MT454 / MT5454 Combinatorics

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(A) Enumeration

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(B) **Generating Functions**: Recurrences and applications to enumeration. Problem sheets will ask you to read the early sections of H. S. Wilf, *generatingfunctionology*.

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(C) Ramsey Theory: 'Complete disorder is impossible'.

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(B) **Generating Functions**: Recurrences and applications to enumeration. Problem sheets will ask you to read the early sections of H. S. Wilf, *generatingfunctionology*.

(C) **Ramsey Theory**: 'Complete disorder is impossible'.

(D) **Probabilistic Methods**: counting *via* discrete probability, lower bounds in Ramsey theory.

Recommended Reading

[1] *A First Course in Combinatorial Mathematics*. Ian Anderson, OUP 1989, second edition.

[2] *Discrete Mathematics.* N. L. Biggs, OUP 1989.

[3] *Combinatorics: Topics, Techniques, Algorithms.* Peter J. Cameron, CUP 1994.

[4] *Concrete Mathematics.* Ron Graham, Donald Knuth and Oren Patashnik, Addison-Wesley 1994.

[5] *Invitation to Discrete Mathematics.* Jiri Matoušek and Jaroslav Nešetřil, OUP 2009, second edition.

[6] *Probability and Computing: Randomized Algorithms and Probabilistic Analysis.* Michael Mitzenmacher and Eli Upfal, CUP 2005.

[7] *generatingfunctionology*. Herbert S. Wilf, A K Peters 1994, second / third edition. Second edition available from `http://www.math.upenn.edu/~wilf/DownldGF.html`.

Permutations

Definition 2.1

A *permutation* of a set X is a bijective function

$$\sigma : X \to X.$$

A *fixed point* of a permutation σ of X is an element $x \in X$ such that $\sigma(x) = x$. A permutation is a *derangement* if it has no fixed points.

Exercise: For $n \in \mathbf{N}_0$, how many permutations are there of $\{1, 2, \ldots, n\}$? How many of these permutations have 1 as a fixed point?

Derangements

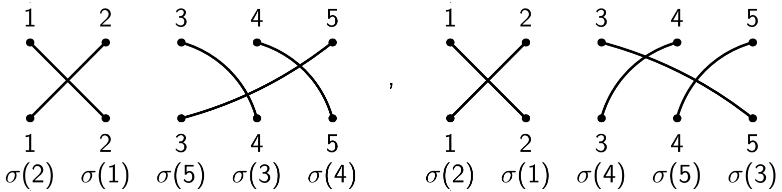
Recall that a derangement is a permutation with no fixed points.

Problem 2.2 (Derangements)

Let X be a set of size n. How many permutations of X are derangements?

Exercise: Show that there are two derangements σ of $\{1, 2, 3, 4, 5\}$ such that $\sigma(1) = 2$ and $\sigma(2) = 1$, but there are three derangements such that $\sigma(1) = 2$ and $\sigma(2) = 3$.

On Monday we found the two derangements such that $\sigma(1) = 2$ and $\sigma(2) = 1$. They are shown below as diagrams, where i in the top row is joined to $\sigma(i)$ in the bottom row.



Derangements: An Ad-hoc Solution

Let d_n be the number of permutations of $\{1, 2, \ldots, n\}$ that are derangements. By definition, although you may regard this as a convention, if you prefer, $d_0 = 1$.

Lemma 2.3

If $n \geq 2$, there are $d_{n-2} + d_{n-1}$ derangements σ of $\{1, 2, \ldots, n\}$ such that $\sigma(1) = 2$.

Theorem 2.4

If $n \geq 2$ then $d_n = (n-1)(d_{n-2} + d_{n-1})$.

Corollary 2.5

For all $n \in \mathbf{N}_0$,

$$d_n = n!\Big(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\Big).$$

Student-Staff Committee Elections

- One student taking the MSci course will be elected today.
- A later election will be held to elect an MSc student.

Two Probabilistic Results on Derangements

Theorem 2.6

(i) *The probability that a permutation of* $\{1, 2, \ldots, n\}$*, chosen uniformly at random, is a derangement tends to* $1/\mathrm{e}$ *as* $n \to \infty$*.*

(ii) *The average number of fixed points of a permutation of* $\{1, 2, \ldots, n\}$ *is* 1*.*

We'll prove more results like these in Part D of the course.

