MT4540 Combinatorics: Sheet 1

Do questions 3 and 4 and at least three other questions.
To be returned to McCrea 280 by 6pm on Monday 11th October 2010, or handed in at the Monday lecture.

Parts of questions marked ($\star$) are optional and slightly harder than the average.

1. Prove that
$$r\binom{n}{r} = n\binom{n-1}{r-1}$$
for $n, r \in \mathbf{N}$ in two ways:

(a) using the formula for a binomial number;

(b) by reasoning with subsets.

2. Prove that
$$\sum_{k=0}^{n} k\binom{m}{k}\binom{n}{k} = n\binom{m+n-1}{n}.$$
[*Hint: use Question 1 and then aim to apply Vandermonde's convolution.*]

3. Prove that
$$\binom{r}{r} + \binom{r+1}{r} + \binom{r+2}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1}$$
in two ways:

(a) by induction on n (for an arbitrary but fixed r);

(b) by reasoning with subsets of $\{1, 2, \ldots, n+1\}$.

4. Read up to the end of Section 1.2 from *generatingfunctionology* by Wilf and do a selection from problems 1, 3, 5 at the end of Chapter 1. Hand in your answer to problem 6(b).

5. A lion-tamer has n cages in a row. Let $g(n, k)$ be the number of ways in which she may accommodate k indistinguishable lions so that no cage contains more than one lion, and no two lions are housed in adjacent cages.

(a) Show that $g(n, k) = g(n-2, k-1) + g(n-1, k)$ if $n \geq 2$ and $k \geq 1$.

(b) Prove by induction that $g(n, k) = \binom{n-k+1}{k}$ for all $n, k \in \mathbf{N}$.

($\star$) Is there a bijective proof of the formula for $g(n, k)$?

6. Let $n \in \mathbf{N}$. How many solutions with $x_i \in \mathbf{N}$ for each i are there to the equation $x_1 + x_2 + \cdots + x_n = k$?

7. Let

$$b_n = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots$$

for $n \in \mathbf{N}_0$.

(a) Find the first members of the sequence $b_0, b_1, b_2, \ldots$.

(b) State and prove a recurrence relating b_{n+2} to b_{n+1} and b_n.

8. (a) What is 11^4? Explain the connection to binomial coefficients.

(b) By considering a suitable binomial expansion, prove that

$$\frac{4^n}{n+1} \le \binom{2n}{n} \le 4^n.$$

9. Here are some further results on derangements.

(a) Let $a_n(k)$ be the number of permutations of $\{1, 2, \ldots, n\}$ with exactly k fixed points. Note that $d_n = a_n(0)$. Use results from lectures to prove that

$$a_n(k) = \frac{n!}{k!}\Big(1 - \frac{1}{1!} + \frac{1}{2!} - \ldots + \frac{(-1)^{n-k}}{(n-k)!}\Big).$$

Hence, or otherwise, give a simple expression for $a_n(0) - a_n(1)$.

(b) Use part (a) to give an alternative proof of Theorem 2.6(ii), that the average number of fixed points of a permutation of $\{1, 2, \ldots, n\}$ is 1.

(c) ($\star$) Let e_n be the number of derangements of $\{1, 2, \ldots, n\}$ which are even permutations, and o_n the number which are odd permutations. By evaluating the determinant of the matrix

$$\begin{pmatrix} 0 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 1 & \ldots & 1 \\ 1 & 1 & 0 & \ldots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \ldots & 0 \end{pmatrix}$$

in two different ways, show that $e_n - o_n = (-1)^{n-1}(n-1)$. [*Hint: find a basis of eigenvectors for this matrix, and hence work out its determinant.*]

10. Assume that any two people are either friends or enemies. Show that in any room containing six people there are either three mutual friends, or three mutual enemies.

[*This is the standard example of a problem in Ramsey Theory.*]

MT4540 Combinatorics: Sheet 2

Do questions 2, 3 and 4 and at least two other questions.
To be returned to McCrea 280 by 6pm on Monday 18th October 2010, or handed in at the Monday lecture.

Parts of questions marked ($\star$) are optional and slightly harder than the average.

1. How many integers between 1 and 2010 are not divisible by either 2 or 3? Illustrate your answer with a Venn diagram.

2. How many numbers in the interval $\{1, 2, \ldots, 100\}$ are not divisible by either 2, 3, 5 or 7? (Use the PIE, making it clear what sets you are considering.) Hence find the number of primes ≤ 100.

3. Read §1.3 from Wilf *generatingfunctionology* and do question 11 at the end of Chapter 1.

4. Euler's φ-function is important in number theory. It is defined by

$$\varphi(n) = \big|\{a \in \mathbf{N} : 1 \leq a \leq n,\ \gcd(a, n) = 1\}\big|.$$

For example, when $n = 10$, the integers a such that $1 \leq a \leq 10$ and $\gcd(a, n) = 1$ are exactly $1, 3, 7, 9$, so $\varphi(10) = 4$. Show that

(a) $\varphi(p) = p - 1$ if p is prime.

(b) Let p, q, r denote distinct primes. Give a formula for $\varphi(pq)$ and one for $\varphi(pqr)$, using the PIE. (Define the sets A_i precisely).

(c) Give a formula for prime powers $\varphi(p^e)$.

(d) Let $\gcd(n_1, n_2) = 1$. Give a formula for $\varphi(n_1 n_2)$ in terms of $\varphi(n_1)$ and $\varphi(n_2)$.

(e) Recall that each integer n has a unique prime factorization $n = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$, where $p_1 < p_2 < \cdots < p_r$ are primes. Give a formula for $\varphi(n)$.

