# Stability of plethysms of symmetric functions

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# Outline

# §1 Motivation: the Wronskian isomorphism

Let $V$ be a vector space.

- $$\begin{aligned} \mathrm{Sym}^2 V &= V^{\otimes 2}/\langle v \otimes w - w \otimes v : v, w \in V \rangle \\ &= \langle vw : v \in V, w \in V \rangle \end{aligned}$$
- $$\begin{aligned} \bigwedge^2 V &= V^{\otimes 2}/\langle v \otimes v : v \in V \rangle \\ &= \langle v \wedge w : v \in V, w \in V \rangle \end{aligned}$$

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Observation. $\mathrm{Sym}^2\,\mathbb{C}^d$ and $\bigwedge^2\mathbb{C}^{d+1}$ both have dimension $\binom{d+1}{2}$.

- Proof. If $v_1, \ldots, v_d$ is a basis for $\mathbb{C}^d$ then $\mathrm{Sym}^2\mathbb{C}^d$ has basis $v_1^2, \ldots, v_d^2, v_1v_2, \ldots, v_{d-1}v_d$, of size $d + \binom{d}{2}$.

Question. Asked by მამუკა ჯიბლაძე on MathOverflow: Is there a natural isomorphism between these vector spaces?

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Answer. Yes!

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Question. Asked by მამუკა ჯიბლაძე on MathOverflow: Is there a natural isomorphism between these vector spaces?

Answer. Yes! Let $E$ be the 2-dimensional natural representation of $\mathrm{SL}_2(\mathbb{C})$. Then $\mathrm{Sym}^{d-1}E$ is $d$-dimensional and

$$\mathrm{Sym}^2\mathrm{Sym}^{d-1}E \cong_{\mathrm{SL}_2(\mathbb{C})} \bigwedge^2 \mathrm{Sym}^d E.$$

# §1 Motivation: the Wronskian isomorphism

## Are there nice isomorphisms $\mathrm{S}^2(k^n) \cong \Lambda^2(k^{n+1})$?

Asked 1 year, 1 month ago Active 1 year, 1 month ago Viewed 349 times

12

This might be forced to migrate to math.SE but let me still risk it.

The spaces $\mathrm{S}^2(k^n)$ and $\Lambda^2(k^{n+1})$ from the title have equal dimensions. Is there a *natural* isomorphism between them?

⋮

share cite edit close flag edited Jan 15 '19 at 10:52

asked Jan 15 '19 at 9:45
მამუკა ჯიბლაძე
13.9k ● 3 ● 50 ● 125

19

Let $E$ be a $2$-dimensional $k$-vector space. The Wronksian isomorphism is an isomorphism of $\mathrm{SL}(E)$-modules $\bigwedge^m \mathrm{S}^{m+r-1}(E) \cong \mathrm{S}^m\mathrm{S}^r(E)$. It is easiest to deduce it from the corresponding identity in symmetric functions (specialized to $1$ and $q$), but it can also be defined explicitly: see for example Section 2.5 of this paper of Abdesselam and Chipalkatti.

In particular, identifying $\mathrm{S}^n(E)$ with the homogeneous polynomial functions on $E$ of degree $n$, their definition becomes the map $\wedge^2\mathrm{S}^n(E) \to \mathrm{S}^2\mathrm{S}^{n-1}(E)$ defined by

$$f \wedge g \mapsto \frac{\partial f}{\partial X}\frac{\partial g}{\partial Y} - \frac{\partial f}{\partial Y}\frac{\partial g}{\partial X}.$$

Now $\mathrm{S}^n(E) \cong k^{n+1}$ and $\mathrm{S}^{n-1}(E) \cong k^n$, so we have the required isomorphism $\mathrm{S}^2 k^n \cong \wedge^2 k^{n+1}$.

share cite edit delete flag edited Jan 15 '19 at 11:49

answered Jan 15 '19 at 11:09
Mark Wildon
8,018 ● 1 ● 32 ● 51

# Action of $\mathrm{SL}_2(F)$ on $\bigwedge^2 \mathrm{Sym}^2 E$ where $E = \langle v, w \rangle$

$$
\begin{array}{c} \begin{array}{cc} v & w \end{array} \\ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \end{array}
\longmapsto
\begin{array}{c} \begin{array}{ccc} v^2 \wedge vw & w^2 \wedge vw & v^2 \wedge w^2 \end{array} \\
\begin{pmatrix} \alpha^3\delta - \alpha^2\beta\gamma & \alpha\beta^2\delta - \alpha\beta^2\gamma & 2\alpha^2\beta\delta - 2\alpha\beta^2\gamma \\ -\alpha\gamma^2\delta + \beta\gamma^3 & \alpha\delta^3 - \beta\gamma\delta^2 & 2\beta\gamma^2\delta - 2\alpha\gamma\delta^2 \\ \alpha^2\gamma\delta - \alpha\gamma^2\beta & \beta^2\gamma\delta - \alpha\beta\delta^2 & \alpha^2\delta^2 - \beta^2\gamma^2 \end{pmatrix} \end{array}
$$

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= \begin{array}{c} \begin{array}{ccc} v^2 \wedge vw & w^2 \wedge vw & v^2 \wedge w^2 \end{array} \\
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- Even after the sign flip, this is not the matrix for $\mathrm{Sym}^2 E$.

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- Even after the sign flip, this is not the matrix for $\mathrm{Sym}^2 E$. The matrices are not even conjugate if $\mathrm{char}\, F = 2$! Instead it is the matrix for $\mathrm{Sym}_2 E = \langle v \otimes v, w \otimes w, v \otimes w + w \otimes v \rangle$
- Thus $(\mathrm{Sym}^2 E)^\star \cong_{\mathrm{SL}_2(F)} \bigwedge^2 \mathrm{Sym}^2 E$ and the duality is critical.

# Duality and the modular Wronskian isomorphism

## Theorem (McDowell–W 2020)

*Let $F$ be any field. Let $E$ be the 2-dimensional natural representation of $\mathrm{SL}_2(F)$. There is an explicit isomorphism*

$$\mathrm{Sym}_r \mathrm{Sym}^\ell E \cong_{\mathrm{SL}_2(F)} \bigwedge^r \mathrm{Sym}^{r+\ell-1} E.$$

Here $\mathrm{Sym}_n V$ is the invariant subspace of $V^{\otimes n}$ under the permutation action of $S_r$ on tensors and $\mathrm{Sym}^n V$ is the usual quotient of $V^{\otimes n}$.

