ON BIDIGARE'S PROOF OF SOLOMON'S THEOREM

1. Introduction

This note gives a version of Bidigare's proof [1] of an important theorem of Solomon [3, Theorem 1] that emphasises certain combinatorial and algebraic features of the proof. There are no essentially new ideas.

To state Solomon's theorem we need the following definitions. A *composition* of $n \in \mathbf{N}_0$ is a sequence $(p_1, \ldots, p_k)$ of natural numbers such that $p_1 + \cdots + p_k = n$. To indicate that p is a composition of n we write $p \models n$. Let S_n denote the symmetric group of degree n and let $\mathbf{Z}S_n$ be the integral group ring of S_n. Given $p \models n$, let $\Xi^p \in \mathbf{Z}S_n$ be the sum of all minimal length coset representatives for the right cosets $S_p \backslash S_n$. Equivalently, if $p = (p_1, \ldots, p_k)$, then Ξ_p is the sum of all $g \in S_n$ such that

$$\mathrm{Des}(g) \subseteq \{p_1, p_1 + p_2, \ldots, p_1 + p_2 + \cdots + p_{k-1}\},$$

where $\mathrm{Des}(g) = \bigl\{x \in \{1, \ldots, n-1\} : xg < (x+1)g\bigr\}$ is the *descent set* of g. Given compositions p, q and r of $\mathbf{N}$ such that p has k parts and q has ℓ parts we define m^r_{pq} to be the number of $k \times \ell$ matrices A with entries in $\mathbf{N}_0$ such that

(i) the ith row sum is p_i for each i,
(ii) the jth column sum is q_j for each j,
(iii) the entries, read in the order $A_{11}, \ldots, A_{1\ell}, \ldots, A_{k1}, \ldots, A_{k\ell}$ with any zero entries ignored, form the composition r.

Theorem 1 (Solomon). *If p, q and r are compositions of $n \in \mathbf{N}_0$ then*

$$\Xi^p \Xi^q = \sum_{r \models n} m^r_{pq} \Xi^r.$$

2. The proof

Define a *set composition* of n to be a tuple $(P_1, \ldots, P_k)$ such that $P_1 \cup \cdots \cup P_k = \{1, \ldots, n\}$ and the sets $P_1, \ldots, P_n$ are disjoint and non-empty. If $|P_i| = p_i$ for each i then we say that $(P_1, \ldots, P_k)$ has *type* $(p_1, \ldots, p_k)$. Let Π_n be the set of all set compositions of n. There is an action of S_n on Π_n defined by

$$(P_1, \ldots, P_k)g = (P_1 g \ldots, P_k g) \quad \text{for } g \in S_n.$$

We define an associative product $\wedge : \Pi_n \times \Pi_n \to \Pi_n$ by

$$(P_1, \ldots, P_k) \wedge (Q_1, \ldots, Q_\ell) = (P_1 \cap Q_1, \ldots, P_1 \cap Q_\ell, \ldots, P_k \cap Q_1, \ldots, P_k \cap Q_\ell)^\star$$

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where the $\star$ indicates that any empty sets in the tuple should be deleted. (Thus in forming $P \wedge Q$ we loop through the sets in Q faster than the sets in P, and reading $P \wedge Q$ in order, if $i < j$ then we see all the elements of P_i before any of the elements of P_j.) We record some further basic properties below.

(1) $\wedge$ is idempotent, i.e. $P \wedge P = P$ for all $P \in \Pi_n$.

(2) $\{1, \ldots, n\}$ is the identity for $\wedge$.

(3) If P has type p and Q has type q then the type of $P \wedge Q$ is a common refinement of p and q.

(4) If P has type (1^n) then $P \wedge Q = P$, for any $Q \in \Pi_n$.

(5) The product $\wedge$ is S_n-invariant. That is, if $g \in S_n$ and P, $Q \in \Pi_n$ then $(P \wedge Q)g = Pg \wedge Qg$.

Thanks to (1) and (2), Π_n is an idempotent semigroup. Note that in (3) 'refinement' allows for some rearrangement of parts in the case of Q: for example $(\{1,2\},\{3\}) \wedge (\{3\},\{1,2\}) = (\{1,2\},\{3\})$ has type $(2,1)$, and is the wedge product of set compositions of types $(2,1)$ and $(1,2)$.

The **Z**-algebra $\mathbf{Z}\Pi_n$ is an associative unital algebra whose product is S_n-invariant. Here are some of its basic properties.

(A) $\mathbf{Z}\Pi_n$ is a right $\mathbf{Z}S_n$-module by linear extension of the action of S_n on Π_n.

(B) Let $\Pi_{(1^n)}$ be the collection of set compositions of type (1^n). Given $P = (\{a_1\}, \ldots, \{a_n\}) \in \Pi_{(1^n)}$, let $\overline{P} \in S_n$ be the permutation sending i to a_i for each i. The map $P \mapsto \overline{P}$ is then a linear isomorphism $\mathbf{Z}\Pi_{(1^n)} \to \mathbf{Z}S_n$ of $\mathbf{Z}S_n$-modules.

(C) By (3) above, $\mathbf{Z}\Pi_{(1^n)}$ is an ideal of $\mathbf{Z}\Pi_n$. Moreover, by (4), each $Q \in \Pi_n$ acts trivially on $\mathbf{Z}\Pi_{(1^n)}$ on the right.