5. How many increasing sequences of length 3 can one create from the set $\{1, 2, \ldots, 8\}$? [*Hint: one approach is to count first the sequences with* 3 *distinct elements, then the sequences like* $(1, 1, 2)$ *with* 2 *distinct elements, and finally the sequences like* $(1, 1, 1)$. *A quicker solution uses Theorem 3.6 on indistinguishable balls in urns of unlimited capacity.*]

6. (a) Explain why there are

$$\binom{11}{4}\binom{7}{4}\binom{3}{2}$$

different ways to arrange the letters of the word 'mississippi'.

(b) How many ways are there to misspell 'abracadabra'?

7. Let $k, n \in \mathbf{N}$. Let X denote the set of all functions from $\{1, 2, \ldots, k\}$ to $\{1, 2, \ldots, n\}$. For $i \in \{1, 2, \ldots, n\}$, define

$$A_i = \bigl\{ f \in X : f(x) \neq i \text{ for any } x \in \{1, 2, \ldots, k\} \bigr\}.$$

(a) What is $|X|$? What is $|A_i|$?

(b) Let $I \subseteq \{1, 2, \ldots, n\}$. What property must a function $f : \{1, 2, \ldots, k\} \to \{1, 2, \ldots, n\}$ satisfy to lie in the set $A_I = \bigcap_{i \in I} A_i$? Hence find $|A_I|$.

(c) Use the Principle of Inclusion and Exclusion to show that the number of surjective functions from $\{1, 2, \ldots, k\}$ to $\{1, 2, \ldots, n\}$ is

$$\sum_{r=0}^{n} \binom{n}{r} (-1)^r (n-r)^k.$$

(d) Explain why the expression above is the number of ways to put k numbered balls into n urns, so that each urn receives *at least* one ball.

8. Given $k, n \in \mathbf{N}_0$, the *Stirling number of the second kind* $\left\{ {k \atop n} \right\}$ is defined to be the number of set partitions of $\{1, 2, \ldots, k\}$ into n disjoint subsets. For example, $\left\{ {4 \atop 3} \right\} = 6$; one of the relevant set partitions is $\bigl\{\{1\}, \{2\}, \{3, 4\}\bigr\}$.

(a) Prove that $\left\{ {k \atop 1} \right\} = 1$, $\left\{ {k \atop 2} \right\} = 2^{k-1} - 1$ and $\left\{ {k \atop k-1} \right\} = \binom{k}{2}$ for all $k \in \mathbf{N}$.

(b) Explain why $\left\{ {k \atop n} \right\}$ is the number of ways to put k numbered balls into n indistinguishable urns, so that each urn receives *at least* one ball. [**Corrected on 15 October.**]

(c) ($\star$) Deduce from question 7 that $\left\{ {k \atop n} \right\} = \frac{1}{n!} \sum_{r=0}^{n} \binom{n}{r} (-1)^r (n-r)^k$ and prove that

$$n^k = \sum_{r=0}^{n} \binom{n}{r} r! \left\{ {k \atop r} \right\}.$$

9. Let $a, b \in \mathbf{N}_0$ and let $m \in \mathbf{N}_0$. By considering the coefficient of x^m in

$$(1+x)^a (1+x)^b = (1+x)^{a+b},$$

give a generating function proof of Vandermonde's convolution identity

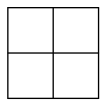
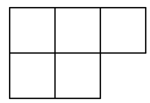
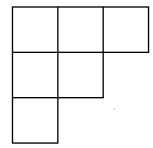
$$\sum_{k=0}^{m} \binom{a}{k} \binom{b}{m-k} = \binom{a+b}{m}.$$

10. Give a bijective proof of the identity $\sum_{k=0}^{n} \binom{n}{k} 2^k = 3^n$. [*Hint: Interpret the left-hand side as the number of ways to choose a subset X of $\{1, 2, \ldots, n\}$, and then to choose a further subset Y such that $Y \subseteq X$. An element of $\{1, 2, \ldots, n\}$ is either in Y, or in X but not in Y, or not in X ...*]

MT4540 Combinatorics: Sheet 3

Do questions 1 and 2 and at least two other questions.
To be returned to McCrea 240 by 6pm on Monday 25th October 2010, or handed in at the Monday lecture.

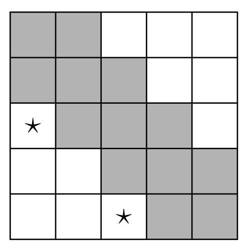
1. Find the rook polynomials of the boards below. For (ii) and (iii) use Lemma 7.6.

(i)  (ii)  (iii) 

2. (a) Let T be the set of all derangements σ of $\{1,2,3,4,5\}$ such that

- $\sigma(i) \neq i+1$ if $1 \leq i \leq 4$;
- $\sigma(i) \neq i-1$ if $2 \leq i \leq 5$.

Explain why $|T|$ is the number of ways to place 5 non-attacking rooks on the board formed by the unshaded squares below. (Give an explicit example of how a permutation corresponds to a rook placement.)



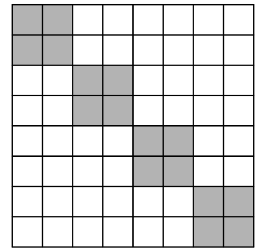
(b) Find the rook polynomial of this board, and hence find $|T|$. [*Hint: consider the four possibilities for the starred squares. For example, if both are occupied, the contribution to the rook polynomial is* $x^2 f_1(x) f_2(x)$ *where* $f_n(x)$ *is the rook polynomial of the* $n \times n$ *square board.*]

(c) Use Theorem 7.9 to find the number of ways to place 5 non-attacking rooks on the shaded squares.

3. Let B be the board in Example 7.1. Show that the complement of B in the 4×4 chessboard has the same rook polynomial as B. [*Hint: for a calculation-free proof, argue that permuting the rows or columns of a board does not change its rook polynomial.*]

4. Find the number of permutations σ of $\{1,2,3,4,5,6\}$ such that $\sigma(m) \neq m$ for any even number m.

5. How many numbers between 100 and 300 can be formed from the digits $1, 2, 3, 4$ if (i) repetition of digits is not allowed, (ii) repetition of digits is allowed?