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As a corollary we obtain a modular version of Hermite reciprocity.

### Corollary (Hermite 1854 over $\mathbb{C}$, McDowell–W 2020)

*Let $F$ be any field. Let $m, \ell \in \mathbb{N}$ and let $E$ be the natural 2-dimensional representation of $\mathrm{GL}_2(F)$. Then*

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*by an explicit map.*

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Question. What other classical $\mathrm{SL}_2(\mathbb{C})$-isomorphisms have modular analogues?

# §2 Decomposition numbers for $S_{2n}$ from $\mathrm{Sym}^n\mathrm{Sym}^2 E$

## Problem (Decomposition numbers)

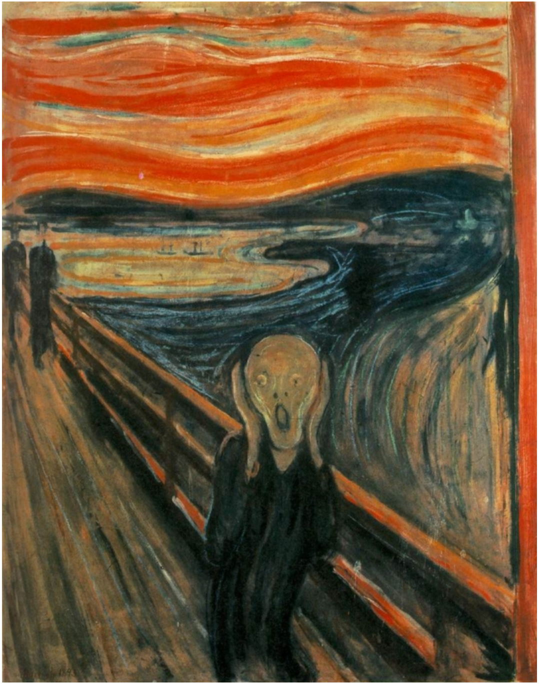
*Determine the composition factors of Specht modules over fields of prime characteristic.*

For instance in characteristic 3 the Specht module $\mathrm{Sp}^{(3,3)}$ has composition factors labelled by $(5,1)$ and $(3,3)$.

| | (6) | (5,1) | (4,2) | (3,3) | (4,1,1) | (3,2,1) | (2,2,1,1) |
|---|---|---|---|---|---|---|---|
| $(6)$ | 1 | | | | | | |
| $(5,1)$ | 1 | 1 | | | | | |
| $(4,2)$ | · | · | 1 | | | | |
| $(3,3)$ | · | 1 | · | 1 | | | |
| $(4,1,1)$ | · | 1 | · | · | 1 | | |
| $(3,2,1)$ | 1 | 1 | · | 1 | 1 | 1 | |
| $(2,2,1,1)$ | · | · | · | · | · | · | 1 |
| $(2,2,2)$ | 1 | · | · | · | · | 1 | · |
| $(3,1,1,1)$ | · | · | · | · | 1 | 1 | · |
| $(2,1,1,1,1)$ | · | · | · | 1 | · | 1 | · |
| $(1,1,1,1,1,1)$ | · | · | · | 1 | · | · | · |

# Decomposition matrix of principal block of $\mathbb{F}_2 S_{10}$

| | (10) | (9,1) | (8,2) | (7,3) | (6,4) | (6,3,1) | (5,3,2) |
|---|---|---|---|---|---|---|---|
| $(10)$ | 1 | | | | | | |
| $(9,1)$ | 1 | 1 | | | | | |
| $(8,2)$ | 1 | 1 | 1 | | | | |
| $(7,3)$ | 1 | · | 1 | 1 | | | |
| $(6,4)$ | · | · | 1 | 1 | 1 | | |
| $(6,3,1)$ | 1 | · | 2 | 1 | 1 | 1 | |
| $(5,3,2)$ | 2 | 1 | 1 | · | 1 | 1 | 1 |
| $(5,5)$ | · | · | 1 | · | 1 | · | · |
| $(8,1,1)$ | 2 | 1 | 1 | · | · | · | · |
| $(6,2,2)$ | 1 | · | 1 | · | · | 1 | · |
| $(4,4,2)$ | 2 | 1 | 1 | · | 1 | · | 1 |
| $(4,3,3)$ | 2 | 1 | · | · | · | · | 1 |
| $(7,1,1,1)$ | 2 | 1 | 1 | 1 | · | · | · |
| $(6,2,1)$ | 2 | 1 | 3 | 1 | 1 | 1 | · |
| $(5,3,1,1)$ | 3 | 1 | 3 | 1 | 2 | 1 | 1 |
| $(4,4,1,1)$ | 2 | 1 | 1 | 1 | 1 | · | 1 |
| $(5,2,2,1)$ | 3 | 1 | 2 | 1 | 1 | 1 | 1 |
| $(6,1,1,1,1)$ | 2 | 1 | 2 | 1 | 1 | · | · |



# $\mathrm{Sym}^n\mathrm{Sym}^2 E$ and even partitions

Let $E = \langle e_1, \ldots, e_d \rangle$ be the $d$-dimensional natural representation of $\mathrm{GL}_n(\mathbb{C})$. For $n \in \mathbb{N}$,

$$\mathrm{Sym}^n\,\mathrm{Sym}^2 E = \sum_{\substack{\lambda\in\mathrm{Par}(n)\\ \ell(\lambda)\leq d}} \nabla^{2\lambda}(E)$$

where $2\lambda$ is the even partition obtained by doubling each part of $\lambda$ and $\nabla^{2\lambda}(E)$ is an irreducible $\mathrm{GL}_n(\mathbb{C})$-representation. Equivalently

$$\mathbb{C}\big\uparrow_{S_2 \wr S_n}^{S_{2n}} = \bigoplus_{\lambda\in\mathrm{Par}(n)} \mathrm{Sp}^{2\lambda}.$$