Part A: Enumeration

§2: Binomial Coefficients and Counting Problems

Notation 3.1

If Y is a set of size k then we say that Y is a *k-set*, and write $|Y| = k$. To emphasise that Y is a subset of some other set X then we may say that Y is a *k-subset of X*.

We shall define binomial coefficients combinatorially.

Definition 3.2

Let n, $k \in \mathbf{N}_0$. Let $X = \{1, 2, \ldots, n\}$. The *binomial coefficient* $\binom{n}{k}$ is the number of k-subsets of X.

Bijective Proofs

We should prove that the combinatorial definition agrees with the usual one.

Lemma 3.3

If n, $k \in \mathbf{N}_0$ *and* $k \leq n$ *then*

$$\binom{n}{k} = \frac{n(n-1)\dots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}.$$

Many of the basic properties of binomial coefficients can be given combinatorial proofs involving explicit bijections. We shall say that such proofs are *bijective*.

Lemma 3.4

If n, $k \in \mathbf{N}_0$ *then*

$$\binom{n}{k} = \binom{n}{n-k}.$$

More Bijective Proofs

Lemma 3.5 (Fundamental Recurrence)

If $n, k \in \mathbf{N}$ then

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

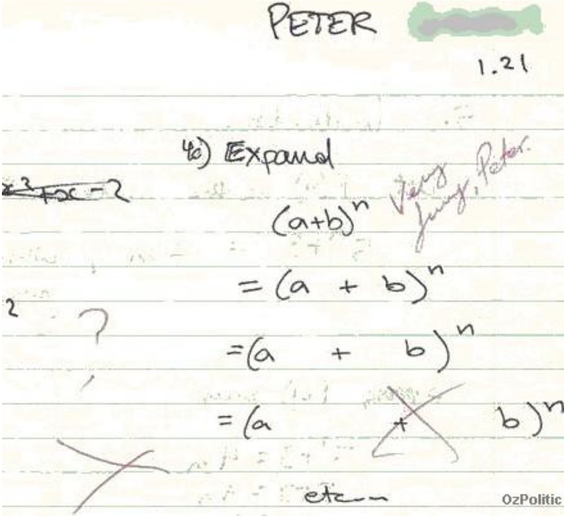
Binomial coefficients are so-named because of the famous binomial theorem. (A binomial is a term of the form $x^r y^s$.)

Theorem 3.6 (Binomial Theorem)

Let $x, y \in \mathbf{C}$. If $n \in \mathbf{N}_0$ then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

How Not to Expand $(x + y)^n$



Recommended Exercises

Exercise: give inductive or algebraic proofs of the previous three results.

Exercise: in New York, how many ways can one start at a junction and walk to another junction 4 blocks away to the east and 3 blocks away to the north?

Balls and Urns

How many ways are there to put k balls into n numbered urns? The answer depends on whether the balls are distinguishable. We may consider urns of unlimited capacity, or urns that can only contain one ball.

	Numbered balls	Indistinguishable balls
≤ 1 ball per urn		
unlimited capacity		

Balls and Urns

How many ways are there to put k balls into n numbered urns?
The answer depends on whether the balls are distinguishable. We may consider urns of unlimited capacity, or urns that can only contain one ball.

	Numbered balls	Indistinguishable balls
≤ 1 ball per urn	$n(n-1)\dots(n-k+1)$	
unlimited capacity		

Balls and Urns

How many ways are there to put k balls into n numbered urns? The answer depends on whether the balls are distinguishable. We may consider urns of unlimited capacity, or urns that can only contain one ball.

	Numbered balls	Indistinguishable balls
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Balls and Urns

How many ways are there to put k balls into n numbered urns? The answer depends on whether the balls are distinguishable. We may consider urns of unlimited capacity, or urns that can only contain one ball.