6. Use Theorem 7.9 to find the number of ways that eight non-attacking rooks can be placed on the unshaded part of the board shown to the right. It may well be helpful to note that

$$(1+4x+2x^2)^4 = 1+16x+104x^2+352x^3+664x^4+704x^5+416x^6+128x^7+16x^8.$$

7. This question gives an alternative proof of the Principle of Inclusion and Exclusion (Theorem 5.3). Fix a set X. For each $A \subseteq X$, define a function $1_A : X \to \{0,1\}$ by

$$1_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

We say that 1_A is the *indicator function* of A.

(a) Show that if $B, C \subseteq X$ then $1_{B\cap C}(x) = 1_B(x)1_C(x)$ for all $x \in X$, and so $1_{B\cap C} = 1_B 1_C$.

(b) Let $A_1, A_2, \ldots, A_n$ be subsets of X. Show that

$$1_{\overline{A_1\cup A_2\cup\cdots\cup A_n}} = (1_X - 1_{A_1})(1_X - 1_{A_2})\ldots(1_X - 1_{A_n}).$$

(c) By multiplying out the right-hand side and using (a), show that

$$1_{\overline{A_1\cup A_2\cup\cdots\cup A_n}} = \prod_{I\subseteq\{1,2,\ldots,n\}} (-1)^{|I|} 1_{A_I}$$

where, as usual, $A_I = \bigcap_{i\in I} A_i$ if $I \neq \varnothing$ and $A_\varnothing = X$. [*Hint: it may be helpful to see how it works when* $n = 3$.]

(d) Deduce the Principle of Inclusion and Exclusion by summing the previous equation over all $x \in X$.

8. (For those who know about group homomorphisms.) Let G denote the set of all permutations of $\{1, 2, \ldots, n\}$, thought of as the symmetric group of degree n. Given $\sigma \in G$, define an $n \times n$ matrix $A(\sigma)$ by

$$A(\sigma)_{ij} = \begin{cases} 1 & \text{if } \sigma(j) = i \\ 0 & \text{otherwise.} \end{cases}$$

Show that the map $\sigma \mapsto A(\sigma)$ is an injective group homomorphism from G into the group of all invertible $n \times n$ real matrices.

9. Recall that d_n is the number of derangements of $\{1, 2, \ldots, n\}$. Use the formula for d_n to prove that if $n > 0$ then d_n is the nearest integer to $n!\,/\,\mathrm{e}$.

MT4540 Combinatorics: Sheet 4

Do questions 2 and 3 and at least two other questions.
To be returned to McCrea 240 by 6pm on Monday 8th November 2010, or handed in at the Monday lecture.

1. Find with proof the generating function for the number of ways to pay n pence using only 5p, 10p and 50p coins.

2. Find an explicit formula for the n-th term of the sequences defined by the following recurrence relations:

(a) $a_n = 6a_{n-2} - a_{n-1}$;

(b) $mb_m = (m+2)b_{m-1}$, $b_0 = 1$.

Using the outline given in lectures, write out a complete proof of Theorem 9.7.

3. (a) What is the convolution of the sequence $a_0, a_1, a_2, \ldots$ with the constant sequence $1, 1, 1, \ldots$?

(b) Let $u_0, u_1, u_2, \ldots$ denote the sequence of Fibonacci numbers, as defined by $u_0 = 0$, $u_1 = 1$ and $u_n = u_{n-1} + u_{n-2}$ for $n \geq 2$. Let $v_n = \sum_{k=0}^{n} u_k$. Show that

$$\sum_{n=0}^{\infty} v_n x^n = \frac{x}{(1-x-x^2)(1-x)}.$$

(c) Find constants A, B, C such that

$$\frac{x}{(1-x-x^2)(1-x)} = \frac{Ax+B}{1-x-x^2} + \frac{C}{1-x}.$$

(d) Hence, or otherwise, prove that $v_n = u_{n+2} - 1$.

4. Let $a, b \in \mathbf{N}_0$ and let $m \in \mathbf{N}_0$. By considering the coefficient of x^m in

$$(1+x)^a(1+x)^b = (1+x)^{a+b},$$

give a generating function proof of Vandermonde's convolution identity

$$\sum_{k=0}^{m} \binom{a}{k}\binom{b}{m-k} = \binom{a+b}{m}.$$

5. Let $c, d \in \mathbf{N}$ be coprime.

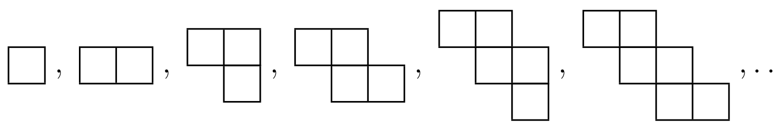
(a) Show that every sufficiently large natural number can be expressed in the form $rc + sd$ with $r, s \in \mathbf{N}_0$.

(b) ($\star$) What is the greatest natural number that cannot be expressed in this form?

6. A Latin square is an $n \times n$ square in which every row and column contains each of the numbers 1, 2, …, n exactly once. Use rook polynomials to find the number of ways to complete the third row of the incomplete Latin square shown below.

1	2	3	4	5
2	3	1	5	4

7. (Problème des Ménages.) Let B_m denote the board with exactly m squares in the sequence shown below.

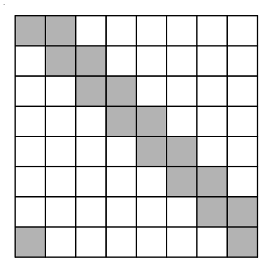


(a) Prove that the rook polynomial of B_m is

$$\sum_k \binom{m-k+1}{k} x^k.$$

[**Corrected from** $\binom{m-k}{k}$ **on 3 November**. *Hint: there is a very short proof using the result on lion caging in Problem 5 of Sheet 1. Alternatively Lemma 7.6 can be used to give an inductive proof.*]

(b) Find the number of ways to place 8 non-attacking rooks on the unshaded squares of the board shown below.