**Example.** Take $d = 4$. Let $\mathcal{F}(V)$ be the $(1,1,1,1)$-weight space of $V$.

$$\mathrm{Sym}^2 E \otimes \mathrm{Sym}^2 E \xrightarrow{\mathcal{F}} \left\langle \begin{array}{ll} e_1e_2 \otimes e_3e_4 & e_3e_4 \otimes e_1e_2 \\ e_1e_3 \otimes e_2e_4 & e_2e_4 \otimes e_1e_3 \\ e_1e_4 \otimes e_2e_3 & e_2e_3 \otimes e_1e_4 \end{array} \right\rangle \xrightarrow{\cong} \underset{\mathrm{Sp}^{(4)}\oplus\mathrm{Sp}^{(3,1)}\oplus\mathrm{Sp}^{(2,2)}}{\mathbb{C}\big\uparrow_{S_2\times S_2}^{S_4}}$$

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Example. Take $d = 4$. Let $\mathcal{F}(V)$ be the $(1,1,1,1)$-weight space of $V$.

$$\begin{tikzcd}
\mathrm{Sym}^2 E \otimes \mathrm{Sym}^2 E \arrow[r, "\mathcal{F}"] \arrow[d, two heads] & \left\langle \begin{matrix} e_1e_2 \otimes e_3e_4 & e_3e_4 \otimes e_1e_2 \\ e_1e_3 \otimes e_2e_4 & e_2e_4 \otimes e_1e_3 \\ e_1e_4 \otimes e_2e_3 & e_2e_3 \otimes e_1e_4 \end{matrix} \right\rangle \arrow[r, "\cong"] \arrow[d, two heads] & \underset{\mathrm{Sp}^{(4)} \oplus \mathrm{Sp}^{(3,1)} \oplus \mathrm{Sp}^{(2,2)}}{\mathbb{C}\big\uparrow_{S_2 \times S_2}^{S_4}} \arrow[d, two heads] \\
\mathrm{Sym}^2\,\mathrm{Sym}^2 E \arrow[r, "\mathcal{F}"] & \left\langle \begin{matrix} (e_1e_2)(e_3e_4) \\ (e_1e_3)(e_2e_4) \\ (e_1e_4)(e_2e_3) \end{matrix} \right\rangle \arrow[r, "\cong"] & \underset{\mathrm{Sp}^{(4)} \oplus \mathrm{Sp}^{(2,2)}}{\mathbb{C}\big\uparrow_{S_2 \wr S_2}^{S_4}}
\end{tikzcd}$$

# From $\mathrm{Sym}^n\mathrm{Sym}^2 E = \bigoplus \nabla^{2\lambda}(E)$ to decomposition numbers

Given a $p$-core $\gamma$, let $\mathcal{E}(\gamma)$ be the set of even partitions obtained from $\gamma$ by adding the least possible number of disjoint $p$-hooks.

- For example if $p = 3$ then $\mathcal{E}\left(\begin{array}{|c|}\hline \phantom{x} \\ \hline \phantom{x} \\ \hline \end{array}\right) = \Big\{(6,2), (4,4), (4,2,2)\Big\}$

## Theorem (Giannelli–W 2014)

*Let $p$ be an odd prime and let $\gamma$ be a $p$-core. Let $\lambda \in \mathcal{E}(\gamma)$ be greatest in the lexicographic order. The column of the decomposition matrix labelled by $\lambda$ has entries $0$ and $1$. Moreover its non-zero entries are in rows labelled by $E(\gamma)$*

Idea of proof. Study the reduction modulo $p$ of the symmetric group module $\mathbb{C}\big\uparrow_{S_2 \wr S_n}^{S_{2n}}$, corresponding to $\mathrm{Sym}^n\mathrm{Sym}^2 E$.

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### Theorem (Giannelli–W 2014)

*Let $p$ be an odd prime and let $\gamma$ be a $p$-core. Let $\lambda \in \mathcal{E}(\gamma)$ be greatest in the lexicographic order. The column of the decomposition matrix labelled by $\lambda$ has entries $0$ and $1$. Moreover its non-zero entries are in rows labelled by $E(\gamma)$*

Idea of proof. Study the reduction modulo $p$ of the symmetric group module $\mathbb{C}\big\uparrow_{S_2 \wr S_n}^{S_{2n}}$, corresponding to $\mathrm{Sym}^n\mathrm{Sym}^2 E$.

- Main step: show that the only summands of $\mathbb{F}_p\big\uparrow_{S_2 \wr S_n}^{S_{2n}}$ in the block of $S_{2n}$ with $p$-core $\gamma$ are projective.
- From the decomposition of $\mathrm{Sym}^n\,\mathrm{Sym}^2 E$, each projective lifts to a direct sum of Specht modules over $\mathbb{C}$ labelled by even partitions.
- By Brauer reciprocity we get information about columns of decomposition matrix.

# Example: 3-block of $S_{12}$ with core $(3,1,1)$

| | $(12,1^2)$ | $(9,4,1)$ | $(9,3,2)$ | $(8,4,2)$ | $(6^2,2)$ | $(6,4^4)$ | $(6,4,2^2)$ | $(6,3,2^2,1)$ | $(5,4,2^2,1)$ | $(4^2,2^2,1^2)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $(12,1^2)=\langle 2\rangle$ | 1 | | | | | | | | | |
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| $(9,3,2)=\langle 2,1\rangle$ | 2 | 1 | 1 | | | | | | | |
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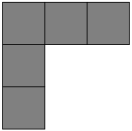
| $(6^2,2)=\langle 1,2\rangle$ | | | | 1 | 1 | | | | | |
| $(6,4^4)=\langle 1,2,2\rangle$ | | | 1 | 1 | 1 | 1 | | | | |
| $(6,4,2^2)=\langle 2,2,2\rangle$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | |
| $(6,3,2^2,1)=\langle 1,1,2\rangle$ | 2 | 1 | 1 | | | | 1 | 1 | | |
| $(5,4,2^2,1)=\langle 1,1\rangle$ | 1 | 1 | 1 | | 1 | 1 | 1 | 1 | 1 | |
| $(4^2,2^2,1^2)=\langle 3\rangle$ | 1 | | 1 | | 1 | 1 | | 1 | 1 | 1 |
| $(9,1^5)=\langle 2,3\rangle$ | | | 1 | | | | | | | |
| $(6,4,1^4)=\langle 2,2,3\rangle$ | | | | | | | 1 | | | |
| $(6,3,2,1^3)=\langle 1,2,3\rangle$ | | | 1 | | | 1 | 1 | 1 | | |
| $(6,2^3,1^2)=\langle 3,2\rangle$ | | | | | | | | 1 | | |
| $(6,1^8)=\langle 2,3,3\rangle$ | | | | | | 1 | | | | |
| $(5,4,2,1^3)=\langle 1,3\rangle$ | | | | | 2 | 1 | 1 | 1 | 1 | |
| $(3^4,1^2)=\langle 3,1\rangle$ | 1 | | 1 | | | 1 | | | | 1 |
| $(3^2,2^4)=\langle 1,1,3\rangle$ | 1 | | | | | | | | | 1 |
| $(3^2,2^2,1^4)=\langle 1,1,1\rangle$ | | | | | 1 | 1 | | | 1 | 1 |
| $(3^2,2,1^6)=\langle 1,3,3\rangle$ | | | | | 2 | 1 | | | 1 | |
| $(3,2^3,1^5)=\langle 3,3\rangle$ | | | | | 1 | | | | 1 | |
| $(3,1^{11})=\langle 3,3,3\rangle$ | | | | | 1 | | | | | |