	Numbered balls	Indistinguishable balls
≤ 1 ball per urn	$n(n-1)\ldots(n-k+1)$	$\binom{n}{k}$
unlimited capacity	n^k	

Balls and Urns

How many ways are there to put k balls into n numbered urns? The answer depends on whether the balls are distinguishable. We may consider urns of unlimited capacity, or urns that can only contain one ball.

	Numbered balls	Indistinguishable balls
≤ 1 ball per urn	$n(n-1)\dots(n-k+1)$	$\binom{n}{k}$
unlimited capacity	n^k	$\binom{n+k-1}{k}$

Unnumbered Balls, Urns of Unlimited Capacity

Theorem 3.7

Let $n \in \mathbf{N}$ and let $k \in \mathbf{N}_0$. The number of ways to place k indistinguishable balls into n urns of unlimited capacity is $\binom{n+k-1}{k}$.

The following reinterpretation of this result can be useful.

Corollary 3.8

Let $n \in \mathbf{N}$ and let $k \in \mathbf{N}_0$. The number of solutions to the equation

$$x_1 + x_2 + \cdots + x_n = k$$

with $x_1, x_2, \ldots, x_n \in \mathbf{N}_0$ is $\binom{n+k-1}{k}$.

Inductive Proof of Theorem 3.7

$n \backslash k$	0	1	2	3	4	$\ldots$
1						
2						
3						
4						
5						
$\vdots$						

Inductive Proof of Theorem 3.7

$n \backslash k$	0	1	2	3	4	$\ldots$
1	1	1	1	1	1	
2	1					
3	1					
4	1					
5	1					
$\vdots$						

Inductive Proof of Theorem 3.7

$n \backslash k$	0	1	2	3	4	...
1	1	1	1	1	1	
2	1	2				
3	1					
4	1					
5	1					
$\vdots$						

Inductive Proof of Theorem 3.7

$n \backslash k$	0	1	2	3	4	...
1	1	1	1	1	1	
2	1	2	3			
3	1	3				
4	1					
5	1					
⋮						

Inductive Proof of Theorem 3.7

$n \backslash k$	0	1	2	3	4	...
1	1	1	1	1	1	
2	1	2	3	4		
3	1	3	6			
4	1	4				
5	1					
$\vdots$						

Inductive Proof of Theorem 3.7

$n \backslash k$	0	1	2	3	4	...
1	1	1	1	1	1	
2	1	2	3	4	5	
3	1	3	6	10		
4	1	4	10			
5	1	5				
$\vdots$						

Inductive Proof of Theorem 3.7

$n \backslash k$	0	1	2	3	4	$\ldots$
1	1	1	1	1	1	
2	1	2	3	4	5	
3	1	3	6	10	15	
4	1	4	10	20		
5	1	5	15			
$\vdots$						

Inductive Proof of Theorem 3.7

$n \backslash k$	0	1	2	3	4	...
1	1	1	1	1	1	
2	1	2	3	4	5	
3	1	3	6	10	15	
4	1	4	10	20	35	
5	1	5	15	35		
⋮						

Inductive Proof of Theorem 3.7

$n \backslash k$	0	1	2	3	4	...
1	1	1	1	1	1	
2	1	2	3	4	5	
3	1	3	6	10	15	
4	1	4	10	20	35	
5	1	5	15	35	70	
⋮						

§4: Further Binomial Identities

Arguments with subsets

Lemma 4.1 (Subset of a subset)

If k, r, $n \in \mathbf{N}_0$ and $k \le r \le n$ then

$$\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}.$$

Lemma 4.2 (Vandermonde's convolution)

If a, $b \in \mathbf{N}_0$ and $m \in \mathbf{N}_0$ then

$$\sum_{k=0}^{m} \binom{a}{k}\binom{b}{m-k} = \binom{a+b}{m}.$$

Corollaries of the Binomial Theorem

Corollary 4.3

If $n \in \mathbf{N}$ then

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n,$$
$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^{n-1}\binom{n}{n-1} + (-1)^n\binom{n}{n} = 0.$$

Corollary 4.4

For all $n \in \mathbf{N}$ there are equally many subsets of $\{1, 2, \ldots, n\}$ of even size as there are of odd size.