(c) At a dinner party eight married couples are to be seated around a circular table. Men and women must sit in alternate places, and no-one may sit next to their spouse. In how many ways can this be done? [*Hint: first seat the women, then use (b) to count the number of ways to seat the men.*]

8. In an election there are two candidates A and B, each of whom gets exactly n votes. If the votes are counted one at a time, what is the probability that A is never behind B? [*Hint: to get a recurrence relation, try splitting ballot counting sequences such as $ABAABB$ at the first point (after the count starts) when the candidates are tied.*]

MT4540 Combinatorics: Sheet 5

Do questions 1 and 6 and at least two other questions.
To be returned to McCrea 240 by 6pm on Monday 15th November 2010, or handed in at the Monday lecture.

1. Complete the last stage of the proof of Theorem 10.5 by using Theorem 8.5 to show that the coefficient of x^{n+1} in $\frac{-1}{2}\sqrt{1-4x}$ is $\binom{2n}{n}/(n+1)$. **Corrected sign 10th November.**

2. Solve the following recurrence relations:

(a) $a_n = a_{n-1} - a_{n-2} + a_{n+3}$;

(b) $a_n = b + \sum_{k=0}^{n-1} a_k$ where $b \in \mathbf{N}_0$ and $a_0 = 0$.

[*Hint: to prove (b) using generating functions, try taking a convolution of* $a_0, a_1, a_2, \ldots$ *with the constant sequence* $1, 1, 1, \ldots$. *What is the generating function of the resulting sequence?*]

3. Let $r \in \mathbf{N}$ and let $\zeta = \exp(2\pi \mathrm{i}/r)$. Show that if $F(x) = \sum_{n=0}^{\infty} a_n x^n$ then

$$F(x) + F(\zeta x) + F(\zeta^2 x) + \cdots + F(\zeta^{r-1} x) = r \sum_{n=0}^{\infty} a_{nr} x^{nr}.$$

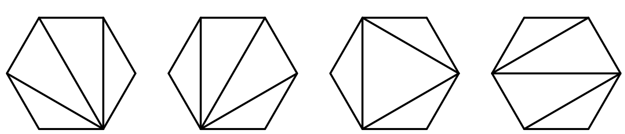
4. Prove that

$$\frac{1}{\sqrt{1-4x}} = \sum_{n=0}^{\infty} \binom{2n}{n} x^n.$$

By squaring both sides prove that

$$\sum_{m=0}^{n} \binom{2m}{m} \binom{2n-2m}{n-m} = 4^n.$$

5. For each $n \geq 3$, let T_n denote the number of ways in which a regular n-gon can be divided into triangles. For example, four of the 14 possible divisions of a hexagon are shown below. (Note that the n-gon sits in a fixed position in the plane: rotations and reflections should *not* be considered in this question.)



(a) Find T_3, T_4 and T_5.

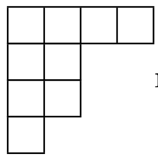
(b) Prove that

$$T_{n+1} = T_n + T_{n-1}T_3 + T_{n-2}T_4 + \cdots + T_3 T_{n-1} + T_n$$

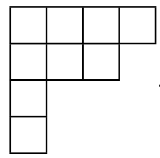
for all $n \geq 3$. Hence prove that $T_n = C_{n-2}$.

6. Which of the 12 partitions of 11 with distinct parts are thick, and which are thin? Check that the maps defined in §12 are mutually inverse bijections between the two classes of partition. What changes for partitions of 12?

7. The *conjugate* of a partition is obtained by reflecting its Young diagram in its major diagonal. For example $(4, 2, 2, 1)$ has conjugate $(4, 3, 1, 1)$ since



reflects to



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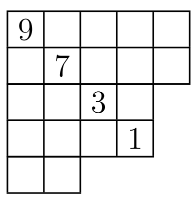
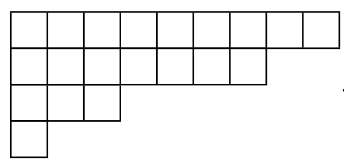
It is usual to write λ' for the conjugate of λ.

(a) Show that λ has exactly k parts if and only if k is the the largest part of λ'.

(b) Show that the number of self-conjugate partitions of n is equal to the number of partitions of n into distinct odd parts. [*Hint: There is a bijective proof based on straightening 'hooks':*



$\longleftrightarrow$



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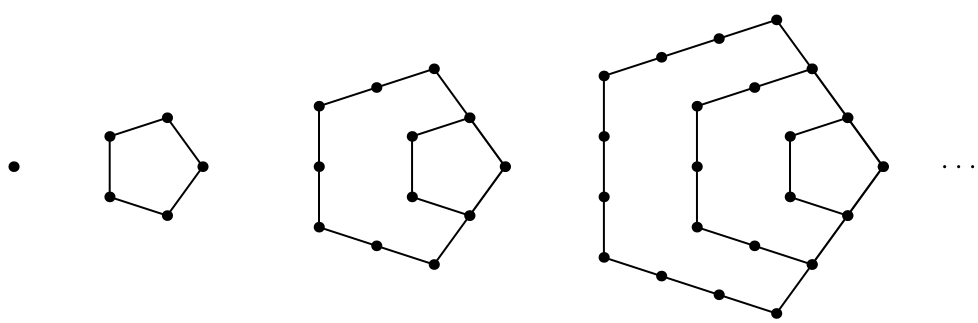
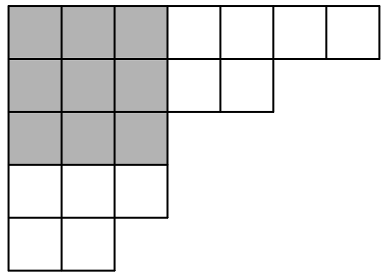
(c) Find a closed form for the generating function for the number of partitions of n that are equal to their conjugate.