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| $(6,4^4)=\langle 1,2,2\rangle$ | | | 1 | 1 | 1 | 1 | | | | |
| $(6,4,2^2)=\langle 2,2,2\rangle$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | |
| $(6,3,2^2,1)=\langle 1,1,2\rangle$ | 2 | 1 | 1 | | | | 1 | 1 | | |
| $(5,4,2^2,1)=\langle 1,1\rangle$ | 1 | 1 | 1 | | 1 | 1 | 1 | 1 | 1 | |
| $(4^2,2^2,1^2)=\langle 3\rangle$ | 1 | | 1 | | 1 | 1 | | 1 | 1 | 1 |
| $(9,1^5)=\langle 2,3\rangle$ | | | 1 | | | | | | | |
| $(6,4,1^4)=\langle 2,2,3\rangle$ | | | | | | | 1 | | | |
| $(6,3,2,1^3)=\langle 1,2,3\rangle$ | | | 1 | | | 1 | 1 | 1 | | |
| $(6,2^3,1^2)=\langle 3,2\rangle$ | | | | | | | | 1 | | |
| $(6,1^8)=\langle 2,3,3\rangle$ | | | | | | 1 | | | | |
| $(5,4,2,1^3)=\langle 1,3\rangle$ | | | | | 2 | 1 | 1 | 1 | 1 | |
| $(3^4,1^2)=\langle 3,1\rangle$ | 1 | | 1 | | | 1 | | | | 1 |
| $(3^2,2^4)=\langle 1,1,3\rangle$ | 1 | | | | | | | | | 1 |
| $(3^2,2^2,1^4)=\langle 1,1,1\rangle$ | | | | | 1 | 1 | | | 1 | 1 |
| $(3^2,2,1^6)=\langle 1,3,3\rangle$ | | | | | 2 | 1 | | | 1 | |
| $(3,2^3,1^5)=\langle 3,3\rangle$ | | | | | 1 | | | | 1 | |
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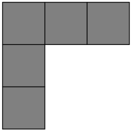

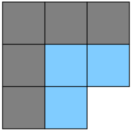

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| $(6^2,2)=\langle 1,2\rangle$ | | | | 1 | 1 | | | | | |
| $(6,4^4)=\langle 1,2,2\rangle$ | | | 1 | 1 | 1 | 1 | | | | |
| $(6,4,2^2)=\langle 2,2,2\rangle$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | |
| $(6,3,2^2,1)=\langle 1,1,2\rangle$ | 2 | 1 | 1 | | | | 1 | 1 | | |
| $(5,4,2^2,1)=\langle 1,1\rangle$ | 1 | 1 | 1 | | 1 | 1 | 1 | 1 | 1 | |
| $(4^2,2^2,1^2)=\langle 3\rangle$ | 1 | | 1 | | 1 | 1 | | 1 | 1 | 1 |
| $(9,1^5)=\langle 2,3\rangle$ | | | 1 | | | | | | | |
| $(6,4,1^4)=\langle 2,2,3\rangle$ | | | | | | | 1 | | | |
| $(6,3,2,1^3)=\langle 1,2,3\rangle$ | | | 1 | | | 1 | 1 | 1 | | |
| $(6,2^3,1^2)=\langle 3,2\rangle$ | | | | | | | | 1 | | |
| $(6,1^8)=\langle 2,3,3\rangle$ | | | | | | 1 | | | | |
| $(5,4,2,1^3)=\langle 1,3\rangle$ | | | | | 2 | 1 | 1 | 1 | 1 | |
| $(3^4,1^2)=\langle 3,1\rangle$ | 1 | | 1 | | | 1 | | | | 1 |
| $(3^2,2^4)=\langle 1,1,3\rangle$ | 1 | | | | | | | | | 1 |
| $(3^2,2^2,1^4)=\langle 1,1,1\rangle$ | | | | | 1 | 1 | | | 1 | 1 |
| $(3^2,2,1^6)=\langle 1,3,3\rangle$ | | | | | 2 | 1 | | | 1 | |
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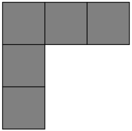

