PRIFYSGOL CYMRU ABERTAWE
UNIVERSITY OF WALES SWANSEA

DEGREE EXAMINATIONS 2007

MODULE MAP363

Combinatorics: SPECIMEN PAPER

Time Allowed — 2 hours

There are SIX questions on the paper.
A candidate's best THREE questions will be used for assessment.

No calculators are permitted.

Each question has equal weight. The maximum possible mark is 75/75.

turn over

1. Let G be a graph with vertex set V and edge list E.

(a) Define the *degree* of a vertex $x \in V$. Define the *degree sequence* of G. What does it mean to say that G is *simple*?
[5 Marks]

(b) State and prove the Handshaking Theorem relating the degree sequence of G to the number of edges of G. [8 Marks]

(c) For each of the sequences below, decide whether or not it is the degree sequence of a simple graph. (If it is, give an explicit example, if not, explain clearly why not.)

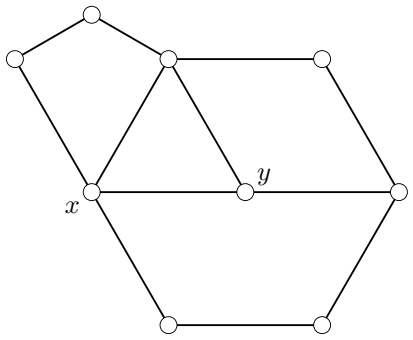
(i) (3,3,3,3,3,3), (ii) (3,3,3,3,3), (iii) (3,1,1,1,0,0), (iv) (4,2,1,1,0)

[12 Marks]

2. (a) What is meant by a *closed Hamiltonian path* in a graph? What does it mean to say that a graph is *bipartite*? [7 Marks]

(b) Supose that G is a bipartite graph with bipartition $\{A, B\}$. Show that if G has a closed Hamiltonian path then $|A| = |B|$.
[8 Marks]

(c) Now let G be the graph shown below.



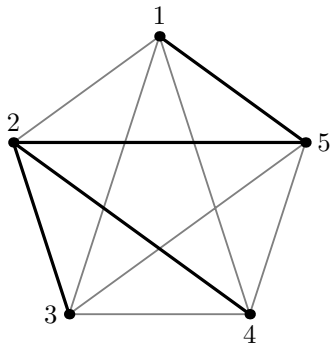
(i) Show that if the edge between vertices x and y is deleted then G becomes bipartite. [4 Marks]

(ii) Deduce that if G has a closed Hamiltonian path then this path must use the edge $\{x, y\}$. Hence, or otherwise, show that G does not have a closed Hamiltonian path. [6 Marks]

3. Let G be a simple graph with vertices labelled by distinct natural numbers.

(a) What does it mean to say that a subgraph of G is a *spanning tree* in G? [4 Marks]

(b) Define the *Prüfer code* of a spanning tree T in G. Illustrate your answer by determining the Prüfer code of the spanning tree shown below.



(In this diagram, grey edges are edges of the complete graph on $\{1, 2, 3, 4, 5\}$ and black edges are edges of a given spanning tree in this graph.) [8 Marks]

(c) Which spanning trees in the complete graph with vertex set $\{1, 2, 3, 4, 5, 6, 7\}$ have Prüfer code of the form (a, a, a, a, b) where $a \neq b$? [9 Marks]

How many such spanning trees are there? [4 Marks]

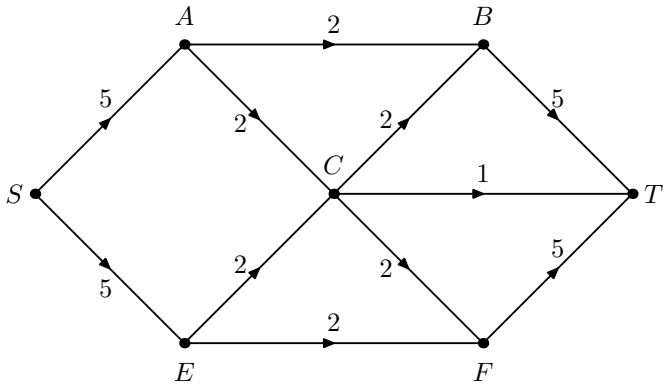
4. Let N be a network with source s and target t. Write $c(x, y)$ for the capacity of the edge (x, y).

(a) What is meant by a *flow* in N? Define the value $\operatorname{val} f$ of a flow f. What does it mean to say that a flow is *maximal*? [5 Marks]

(b) What does it mean to say that (S, T) is a *cut* of N? Define the *capacity* of a cut. [3 Marks]

(c) Suppose that f is a flow in N and (S, T) is a cut of N such that if (x, y) is an edge of N with $x \in S$ and $y \in T$ then $f(x, y) = c(x, y)$, and if (y, x) is an edge of N with $x \in S$ and $y \in T$ then $f(y, x) = 0$. Explain why f is a maximal flow. [7 Marks]

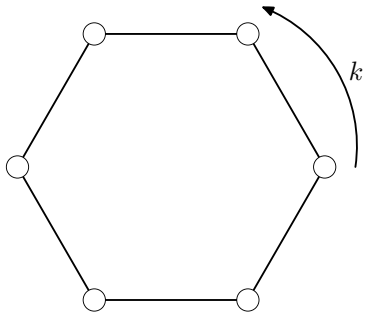
(d) The diagram below shows a network of water pipes between cities in the land of Erewhon. The source is Swaniff (marked S), and the target is Tropwen (marked T). The numbers on the edges give their capacities.



Find with proof a maximal flow in this network. What is its value? [6 Marks]

At fabulous expense, the regional assembly of Erewhon builds a new pipe of capacity 10 units, running from Swaniff to the city marked C. Find the new maximal flow value. [4 Marks]

5. Let k be a rotation by 60° of the regular hexagon. Let K be the subgroup of the symmetry group of the hexagon generated by k. Let e be the identity element of K.



Let X be the set of vertex colouring of the hexagon using the colours red and blue. Explain how K may be regarded as a subgroup of $\mathrm{Sym}(X)$. [5 Marks]

For $h \in K$, let $\mathrm{Fix}\, h$ be the set of colourings fixed by h. Show that $|\mathrm{Fix}\, e| = 2^6 = 64$ and that $|\mathrm{Fix}\, k| = 2$. Find the number of fixed points of the remaining elements of K in their action on X. [8 Marks]

How many ways are there to colour the vertices of a regular hexagon red and blue if two colourings are regarded as the same if one can be rotated into the other? [6 Marks]

[*You may assume that the number of orbits of* K *in its action on* X *is given by*

$$\frac{1}{|K|}\sum_{h\in K} |\mathrm{Fix}\, h|.]$$

How does your answer change if reflections are also taken into account? [6 Marks]

6. (a) Define the *generating function* associated to a function $f : \mathbb{N}_0 \to \mathbb{N}_0$. [3 Marks]

(b) Let $f(n)$ be the number of ways to write $n \in \mathbb{N}_0$ as a sum of the form $3a + 5b$ where $a, b \in \mathbb{N}_0$.

Show that $f(15) = 2$ and find $f(14)$ and $f(16)$. [7 Marks]

Let F be the generating function of f. Show that

$$F(x) = \frac{1}{(1 - x^3)(1 - x^5)}.$$

[8 Marks]

Hence, or otherwise, show that f satisfies the recurrence

$$f(n) = f(n - 3) + f(n - 5) - f(n - 8) \quad \text{if } n \geq 8.$$

[7 Marks]