8. Show that the number of dots in the n-th figure in the sequence below is $n(3n-1)/2$.



Show that if the sequence $a_n = n(3n - 1)/2$ is defined for all $n \in \mathbf{Z}$ then every pentagonal number is obtained.

9. Given a non-empty partition λ, let $r(\lambda)$ denote the greatest r such that $\lambda_r \geq r$. The *Durfee square* of λ consists of all boxes in the Young diagram of λ that are in both the first $r(\lambda)$ rows and in the first $r(\lambda)$ columns of λ. For example, if $\lambda = (7, 5, 3, 3, 2)$ then $r(\lambda) = 3$ and the Durfee square consists of the shaded boxes in the Young diagram below.



(a) What is the generating function for the number of partitions with largest part of size $\leq r$?

(b) Prove the identity

$$\frac{1}{(1-q)(1-q^2)(1-q^3)\ldots} = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)^2(1-q^2)^2\ldots(1-q^n)^2}.$$

10. A *set partition* of a set X is collection of non-empty disjoint subsets of X whose union is X. The *Bell number* B_n is defined to be the number of set partitions of $\{1, 2, \ldots, n\}$. For example, one of the set partitions counted by B_5 is $\big\{\{1, 4\}, \{2, 5\}, \{3\}\big\}$. There is no simple formula for the Bell numbers, but their exponential generating function has a simple form.

(a) Show by listing set partitions that $B_0 = 1$, $B_1 = 1$, $B_2 = 2$, $B_3 = 5$. Show that $B_4 = 15$.

(b) By consider the subset containing n in a set partition of $\{1, 2, \ldots, n\}$, prove that

$$B_n = \sum_{r=1}^{n} \binom{n-1}{r-1} B_{n-r}$$

for all $n \in \mathbf{N}$. Deduce that

$$\frac{B_n}{(n-1)!} = \sum_{r=1}^{n} \frac{1}{(r-1)!} \frac{B_{n-r}}{(n-r)!}$$

for all $n \in \mathbf{N}$.

(c) Let $F(x) = \sum_{n=0}^{\infty} B_n x^n/n!$ be the exponential generating function for the Bell numbers. Show that $F'(x) = \sum_{n=1}^{\infty} B_n x^n/(n-1)!$ and hence that $F(x) = \exp(\exp(x) - 1)$.

11. This question gives an example of Wilf's 'snake-oil' method for proving identities involving binomial coefficients. For an alternative way to do this problem and more examples, see §4.3 of *generatingfunctionology*.

(a) Use Theorem 8.4 to show that

$$\frac{x^r}{(1-x)^{r+1}} = \sum_{m=0}^{\infty} \binom{m}{r} x^m$$

and deduce that

$$\sum_{m=r}^{\infty} \binom{m-r}{r} x^m = \frac{x^{2r}}{(1-x)^{r+1}}$$

for all $r \in \mathbf{N}_0$.

(b) Let

$$b_m = \binom{m}{0} + \binom{m-1}{1} + \binom{m-2}{2} + \cdots$$

for $m \in \mathbf{N}_0$. Show that

$$\sum_{m=0}^{\infty} b_m x^m = \frac{1}{1-x-x^2}.$$

(c) Hence relate the sequence $b_0, b_1, b_2, \ldots$ to the sequence of Fibonaaci numbers, as defined in Question 3(b) of Sheet 4.

MT4540 Combinatorics: Sheet 6

Do questions 2 and 6 and at least two other questions.
To be returned to McCrea 240 by 6pm on Tuesday 22th November 2010, or handed in at the Tuesday lecture.

Parts of questions marked ($\star$) are optional and slightly harder than the average.

1. Show that there is a two-colouring of K_5 with no monochromatic triangle.

2. Prove that $R(4,4) \leq 18$. You may assume that $R(4,3) \leq 9$.

3. Let $X = \{0, 1, 2, \ldots, 16\}$ be the set of residues mod 17 and let G be the complete graph on X. Given $x, y \in X$ with $x < y$, colour the edge $\{x, y\}$ red if $y - x$ is equal to a square number modulo 17, and blue otherwise. (For example, $\{2, 10\}$ is red because $10 - 2 \equiv 5^2 \bmod 17$.)

(a) Find all square numbers modulo 17.

(b) Show that if $x, y, u \in X$ then $\{x, y\}$, $\{x + u, y + u\}$ and $\{xu^2, yu^2\}$ all have the same colour.

(c) Prove that G has no monochromatic 4-set. [*Hint: use (b) to reduce the number of cases that have to be considered.*]

(d) What does this imply about $R(4, 4)$?

4. Let G be a graph with vertex set $\{1, 2, \ldots, n\}$ and edge set $E(G)$. Let G' be the graph on $\{1, 2, \ldots, n\}$ with edge set $E(G')$ defined by

$$\{i, j\} \in E(G') \quad \text{if and only if} \quad \{i, j\} \notin E(G).$$

(a) Show that at least one of G and G' is connected.

(b) Deduce that in any red-blue colouring of a complete graph, either the red edges form a connected graph, or the blue edges form a connected graph.

(c) Can both G and G' be connected?

5. By comparing $\int_1^n \log x \,\mathrm{d}x$ with $\log n!$, prove that there exist constants A, $B \in \mathbf{R}$ such that

$$A\left(\frac{n}{\mathrm{e}}\right)^n \leq n! \leq B\left(\frac{n}{\mathrm{e}}\right)^{n+1}$$

for all $n \in \mathbf{N}$.