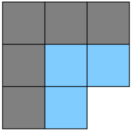

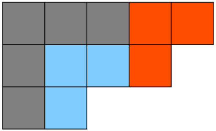

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| $(12,1^2) = \langle 2\rangle$ | 1 | | | | | | | | | |
| $(9,4,1) = \langle 2,2\rangle$ | 1 | 1 | | | | | | | | |
| $(9,3,2) = \langle 2,1\rangle$ | 2 | 1 | 1 | | | | | | | |
| $(8,4,2) = \langle 1\rangle$ | 1 | 1 | 1 | 1 | | | | | | |
| $(6^2,2) = \langle 1,2\rangle$ | | | | 1 | 1 | | | | | |
| $(6,4^4) = \langle 1,2,2\rangle$ | | | 1 | 1 | 1 | 1 | | | | |
| $(6,4,2^2) = \langle 2,2,2\rangle$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | |
| $(6,3,2^2,1) = \langle 1,1,2\rangle$ | 2 | 1 | 1 | | | | 1 | 1 | | |
| $(5,4,2^2,1) = \langle 1,1\rangle$ | 1 | 1 | 1 | | 1 | 1 | 1 | 1 | 1 | |
| $(4^2,2^2,1^2) = \langle 3\rangle$ | 1 | | 1 | | 1 | 1 | | 1 | 1 | 1 |
| $(9,1^5) = \langle 2,3\rangle$ | | | 1 | | | | | | | |
| $(6,4,1^4) = \langle 2,2,3\rangle$ | | | | | | | 1 | | | |
| $(6,3,2,1^3) = \langle 1,2,3\rangle$ | | | 1 | | | 1 | 1 | 1 | | |
| $(6,2^3,1^2) = \langle 3,2\rangle$ | | | | | | | | 1 | | |
| $(6,1^8) = \langle 2,3,3\rangle$ | | | | | | 1 | | | | |
| $(5,4,2,1^3) = \langle 1,3\rangle$ | | | | | 2 | 1 | 1 | 1 | 1 | |
| $(3^4,1^2) = \langle 3,1\rangle$ | 1 | | 1 | | | 1 | | | | 1 |
| $(3^2,2^4) = \langle 1,1,3\rangle$ | 1 | | | | | | | | | 1 |
| $(3^2,2^2,1^4) = \langle 1,1,1\rangle$ | | | | | 1 | 1 | | | 1 | 1 |
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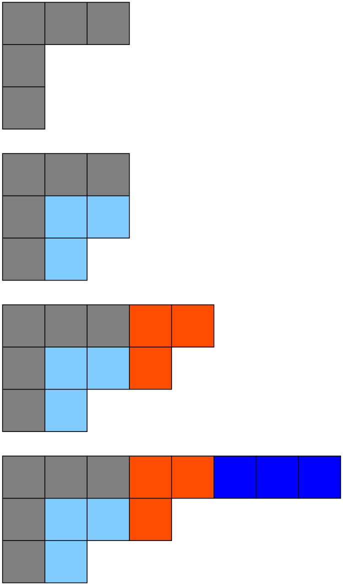

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| $(6,4^4)=\langle 1,2,2\rangle$ | | | 1 | 1 | 1 | 1 | | | | |
| $(6,4,2^2)=\langle 2,2,2\rangle$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | | |
| $(6,3,2^2,1)=\langle 1,1,2\rangle$ | 2 | 1 | 1 | | | | 1 | 1 | | |
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| $(3^4,1^2)=\langle 3,1\rangle$ | 1 | | 1 | | | 1 | | | | 1 |
| $(3^2,2^4)=\langle 1,1,3\rangle$ | 1 | | | | | | | | | 1 |
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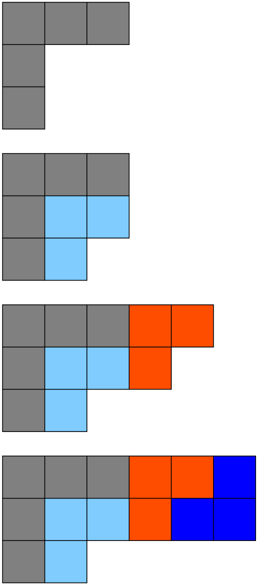

## §3 Polynomial representations and plethysms of Schur functions

- Polynomial representations of $\mathrm{GL}(E)$ with $E = \langle e_1, e_2, e_3 \rangle \cong \mathbb{C}^3$.

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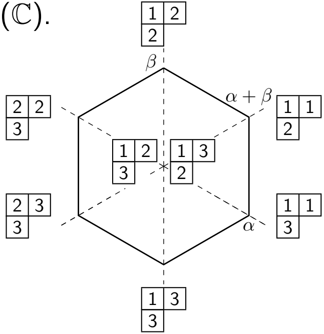
  Here $\nabla^{(2,1)}(E)$ has a basis $F(t)$ for $t$ a semistandard tableaux of shape $(2,1)$ with entries from $\{1,2,3\}$:

$$F\left(\begin{array}{|c|c|}\hline a & b \\ \hline c & \\ \cline{1-1}\end{array}\right) = e_a e_b \otimes e_c - e_c e_b \otimes e_a \in \mathrm{Sym}^2 E \otimes E.$$

  You might also know it as the adjoint representation of the Lie algebra $\mathrm{sl}_3(\mathbb{C})$.



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Now take $E = \langle e_1, e_2 \rangle \cong \mathbb{C}^2$

- Tensor product: $\mathrm{Sym}^2 E \otimes \mathrm{Sym}^2 E$
- Symmetric power of symmetric power: $\mathrm{Sym}^2 \mathrm{Sym}^2 E$ with basis $(e_1^2)(e_1^2)$, $(e_1^2)(e_2^2)$, $(e_1^2)(e_1 e_2)$, $(e_2^2)(e_2^2)$, $(e_2^2)(e_1 e_2)$, $(e_1 e_2)(e_1 e_2)$

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- Symmetric functions
  - $s_{(2)}(y_1, y_2, y_3) \;\; = y_1^2 + y_2^2 + y_3^3 + y_1 y_2 + y_1 y_3 + y_2 y_3$

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- Symmetric functions
  - $s_{(2)}(y_1, y_2, y_3) = y_1^2 + y_2^2 + y_3^3 + y_1 y_2 + y_1 y_3 + y_2 y_3$
  - $$\begin{aligned} s_{(2,1)}(x_1, x_2, x_3) &= x^{\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}} + x^{\begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array}} + x^{\begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}} + x^{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}} + x^{\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}} + \cdots + x^{\begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1}\end{array}} + x^{\begin{array}{|c|c|}\hline 2&3\\\hline 3\\\cline{1-1}\end{array}} \\ &= x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + 2 x_1 x_2 x_3 + \cdots + x_2^2 x_3 + x_2 x_3^2 \end{aligned}$$