Corollary 4.5

If $n \in \mathbf{N}_0$ and $b \in \mathbf{N}$ then

$$\binom{n}{0}b^n + \binom{n}{1}b^{n-1} + \cdots + \binom{n}{n-1}b + \binom{n}{n} = (1+b)^n.$$

Some Identities Visible in Pascal's Triangle

Lemma 4.6 (Alternating row sums)

If $n \in \mathbf{N}$, $r \in \mathbf{N}_0$ and $r \leq n$ then

$$\sum_{k=0}^{r} (-1)^k \binom{n}{k} = (-1)^r \binom{n-1}{r}.$$

Perhaps surprisingly, there is no simple formula for the unsigned row sums $\sum_{k=0}^{r} \binom{n}{k}$.

Lemma 4.7 (Diagonal sums, a.k.a. parallel summation)

If $n \in \mathbf{N}$, $r \in \mathbf{N}_0$ then

$$\sum_{k=0}^{r} \binom{n+k}{k} = \binom{n+r+1}{r}$$

§5: Principle of Inclusion and Exclusion

Example 5.1

If A, B, C are subsets of a finite set X then

$$|A \cup B| = |A| + |B| - |A \cap B|$$
$$|\overline{A \cup B}| = |X| - |A| - |B| + |A \cap B|$$

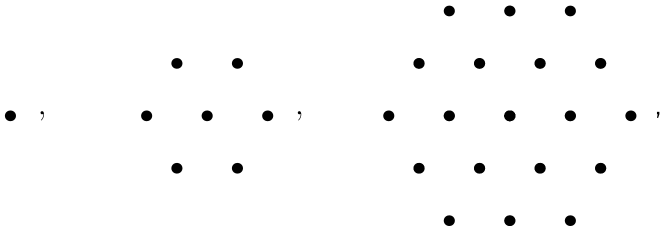
and

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| \\ &\quad - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C| \\ |\overline{A \cup B \cup C}| &= |X| - |A| - |B| - |C| \\ &\quad + |A \cap B| + |B \cap C| + |C \cap A| - |A \cap B \cap C| \end{aligned}$$

Hexagonal Numbers

Example 5.2

The formula for $|A \cup B \cup C|$ gives a nice way to find a formula for the (centred) hexagonal numbers.



It is easier to find the sizes of the intersections of the three rhombi making up each hexagon than it is to find the sizes of their unions. Whenever this situation occurs, the PIE is likely to work well.

Principle of Inclusion and Exclusion

In general we have finite universe set X and subsets $A_1, A_2, \ldots, A_n \subseteq X$. For each non-empty subset $I \subseteq \{1, 2, \ldots, n\}$ we define

$$A_I = \bigcap_{i \in I} A_i.$$

By convention we set $A_\emptyset = X$.

Theorem 5.3 (Principle of Inclusion and Exclusion)

If $A_1, A_2, \ldots, A_n$ are subsets of a finite set X then

$$|\overline{A_1 \cup A_2 \cup \cdots \cup A_n}| = \sum_{I \subseteq \{1,2,\ldots,n\}} (-1)^{|I|} |A_I|.$$

Exercise: Check that Theorem 5.3 holds when $n = 1$ and check that it agrees with Example 5.1 when $n = 2$ and $n = 3$.

Application: Counting Prime Numbers

Example 5.5

Let $X = \{1, 2, \ldots, 48\}$. We define three subsets of X:

$$B(2) = \{m \in X,\ m \text{ is divisible by } 2\}$$
$$B(3) = \{m \in X,\ m \text{ is divisible by } 3\}$$
$$B(5) = \{m \in X,\ m \text{ is divisible by } 5\}$$

Any composite number ≤ 48 is divisible by either 2, 3 or 5. So

$$\overline{B(2) \cup B(3) \cup B(5)} = \{1\} \cup \{p : 5 < p \leq 48,\ \ p \text{ is prime}\}.$$

Application: Counting Derangements

Let $n \in \mathbf{N}$. Let X be the set of all permutations of $\{1, 2, \ldots, n\}$ and let

$$A_i = \{\sigma \in X : \sigma(i) = i\}.$$

To apply the PIE to count derangements we need this lemma.