6. Let $s, t \in \mathbf{N}$. By constructing a suitable red-blue colouring of $K_{(s-1)(t-1)}$, prove that

$$R(s,t) > (s-1)(t-1).$$

7. Three applications of the Pigeonhole Principle.

(a) Making any reasonable assumptions, prove that there are two students at British universities whose bank balances agree to the nearest penny.

(b) Prove that if five points are chosen inside an equilateral triangle of size 1 then there is a pair of points whose distance is $\leq 1/2$.

(c) ($\star$) Show that in any sequence of n integers, there is a consecutive subsequence whose sum is divisible by n. (For example, in $1, 4, 5, 1, 2, 2, 1$, the sum of $4, 5, 1, 2, 2$ is divisible by 7.)

8. Let $x_1, x_2, x_3, \ldots, x_N$ be a sequence of distinct integers. Prove that provided N is sufficiently large, there is either an increasing subsequence of length 2010 or a decreasing subsequence of length 2010. [*Hint: given i and j such that $1 \leq i < j \leq n$, colour the edge $\{i, j\}$ of K_N red if $x_i < x_j$ and blue if $x_i > x_j$.*]

9. Let $\ell \in \mathbf{N}$. Say that a partition is *ℓ-regular* if it has at most $\ell - 1$ parts of any given size. For example the partition $(6, 3, 3, 2, 2, 1)$ is 3-regular, but not 2-regular.

(a) Show that the generating function for ℓ-regular partitions is

$$\prod_{k=1}^{\infty}(1 + x^k + x^{2k} + \cdots + x^{(\ell-1)k}).$$

(b) Find the generating function for partitions with no part divisible by ℓ.

(c) Use (a) and (b) to prove a generalization of Theorem 11.6.

10. ($\star$) Here is an elegant way to obtain a fairly strong upper bound on $p(n)$, due to the Dutch mathematician van Lint. Let $P(x) = \sum_{n=0}^{\infty} p(n)x^n$.

(a) Show that

$$\log P(x) = \sum_{r=1}^{\infty} \frac{x^r}{r(1 - x^r)}.$$

(b) By replacing x with e^{-y} prove that $\log P(\mathrm{e}^{-y}) \leq \frac{\pi^2}{6y}$.

(c) Hence show that if $n \in \mathbf{N}$ then $\log p(n) \leq ny + \frac{\pi^2}{6y}$. By making a strategic choice of y, prove that

$$\log p(n) \leq 2\sqrt{\frac{\pi^2}{6}}\sqrt{n}$$

for all $n \in \mathbf{N}$.

MT4540 Combinatorics: Sheet 7

Do question 3 and at least three other questions.
To be returned to McCrea 240 by 6pm on Tuesday 30th November 2010, or handed in at the Tuesday lecture.

Parts of questions marked ($\star$) are optional and slightly harder than the average.

1. Find an explicit n such that if the edges of the complete graph on $\{1, 2, \ldots, n\}$ are coloured red, blue, green, and yellow then there exists a monochromatic K_4.

2. Let $s \in \mathbf{N}$. Let $G(s)$ denote the complete graph on $\{1, 2, \ldots, 3(s-1)-1\}$ coloured so that the edge $\{x, y\}$ is red if $|x - y| \equiv 1 \bmod 3$, and blue if $|x - y| \equiv 0$ or $2 \bmod 3$.

(a) Draw diagrams showing the red and blue edges in $G(2)$ and $G(3)$.

(b) Prove that $G(s)$ has no red K_3.

(c) Suppose that $X \subseteq \{1, 2, \ldots, n\}$ is a blue K_s in $G(s)$. Let $X = \{x_1, \ldots, x_s\}$ where $x_1 < x_2 < \ldots < x_s$. By considering the differences $x_{i+1} - x_i$, or otherwise, get a contradiction.

(d) Deduce that $R(3, s) \geq 3(s-1)$.

3. Suppose that the edges of the complete graph on $\{1, 2, \ldots, 17\}$ are coloured red, blue and green. By looking at the edges coming out of vertex 1 and adapting the argument used several times in lectures, show that there must exist a monochromatic triangle. (This is the alternative way to prove Theorem 15.4.)

4. (a) Use Lemma 15.1 to prove that $R(3, s) \leq s(s+1)/2$ for all $s \in \mathbf{N}$.

(b) Use part (a) together with the result of question 2 to give upper and lower bound for $R(3, 6)$ and $R(3, R(3, 6))$.

5. Given $s, t \in \mathbf{N}$, let $T(s, t)$ denote the smallest n (if one exists) such that whenever the 3-subsets of $\{1, 2, \ldots, n\}$ are coloured red and blue, then *either*

- there is an s-subset $X \subseteq \{1, 2, \ldots, n\}$ such that all 3-subsets of X are red, *or*
- there is a t-subset $Y \subseteq \{1, 2, \ldots, n\}$ such that all 3-subsets of Y are blue.

(a) Prove that $T(3, r) = T(r, 3) = r$ for all $r \in \mathbf{N}$

(b) Prove that $T(4, 4) \leq R(4, 4) + 1 = 19$. [*Hint: to mimic the usual argument, consider the colouring induced on the 2-subsets of* $\{2, 3, \ldots, 19\}$ *by giving* $\{x, y\}$ *the colour of* $\{1, x, y\}$.]

(c) ($\star$) Prove that $T(4, 5) \leq R(5, 19) + 1$.

(d) ($\star$) Give an explicit upper bound for $T(5, 5)$.

6. Let $m, n \in \mathbf{N}$. A platoon of mn soldiers is arranged in m rows of n soldiers. The sergeant rearranges the soldiers in each row in decreasing order of height, and then does the same to the columns.

(a) Show that the tallest soldier is now in the first row and the first column.