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  Now take $E = \langle e_1, e_2 \rangle \cong \mathbb{C}^2$
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- Symmetric functions
  - $s_{(2)}(y_1, y_2, y_3) \;= y_1^2 + y_2^2 + y_3^3 + y_1 y_2 + y_1 y_3 + y_2 y_3$
  - $$\begin{aligned} s_{(2,1)}(x_1, x_2, x_3) &= x^{\ytableaushort{11,2}} + x^{\ytableaushort{11,3}} + x^{\ytableaushort{12,2}} + x^{\ytableaushort{12,3}} + x^{\ytableaushort{13,2}} + \cdots + x^{\ytableaushort{22,3}} + x^{\ytableaushort{23,3}} \\ &= x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + 2 x_1 x_2 x_3 + \cdots + x_2^2 x_3 + x_2 x_3^2 \end{aligned}$$
  - Multiplication: $s_{(2)}(x_1, x_2)^2 = (x_1^2 + x_2^2 + x_1 x_2)^2$
  - Evaluate $s_{(2)}(y_1, y_2, y_3)$ at monomials in $s_{(2)}(x_1, x_2)$ to get

$$s_{(2)}(x_1^2, x_2^2, x_1 x_2) = (x_1^2)(x_1^2) + (x_1^2)(x_2^2) + (x_1^2)(x_1 x_2) + \cdots + (x_1 x_2)(x_1 x_2).$$

    This is the plethysm $(s_{(2)} \circ s_{(2)})(x_1, x_2)$, obtained by evaluating $s_{(2)}$ at the monomials $x_1^2$, $x_2^2$, $x_1 x_2$ in $s_{(2)}(x_1, x_2)$.

# Combinatorial definition of plethysm

Given a tableau $t$ let $x^t = x_1^{a_1} x_2^{a_2} \ldots$ where $a_i$ is the number of entries of $t$ equal to $i$. We say $t$ has *weight* $(a_1, a_2, \ldots)$.

## Definition (Schur function)

Let $\mu$ be a partition. The *Schur function* $s_\mu$ is the generating function enumerating semistandard tableaux of shape $\mu$ by weight:

$$s_\mu = \sum_{t \in \mathrm{SSYT}(\mu)} x^t.$$

For instance

$$\begin{aligned} s_{(2)}(x_1, x_2, \ldots) &= x^{\boxed{1}\boxed{1}} + x^{\boxed{1}\boxed{2}} + x^{\boxed{2}\boxed{2}} + x^{\boxed{1}\boxed{3}} + \cdots \\ &= x_1^2 + x_1 x_2 + x_2^2 + x_1 x_3 + \cdots \end{aligned}$$

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Equivalently, $s_\mu(x_1, \ldots, x_d)$ is the trace of $\mathrm{diag}(x_1, \ldots, x_n)$ acting on $\nabla^\mu(E)$. For instance $s_{(n)}(x_1, \ldots, x_d)$ is the character of $\mathrm{Sym}^n E$.

## Definition (Plethysm of Schur functions)

Let $\mu$ and $\nu$ be partitions. Let $\mathrm{SSYT}(\mu) = \{t(1), t(2), \ldots\}$. The *plethystic product* of $s_\nu$ and $s_\mu$ is $s_\nu \circ s_\mu = s_\nu(x^{t(1)}, x^{t(2)}, \ldots)$.

By definition of the Hall inner product, $\langle f, s_\lambda \rangle$ is the multiplicity of $s_\lambda$ as a summand of the symmetric function $f$.

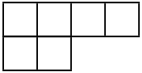
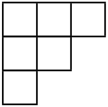
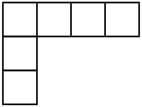
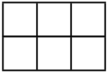
### Problem (Stanley's Problem 9, 2000)

*Find a combinatorial interpretation of the plethysm coefficients $\langle s_{(n)} \circ s_{(m)}, s_\lambda \rangle$ that makes it clear they are non-negative.*

Equivalently, find a combinatorial interpretation for the multiplicity of the irreducible $\mathrm{GL}_d(\mathbb{C})$-module $\nabla^\lambda(E)$ in $\mathrm{Sym}^n \mathrm{Sym}^m E$.

# §4: Maximal summands in plethysms

A partition $\lambda$ *dominates* a partition $\kappa$ if the Young diagram of $\kappa$ can be obtained from the Young diagram of $\lambda$ by repeatedly moving boxes downwards. For instance

 $\trianglerighteq$      $\not\trianglerighteq \not\trianglelefteq$ .

Quiz. Choose partitions $\kappa$ and $\lambda$ of $n$ (a very large number) uniformly at random. What, roughly, is the chance that $\kappa$ and $\lambda$ are comparable in the dominance order?

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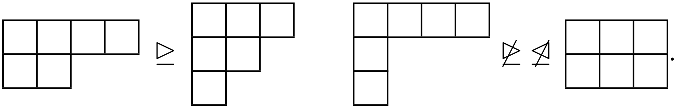


Quiz. Choose partitions $\kappa$ and $\lambda$ of $n$ (a very large number) uniformly at random. What, roughly, is the chance that $\kappa$ and $\lambda$ are comparable in the dominance order?

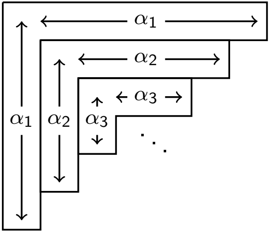
Answer. Asymptotically 0, by a theorem of Pittel (1997).

| $n$ | 5 | 6 | 10 | 20 | 30 | 35 |
|---|---|---|---|---|---|---|
| $p_{\mathrm{comparable}}$ | 1 | 0.967 | 0.904 | 0.782 | 0.716 | 0.694 |

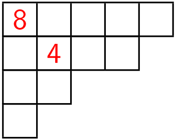
But no problem if you guessed something else: the convergence is very slow, and the small cases are misleading.

Most plethysms have many different maximal summands.