(D) By (5), the fixed point space $(\mathbf{Z}\Pi_n)^{S_n}$ is a subalgebra of $\mathbf{Z}\Pi_n$. Given $q \models n$, let X^q be the sum of all set compositions of type q. Then $\{X^q : q \models n\}$ is a basis of $(\mathbf{Z}\Pi_n)^{S_n}$. If q has ℓ parts then X^q is the orbit sum under the action of S_n for the set composition

$$T^q = (\{1 \ldots q_1\}, \ldots, \{q_1 + \cdots + q_{\ell-1} + 1, \ldots, n\}).$$

Let $\mathbf{I} = (\{1\}, \ldots, \{n\}) \in \Pi_n$. By (3) above $P \wedge \mathbf{I} \in \Pi_{(1^n)}$ for each $P \in \Pi_n$. The main step in Bidigare's proof is the following theorem.

Theorem 2. *The map $f \mapsto \overline{f}$ from $(\mathbf{Z}\Pi_n)^{S_n}$ to $\mathbf{Z}S_n$ defined by linear extension of $P \mapsto \overline{P \wedge \mathbf{I}}$ is a **Z**-algebra homomorphism such that $\overline{X^p \wedge \mathbf{I}} = \Xi_p$.*

The final claim concerning Ξ_p is clear. The first part is a corollary of the following stronger proposition.

Proposition 3. *If $f \in (\mathbf{Z}\Pi_n)^{S_n}$ and $x \in \mathbf{Z}\Pi_n$ then $\overline{f \wedge \mathbf{I}}\;\overline{x \wedge \mathbf{I}} = \overline{f \wedge x \wedge \mathbf{I}}$.*

Proof. Let p be a composition with k parts. It suffices to prove the proposition when $f = X^p$, the sum of all set compositions of type p, and $x = Q$, an arbitrary set composition.

Suppose that Q has type q where q has ℓ parts. Let $g = \overline{Q \wedge \mathbf{I}} \in S_n$; equivalently, g is the permutation of minimal length such that $T^q g = Q$. We have

$$\overline{X^p \wedge \mathbf{I}}\; \overline{Q \wedge \mathbf{I}} = \sum_P \overline{P \wedge \mathbf{I}}\, g$$

where the sum is over all $P \in \Pi_n$ of type p. Fix such a P. Set $d_i = p_1 + \cdots + p_{i-1}$ for $1 \le i < k$. *Claim:* $(P \wedge \mathbf{I})g = (P \wedge T_q)g \wedge \mathbf{I}$. *Proof of claim:* Since $T_q \wedge \mathbf{I}$ has increasing entries, the singleton sets in positions $d_i + 1, \ldots, d_i + p_i$ on both sides are obtained by taking the entries of P_i in increasing order, and applying g to each. || Hence

$$\begin{aligned}
\overline{X^p \wedge \mathbf{I}}\; \overline{Q \wedge \mathbf{I}} &= \sum_P \overline{P \wedge \mathbf{I}}\, g \\
&= \overline{\sum_P (P \wedge T_q)g \wedge \mathbf{I}} \\
&= \overline{(X^p \wedge T_q)g \wedge \mathbf{I}} \\
&= \overline{(X^p \wedge Q) \wedge \mathbf{I}}
\end{aligned}$$

as required. □

It follows from Theorem 2 that the span of the Ξ^p for $p \models n$ is a subalgebra of $\mathbf{Z}S_n$ isomorphic to $(\mathbf{Z}\Pi_n)^{S_n}$. To complete the proof of Theorem 1 we compute the structure constants for this algebra. The following definition will be helpful: say that $T \in \Pi_n$ is *increasing* if whenever $1 \le i < i' \le \ell$ and $x \in T_i$, $x' \in T_{i'}$, we have $x < x'$. (Equivalently, T is increasing if and only if $T = T^p$ for some $p \models n$.)

Proposition 4. *Let p, q and r be compositions of n. Then the coefficient of X^r in $X^p \wedge X^q$ is m^r_{pq}.*

Proof. It is equivalent to show that the coefficient of T^r in $X^p \wedge X^q$ is m^r_{pq}. If $T^r = P \wedge Q$ where P and Q are set compositions then, since T^r is increasing, P must also be increasing. Therefore it suffices to show that if

$$\mathcal{Q} = \{Q \in \Pi_n : T^p \wedge Q = T^r,\ Q \text{ has type } q\}$$

then $|\mathcal{Q}| = m^r_{pq}$. Suppose that p has k parts, q has ℓ parts and that r has m parts. Given $Q \in \mathcal{Q}$ define $M(Q)$ to be the $k \times \ell$ matrix such that

$$M(Q)_{ij} = |T^p_i \cap Q_j| \quad \text{for } 1 \le i \le k,\ 1 \le j \le \ell.$$

The ith row sum of $M(Q)$ is $|T^p_i| = p_i$ and the jth column sum of $M(Q)$ is $|Q_j| = q_j$. Moreover, reading the non-zero entries of $M(Q)$ in the order specified in (iii) gives the composition r. *Claim:* Conversely, given a matrix M satisfying these conditions, there is a unique Q such that $M(Q) = M$

and $T^p \wedge Q = T^r$. *Proof of claim*: fix a row i and suppose inductively that we have allocated the elements of T_i^p up to and including a to the sets $Q_1, \ldots, Q_{j-1}$. (For the base case $j = 1$, take $a = p_1 + \cdots + p_{i-1}$.) Then we must put $a+1, \ldots, a+M_{ij}$ into the set Q_j to have $T^p \wedge Q$ increasing. $||$ $\square$

References

[1] Bidigare, T. P. *Hyperplane Arrangement Face Algebras and their Associated Markov Chains*. PhD thesis, University of Michigan, 1997.

[2] Schocker, M. The descent algebra of the symmetric group. In *Representations of finite dimensional algebras and related topics in Lie theory and geometry*, vol. 40 of *Fields Inst. Commun.* Amer. Math. Soc., Providence, RI, 2004, pp. 145–161.

[3] Solomon, L. A Mackey formula in the group ring of a Coxeter group. *J. Algebra 41*, 2 (1976), 255–264.