(b) Show that the rows are still arranged in decreasing order of height. [*Hint: there is an argument using the pigeonhole principle.*]

7. Prove Lemma 15.6. [*Hint: colour the edge $\{x, y\}$ of the complete graph on $\{1, 2, 3, 4, 5, 6\}$ red if $|x - y| \in Y$, and blue if $|x - y| \in Z$.*]

8. Suppose that we independently roll two fair dice. Let X and Y be the numbers of the two dice, and let $S = X + Y$. Let $Z = |X - Y|$. Find

(i) $\mathbf{E}[S]$ and $\mathbf{E}[Z]$,

(ii) $\mathbf{E}[X|S = 10]$ and $\mathbf{E}[Z|S = 7]$.

9. At the University of Erewhon, whenever any of its n employees has a birthday, the university closes and everyone takes the day off. Apart from this there are no holidays whatsoever. Local laws require that people are appointed without regard to their date of birth (and there are no leap years).

(a) Show that the probability that the university is open on 25th December is $\left(1 - \frac{1}{365}\right)^n$.

(b) Prove, using linearity of expectation, that the expected number of days of the year when the university is open is

$$365\left(1 - \frac{1}{365}\right)^n.$$

(c) The Pro-Vice Chancellor for Administrative Affairs is keen to maximize the number of person-days worked over the year. Advise him on an optimal choice for n.

10. Let $p \in \mathbf{R}$ and let $n \in \mathbf{N}$. Suppose that a coin biased to land heads with probability p is tossed n times. Let X be the number of times the coin lands heads.

(a) Describe a suitable probability space Ω and define X as a function $\Omega \to \mathbf{R}$.

(b) Show that $\mathbf{P}[X = k] = \binom{n}{k} p^k (1 - p)^{n-k}$.

(c) Find $\mathbf{E}[X]$ and $\mathbf{Var}\,[X]$. [*Hint: the calculations can be simplified using linearity of expectation.*]

11. Show that if $m \in \mathbf{N}$ then

$$\binom{2m}{m} = \frac{4^m}{m!}\left(\frac{1}{2} \times \frac{3}{2} \times \cdots \times \frac{2m - 1}{2}\right).$$

Deduce that $\binom{2(s-1)}{s-1} \leq 4^{s-1}$ for all $s \in \mathbf{N}$, as required by Corollary 15.3.

MT4540 Combinatorics: Sheet 8

Do question 1 and at least one other question.
To be returned to McCrea 240 by 6pm on Tuesday 7th December 2010, or handed in at the Tuesday lecture.

The question marked ($\star$) is slightly harder than the average.

1. (a) Show, by counting permutations, that the probability that 1 and 2 lie in the same cycle of a random permutation of $\{1, 2, 3, 4\}$ is $1/2$.

(b) Let $\sigma = (1, 2, 3, 4, 5, 6)$ and let $\tau = (3, 5)$. Write the composition $\tau \circ \sigma$ as a product of disjoint cycles.

2. (a) Let $n \in \mathbf{N}$ and let $1 \leq x < y \leq n$. Let τ be the transposition (x, y). Show that if σ is a permutation of $\{1, 2, \ldots, n\}$ then x and y lie in the same cycle of σ if and only if x and y lie in different cycles of $\tau \circ \sigma$.

(b) Hence, or otherwise, find the probability that 1 and n lie in the same cycle of a randomly chosen permutation of $\{1, 2, \ldots, n\}$.

3. A lion-tamer has n numbered cages, arranged in a line, and k indistinguishable lions. Each cage can accommodate at most one lion.

(a) Let $1 \leq r < n$. If the lion-tamer puts the lions into the cages at random, what is the probability that both cages r and $r + 1$ are occupied?

(b) On average, how many pairs of adjacent cages will both contain lions? [*Hint: use linearity of expectation.*]

4. Let Ω be the probability space of all permutations of $\{1, 2, 3, 4, 5, 6\}$, in which each permutation has probability $1/6!$. Define

$$\begin{aligned} A &= \{\sigma \in \Omega : \sigma(2) < \sigma(1) < \sigma(4)\} \\ B &= \{\sigma \in \Omega : \sigma(6) < \sigma(1) < \sigma(2)\} \\ C &= \{\sigma \in \Omega : \sigma(6) < \sigma(1) < \sigma(4)\}. \end{aligned}$$

(a) Show that $\mathbf{P}[A] = \mathbf{P}[B] = \mathbf{P}[C] = 1/3!$. [*Hint: in a permutation of* $\{1, 2, \ldots, 6\}$*, there are* $3!$ *possible relative orders for* $\sigma(2), \sigma(1), \sigma(4)$.]

(b) Show that $\mathbf{P}[A \cap B] = 0$ and that $\mathbf{P}[A \cap C] = \mathbf{P}[B \cap C] = 2/4!$.

(c) Using the Principle of Inclusion and Exclusion, find the number of ways in which the letters A, B, C, D, E, F may be arranged so that none of the words BAD, FAB, FAD can be obtained by crossing out some of the letters.

5. Let Ω be a probability space and let $X : \Omega \to \mathbf{N}_0$ be a random variable. Prove, using the formula after Definition 16.7, that

$$\mathbf{E}[X] = \sum_{k=1}^{\infty} \mathbf{P}[X \geq k].$$

Deduce *Markov's inequality*, that $\mathbf{P}[X \geq k] \leq \frac{\mathbf{E}[X]}{k}$ for each $k \in \mathbf{N}$.

6. An aircraft has exactly 100 seats for passengers, and 100 people are due to travel on it. The first person in the queue to get on the plane has forgotten their seat number, and so sits in one of the seats at random. The remaining 99 people all know their seat numbers and so if their seat is not taken they sit in it. If their seat is taken, they are too shy to complain and so they sit in a free seat which they choose at random.