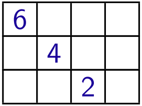
Extreme example: $s_{(1^n)} \circ s_{(2)}$. Let $n \in \mathbb{N}$. Given a partition $\alpha$ of $n$ with distinct parts, let $2[\alpha]$ be the partition of $2n$ with leading diagonal hook lengths $2\alpha_1, 2\alpha_2, \ldots$




$2[4,2]$




$2[3,2,1]$




The plethysm $s_{(1^n)} \circ s_{(2)}$ corresponding to $\bigwedge^n \mathrm{Sym}^2 E$ is

$$s_{(1^n)} \circ s_2 = \sum_{\alpha \in \mathrm{Par}_{\mathrm{distinct}(n)}} s_{2[\alpha]}.$$

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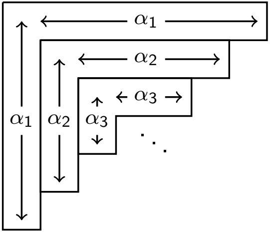


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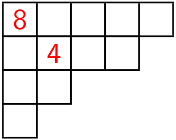


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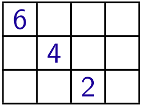


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For instance, if $n = 6$ then

$$s_{(1^6)} \circ s_2 = s_{(7,1^5)} + s_{(6,3,1^3)} + s_{(5,4,2,1)} + s_{(4,4,4)}.$$

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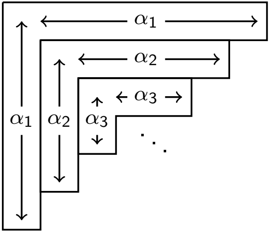
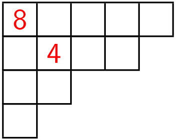



2[4, 2]

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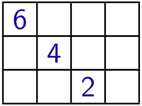


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$$s_{(1^n)} \circ s_2 = \sum_{\alpha \in \mathrm{Par}_{\mathrm{distinct}(n)}} s_{2[\alpha]}.$$

Exercise. Show that if $\alpha, \beta \in \mathrm{Par}_{\mathrm{distinct}}(n)$ are different partitions then $2[\alpha]$ and $2[\beta]$ are incomparable.

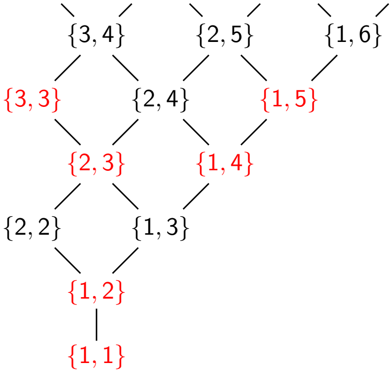
Thus every constituent of $s_{(1^n)} \circ s_{(2)}$ is both maximal *and* minimal. All of them are determined by our theorem.

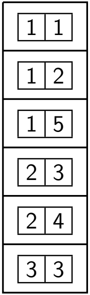
## Theorem (Paget–W 2018)

*The maximal constituents of the plethysm $s_\nu \circ s_\mu$ are precisely the maximal weights of the plethystic semistandard tableaux of outer shape $\nu$ and inner shape $\mu$.*

A plethystic semistandard tableaux of outer shape $(1^n)$ and inner shape $(m)$ is the same as a set of $n$ distinct $m$-multisets of $\mathbb{N}$, ordered by the majorization order.

Taking $m = 2$ we get the decomposition of $s_{(1^n)} \circ s_{(2)}$. For $n = 6$:



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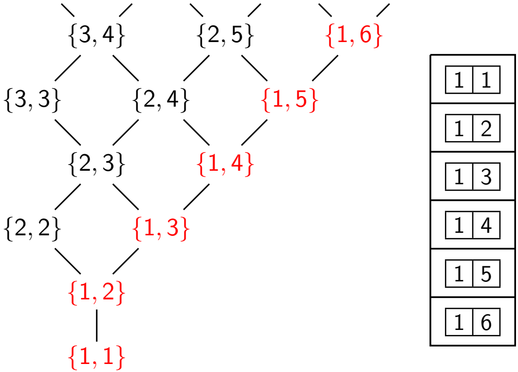


$(7, 1, 1, 1, 1, 1)$

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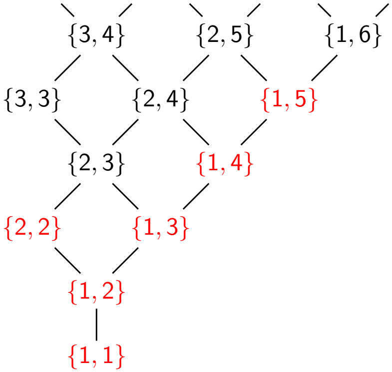


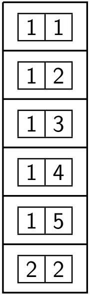


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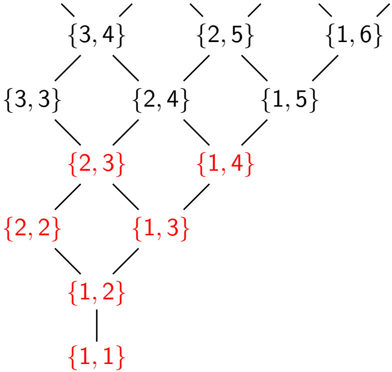


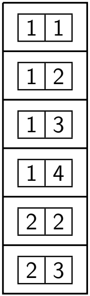


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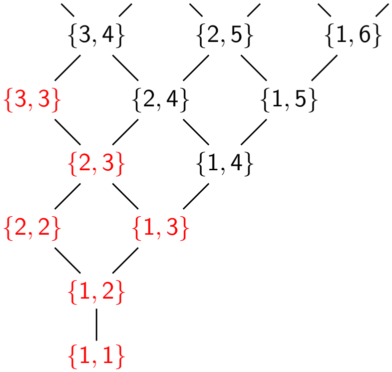


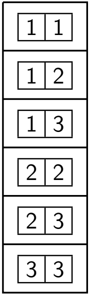


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$(4, 4, 4)$

### Theorem (Paget–W 2018)

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A plethystic semistandard tableaux of outer shape $(1^n)$ and inner shape $(m)$ is the same as a set of $n$ distinct $m$-multisets of $\mathbb{N}$, ordered by the majorization order.

- The 2018 proof uses the symmetric group.
- In 2020 with Melanie de Boeck we gave a shorter proof using polynomial representations of $\mathrm{GL}_n(\mathbb{C})$.
- Our recent work in 2022 gives a still shorter combinatorial proof, with an explicit 'gap' result on the separation between maximal and minimal summands.