Lemma 5.4

(i) *A permutation $\sigma \in X$ is a derangement if and only if*

$$\sigma \in \overline{A_1 \cup A_2 \cup \cdots \cup A_n}.$$

(ii) *If $I \subseteq \{1, 2, \ldots, n\}$ then A_I consists of all permutations of $\{1, 2, \ldots, n\}$ which fix the elements of I. If $|I| = k$ then*

$$|A_I| = (n-k)!.$$

Counting Prime numbers

Lemma 5.5

Let r, $M \in \mathbf{N}$. There are exactly $\lfloor M/r \rfloor$ numbers in $\{1, 2, \ldots, M\}$ that are divisible by r.

Theorem 5.6

Let p_1, ..., p_n be distinct prime numbers and let $M \in \mathbf{N}$. The number of natural numbers $\leq M$ that are not divisible by any of primes $p_1, \ldots, p_n$ is

$$\sum_{I \subseteq \{1,2,\ldots,n\}} (-1)^{|I|} \left\lfloor \frac{M}{\prod_{i \in I} p_i} \right\rfloor .$$

Example 5.7

Let $M = pqr$ where p, q, r are distinct prime numbers. The numbers of natural numbers $\leq pqr$ that are coprime to M is

$$M\Big(1 - \frac{1}{p}\Big)\Big(1 - \frac{1}{q}\Big)\Big(1 - \frac{1}{r}\Big).$$

§6: Rook Polynomials

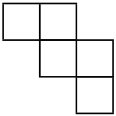
Definition 6.1

A *board* is a subset of the squares of an $n \times n$ grid. Given a board B, we let $r_k(B)$ denote the number of ways to place k rooks on B, so that no two rooks are in the same row or column. Such rooks are said to be *non-attacking*. The *rook polynomial* of B is defined to be

$$f_B(x) = r_0(B) + r_1(B)x + r_2(B)x^2 + \cdots + r_n(B)x^n.$$

Example 6.2

The rook polynomial of the board B below is $1 + 5x + 6x^2 + x^3$.



Examples

Exercise: Let B be a board. Check that $r_0(B) = 1$ and that $r_1(B)$ is the number of squares in B.

Example 6.3

After the recent spate of cutbacks, only four professors remain at the University of Erewhon. Prof. W can lecture courses 1 or 4; Prof. X is an all-rounder and can lecture 2, 3 or 4; Prof. Y refuses to lecture anything except 3; Prof. Z can lecture 1 or 2. If each professor must lecture exactly one course, how many ways are there to assign professors to courses?

Examples

Exercise: Let B be a board. Check that $r_0(B) = 1$ and that $r_1(B)$ is the number of squares in B.

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Example 6.4

How many derangements σ of $\{1,2,3,4,5\}$ have the property that $\sigma(i) \neq i + 1$ for $1 \leq i \leq 4$?

Square Boards

Lemma 6.5

The rook polynomial of the $n \times n$-board is

$$\sum_{k=0}^{n} k! \binom{n}{k}^2 x^k.$$

Administration

- You should find a circled letter at the top of your work. This is for use as an anonymous identifier.
- Please take Problem Sheet 3 if you don't already have it. The deadline is next Friday. Question 2 will be peer-marked: please mark it over next weekend, and return **to the lecturer** in the Monday lecture. A detailed marking scheme will be issued.

Administration

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- Please take Problem Sheet 3 if you don't already have it. The deadline is next Friday. Question 2 will be peer-marked: please mark it over next weekend, and return **to the lecturer** in the Monday lecture. A detailed marking scheme will be issued.
- Problem Sheet 4 is now available from Moodle: see Question 6 for the rook polynomials solution to the *Probléme des Menages*.

