethystic isomorphism for complements

Theorem (McDowell–W 2020)

Let F be an infinite field and let λ be a partition with at most $\ell + 1$ parts. Let $\lambda^\bullet$ be the complement of λ in a box with $\ell + 1$ rows. Then

$$\nabla^\lambda \mathrm{Sym}^\ell E \cong \nabla^{\lambda^\bullet} \mathrm{Sym}_\ell E.$$

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Some ideas in the proof

- If V is a polynomial representation of $\mathrm{SL}_2(F)$ of dimension d then $\bigwedge^r V \cong \bigwedge^{d-r} V^\star$
  - This is almost general nonsense:
  $$\bigwedge^r V \times \bigwedge^{d-r} V \longrightarrow \bigwedge^d V = \det = F;$$
  - Hence $\bigwedge^r V \otimes \bigwedge^{d-r} V$ has a trivial top composition factor;
  - Use $U^\star \otimes W \cong \mathrm{Hom}_F(U, W)$ to get a candidate homomorphism;
  - Some more work is needed to see it's an isomorphism.
- Use the explicit construction of $\nabla^\lambda V$ in deBoeck–Paget–W *Plethysms of symmetric functions and highest weight representations*, arXiv:1810.03448, Oct 2018 and generalized Garnir relations.

§4: Obstruction to modular plethysms

Theorem (King 1985)

Let E be the natural representation of $\mathrm{SL}_2(\mathbb{C})$. For a large class of partitions λ, there is an isomorphism

$$\nabla^\lambda \mathrm{Sym}^\ell E \cong_{\mathrm{SL}(E)} \nabla^{\lambda'} \mathrm{Sym}^{\ell+\ell(\lambda')-\ell(\lambda)} E.$$

- In particular, King's result holds when λ is a hook; that is $\lambda = (a+1, 1^b)$ for some $a, b \in \mathbb{N}_0$.
- In Paget–W 2019 we showed that King's Theorem gives all plethystic isomorphisms relating $\nabla^\lambda \mathrm{Sym}^\ell E$ and $\nabla^{\lambda'} \mathrm{Sym}^m E$.
- King's result was (independently) reproved by Cagliero and Penazzi 2016.
- The special case of King's Theorem when λ is a rectangle is an instance of a theorem of Manivel 2007.

Obstruction to a modular generalization

Let F be an infinite field of prime characteristic p and let E be the natural representation of $\mathrm{SL}_2(F)$.

Theorem (McDowell–W 2020)

There exist infinitely many pairs (a, b) such that, provided e is sufficiently large, the eight representations of $\mathrm{SL}_2(F)$ obtained from $\nabla^{(a+1,1^b)}\mathrm{Sym}^{p^e+b}E$ by

- *Replacing ∇ with Δ (duality)*
- *Replacing $(a+1, 1^b)$ with $(b+1, 1^a)$ and $p^e + b$ with $p^e + a$ (King conjugation);*
- *Replacing $\mathrm{Sym}^\ell E$ with $\mathrm{Sym}_\ell E$ (another duality);*

are all non-isomorphic.

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Problem

What plethystic isomorphisms of representations of $\mathrm{SL}_2(\mathbb{C})$ have modular analogues?