## §5: Plethysms stability

**Theorem**

*Let $\gamma$ be a partition, and let $(mn - |\gamma|; \gamma)$ denote the partition $(mn - |\gamma|, \gamma_1, \ldots, \gamma_\ell)$. The plethysm coefficient*

$$\left\langle s_{(n)} \circ s_{(m)}, s_{(mn-|\gamma|;\gamma)} \right\rangle$$

*is constant for all m and n sufficiently large.*

Proved by

- Carré and Thibon (1992): vertex operators
- Brion (1993): dominant maps of algebraic varieties
- Manivel (1997): stable embeddings of varieties
- Bowman and Paget (2018): partition algebra
- Paget and W (2022): plethystic semistandard tableaux

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The Bowman–Paget proof is notable as the only one to give an explicit (if intricate) formula for the multiplicity that is clearly non-negative. This is a significant step towards the solution of Stanley's Problem 9.

Using combinatorial arguments with signed plethystic semistandard tableaux Paget and I have given unified proofs of every stability result in the literature we know about.

Here are three representative examples.

**Update:** three days after my talk Law and Okitani published *Some stable plethysms* arXiv:2214.06964. It has a generalization of their theorem (next slide). It is not yet clear if we can adapt our methods to prove it.

Using combinatorial arguments with signed plethystic semistandard tableaux Paget and I have given unified proofs of every stability result in the literature we know about.

Here are three representative examples.

### Theorem (Brion 1993)

*Let $\nu \in \mathrm{Par}(n)$, $\mu \in \mathrm{Par}(m)$, $\lambda \in \mathrm{Par}(mn)$. Let $r \in \mathbb{N}$. The plethysm coefficient*

$$\langle s_\nu \circ s_{\mu + N(1^r)}, s_{\lambda + N(n^r)} \rangle$$

*is constant for all $N$ sufficiently large, with an explicit bound.*

### Theorem (Paget–W 2022)

*Let $\nu/\nu^\star \in \mathrm{SkewPar}(n)$, $\mu/\mu^\star \in \mathrm{SkewPar}(m)$, $\lambda \in \mathrm{Par}(mn)$. Let $r \in \mathbb{N}$. The plethysm coefficient*

$$\langle s_{\nu/\nu^\star} \circ s_{\mu + N(1^r)/\mu^\star}, s_{\lambda + N(n^r)} \rangle$$

*is constant for all $N$ sufficiently large, with an explicit bound.*

### Theorem (Law–Okitani 2021: Proposition 5.3)

*Let $\nu \in \mathrm{Par}(n)$ and $\lambda \in \mathrm{Par}(mn)$. The plethysm coefficient*

$$\left\langle s_{\nu \sqcup (1^N)} \circ s_{(2)}, s_{\lambda + (N) \sqcup (1^N)} \right\rangle$$

*is constant for $N$ sufficiently large.*

The generalization replacing 2 with an arbitrary $m \in \mathbb{N}$ and $\lambda + (N) \sqcup (1^N)$ with $\lambda + (m-1)N \sqcup (1^N)$ was announced at Oberwolfach in September 2022.

Our methods generalize the Law–Okitani result further, from $(m)$ to an arbitrary rectangular partition. The proof requires signed plethystic semistandard tableaux with negative entries.

### Theorem (Paget–W 2022)

*Let $\nu \in \mathrm{Par}(n)$, let $a, b \in \mathbb{N}$ and let $\lambda \in \mathrm{Par}(abn)$. The plethysm coefficient*

$$\left\langle s_{\nu \sqcup (1^N)} \circ s_{(a^b)}, s_{\lambda + N(a^{b-1}, a-1) \sqcup (1^N)} \right\rangle$$

*is constant for $N$ sufficiently large, with an explicit bound on $N$.*

## Theorem (Brion 1993)

*Let* $\nu \in \mathrm{Par}(n)$, $\mu \in \mathrm{Par}(m)$, $\lambda \in \mathrm{Par}(mn)$. *The plethysm coefficient* $\langle s_{\nu+(N)} \circ s_\mu, s_{\lambda+N\mu} \rangle$ *is constant for* $N$ *sufficiently large.*

## Theorem (Briand–Orrelana–Rosas 2014)

*Let* $\nu \in \mathrm{Par}(n)$, $\mu \in \mathrm{Par}(m)$, $\lambda \in \mathrm{Par}(mn)$. *Let* $r$ *be the total number of semistandard tableaux of shape* $\mu$ *with entries from* $\{1, \ldots, d\}$ *and let* $q = r|\nu|/d$. *The plethysm coefficient*

$$\left\langle s_{\nu+N(1^r)} \circ s_\mu, s_{\lambda+N(q^d)} \right\rangle$$

*is constant for* $N$ *sufficiently large.*

Common generalization, by replacing the set of semistandard tableaux of shape $\mu$ with entries from $\{1, \ldots, d\}$ with certain maximal tableau families, such as

$$\boxed{1\,|\,1},\ \boxed{1\,|\,2},\ \boxed{1\,|\,3},\ \boxed{1\,|\,4},\ \boxed{1\,|\,5},\ \boxed{1\,|\,6}$$

seen earlier. The family has size 1 for Brion, and size $r$ for Briand–Orrelana–Rosas.

### Theorem (Brion 1993)

*Let* $\nu \in \mathrm{Par}(n)$, $\mu \in \mathrm{Par}(m)$, $\lambda \in \mathrm{Par}(mn)$. *The plethysm coefficient* $\langle s_{\nu+(N)} \circ s_\mu, s_{\lambda+N\mu} \rangle$ *is constant for* $N$ *sufficiently large.*

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### Theorem (Paget–W 2022)

*Let* $\nu \in \mathrm{Par}(n)$, $\mu \in \mathrm{Par}(m)$, $\lambda \in \mathrm{Par}(mn)$. *Let* $\omega$ *be the weight of a strongly maximal tableau family of size* $r$. *The plethysm coefficient*

$$\left\langle s_{\nu+N(1^r)} \circ s_\mu, s_{\lambda+N\omega} \right\rangle$$

*is constant for* $N$ *sufficiently large, with an explicit bound on* $N$.

Thank you! Any questions?

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