Find the probability that person 100 sits in their own seat. [*Hint: Question 2 is relevant.*]

7. ($\star$) In a room there are 100 numbered lockers. Each locker contains a piece of paper numbered between 1 and 100 so that each number is used exactly once. A team of 100 numbered people are let into the room, one at a time in numerical order. Each person is allowed to open up to 50 lockers before leaving the room. If every team members finds the piece of paper with their number on it, the team succeeds, otherwise they fail.

(After each visit the room is returned to its original state, and once someone has visited the room, they cannot communicate with their colleagues.)

Find a good strategy for the team. What is its chance of success?

8. (a) Give a list of all the proofs you have seen that the number of derangements of $\{1, 2, \ldots, n\}$ is

$$n! - \frac{n!}{1!} + \frac{n!}{2!} - \frac{n!}{3!} + \cdots + (-1)^n \frac{n!}{n!}.$$

(A one- or two-line description of each proof will suffice.)

(b) Which proof would you use if a keen first year undergraduate asked you to prove the result? Is this also your favourite proof?

MT4540 Combinatorics: Sheet 9

Do questions 2 and 5 and at least two other questions.
To be returned by Wednesday on the first week of Spring term.

1. Suppose that a permutation of $\{1, 2, \ldots, n\}$ is chosen uniformly at random. Use Theorem 17.4 to find the average length of the cycle containing 1.

2. Let Ω be the probability space of all permutations of $\{1, 2, \ldots, n\}$, where each permutation is equally probable. Fix k such that $1 \leq k \leq n$. For each $x \in \{1, 2, \ldots, n\}$, let

$$Z_x(\sigma) = \begin{cases} 1 & \text{if } x \text{ is in a } k\text{-cycle of } \sigma \\ 0 & \text{otherwise.} \end{cases}$$

Let $Z = \sum_{x=1}^{n} Z_x$.

(a) If $k = 3$ and $\tau = (13)(245)(789)$ find the set of x such that $Z_x(\tau) = 1$. What is $Z(\tau)$?

(b) Show that if $\sigma \in \Omega$ then $Z(\sigma)/k$ is the number of k-cycles in σ.

(c) Use Theorem 17.4 to show that $\mathbf{E}[Z_x] = 1/n$ for each $x \in \{1, 2, \ldots, n\}$. Hence show that $\mathbf{E}[Z] = 1$.

(d) Use (b) and (c) to prove Theorem 17.7, i.e. if $1 \leq k \leq n$ then the average number of k-cycles in a permutation of $\{1, 2, \ldots, n\}$ is $1/k$.

3. Let $H_n = 1 + 1/2 + 1/3 + \cdots + 1/n$. Use Theorem 17.7 to show that the average number of cycles in a permutation of $\{1, 2, \ldots, n\}$ is H_n. [*Hint: use linearity of expectation.*] Check your answer directly for some small values of n.

4. Let Ω be a probability space. The *probability generating function* for a random variable $X : \Omega \to \mathbf{N}_0$ is defined by

$$f_X(t) = \sum_{n=0}^{\infty} \mathbf{P}[X = n] t^n.$$

(a) Show that $\mathbf{E}[X] = f_X'(1)$ and that $\mathbf{Var}[X] = f_X''(1) + f_X'(1) - f_X'(1)^2$.

(b) Show that if $X, Y : \Omega \to \mathbf{N}_0$ are random variables then $f_{X+Y} = f_X f_Y$.

(c) Suppose that an coin biased to land heads with probability p is flipped n times. Let Z be the number of heads obtained. Show that the probability generating function of Z is $((1-p) + pt)^n$. Hence find $\mathbf{E}[Z]$ and $\mathbf{Var}[Z]$.

5. Suppose that the edges of the complete graph on $\{1, 2, \ldots, n\}$ are coloured red, blue and green. Adapt the proof of Theorem 18.5 to show that if

$$\binom{n}{s} 3^{1-\binom{s}{2}} < 1$$

then there is a colouring with no monochromatic K_s. If $s = 10$, what is the largest n that can be taken?

6. Let $n \in \mathbf{N}$ and let G be the complete graph on $\{1, 2, \ldots, n\}$. Suppose that a subset A of $\{1, 2, \ldots, n\}$ is chosen uniformly at random. What is the probability that the cut given by A and $\{1, 2, \ldots, n\} \backslash A$ has capacity $\geq \frac{n(n-1)}{4}$?

7. Let p_n be the probability that a permutation of $\{1, 2, \ldots, n\}$ chosen uniformly at random is a derangement. Assuming the recurrence

$$p_n = \frac{p_{n-1}}{n} + \frac{p_{n-2}}{n} + \cdots + \frac{p_1}{n} + \frac{p_0}{n}$$

proved in Lemma 17.5, write out a careful proof, following the usual three step plan, that

$$p_n = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}.$$

8. Prove Theorem 14.3. [For many further applications of the Pigeonhole Principle, see `www.cut-the-knot.org/do_you_know/pigeon.shtml`.]

9. Let $n, s \in \mathbf{N}$. Prove that

$$\binom{n}{s} 2^{1-\binom{s}{2}} \leq \left(\frac{n}{2^{(s-1)/2}}\right)^s \frac{2}{s!}.$$

Hence deduce Corollary 18.6 from Theorem 18.5. [**Corrected $=$ to $\leq$ 21 December.**]

10. There are 10 pirates who have recently acquired a bag with 100 coins. The leader, number 1, must propose a way to divide up the loot. For instance he might say 'I'll take 91 coins and the rest of you can have one each'. A vote is then taken. If the leader gets *half or more* of the votes (the leader getting one vote himself), the loot is so divided. Otherwise he is made to walk the plank by his dissatisfied subordinates, and number 2 takes over, with the same responsibility to propose an acceptable division.

Assuming that the pirates are all greedy, untrustworthy, and capable mathematicians, what happens?

[*Hint: try thinking about a smaller* 2 *or* 3 *pirate problem to get started.*]