Lemmas for Calculating Rook Polynomials

Lemma 6.6

Let B be a board. Suppose that the squares in B can be partitioned into sets C and D so that no square in C lies in the same row or column as a square of D. Then

$$f_B(x) = f_C(x) f_D(x).$$

Lemma 6.7

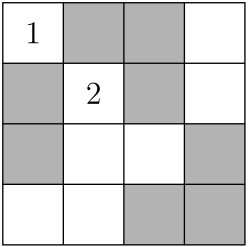
Let B be a board and let s be a square in B. Let C be the board obtained from B by deleting s and let D be the board obtained from B by deleting the entire row and column containing s. Then

$$f_B(x) = f_C(x) + x f_D(x).$$

Example of Lemma 6.7

Example 6.8

The rook-polynomial of the boards in Examples 6.3 and 6.4 can be found using Lemma 6.7. For the board in Example 6.3 it works well to apply the lemma first to the square marked 1, then to the square marked 2 (in the new boards).



Placements on the Complement

Lemma 6.9

Let B be a board contained in an $n \times n$ grid and let $0 \leq k \leq n$ The number of ways to place k red rooks on B and $n-k$ blue rooks anywhere on the grid, so that the n rooks are non-attacking, is $r_k(B)(n-k)!$.

Theorem 6.10

Let B be a board contained in an $n \times n$ grid. Let $\bar{B}$ denote the board formed by all the squares in the grid that are not in B. The number of ways to place n non-attacking rooks on $\bar{B}$ is

$$n! - (n-1)!\, r_1(B) + (n-2)!\, r_2(B) - \cdots + (-1)^n r_n(B).$$

Part B: Generating Functions

§7: Introduction to Generating Functions

Definition 7.1

The *ordinary generating function* associated to the sequence $a_0, a_, a_2, \ldots$ is the power series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots .$$

Usually we shall drop the word 'ordinary' and just write 'generating function'.

The sequences we deal with usually have integer entries, and so the coefficients in generating functions will usually be integers.

Sums and Products of Formal Power Series

Let $F(x) = \sum_{n=0}^{\infty} a_n x^n$ and $G(x) = \sum_{n=0}^{\infty} b_n x^n$. Then

- $F(x) + G(x) = \sum_{n=0}^{\infty}(a_n + b_n)x^n$
- $F(x)G(x) = \sum_{n=0}^{\infty} c_n x^n$ where $c_n = \sum_{m=0}^{n} a_m b_{n-m}$.
- $F'(x) = \sum_{n=0}^{\infty} n x^{n-1}$.

It is also possible to define the reciprocal $1/F(x)$ whenever $a_0 \neq 0$. By far the most important case is the case $F(x) = 1 - x$, when

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

is the usual formula for the sum of a geometric progression.

Analytic and Formal Interpretations.

We can think of a generating function $\sum_{n=0}^{\infty} a_n x^n$ in two ways. Either:

- As a formal power series with x acting as a place-holder. This is the 'clothes-line' interpretation (see Wilf *generatingfunctionology*, page 4), in which we regard the power-series merely as a convenient way to display the terms in our sequence.

- As a function of a real or complex variable x convergent when $|x| < r$, where r is the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$.

Administration

- Please hand in your answers to Question 2 on Sheet 3 **now**.
- Please hand in your answers to the other questions at the end of the lecture.
- Mark answers to Question 2 using the supplied marking scheme. Put your reviewer letter at the top, and return to the lecturer at the Monday lecture.

  You will get your work back for Question 2 on Tuesday, and for other questions on Monday.

Administration

- Please return marked answers to Sheet 3, Question 2. (You will get your own work back tomorrow.)
- Pure Mathematics Seminars this week are on combinatorial topics and should be relatively accessible:
  - Tuesday 2pm, C219, Steven Noble, The Merino-Welsh conjecture: an inequality for Tutte polynomials
  - Wednesday 3pm, ABLT2, Sergei Chmutov: Beraha numbers and graph polynomials

Examples

Example 7.2

How many ways are there to tile a $2 \times n$ path with bricks that are either 1×2 or 2×1?

Example 7.3

Let $k \in \mathbf{N}$. How many n-tuples $(x_1, \ldots, x_n)$ are there such that $x_i \in \mathbf{N}_0$ for each i and $x_1 + \cdots + x_n = k$? (Such an n-tuple is called a *composition* of k.)

To complete the example we need the power series

$$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k$$

found on Question 5 of Sheet 3.

General Binomial Theorem

Theorem 7.4

If $\alpha \in \mathbf{R}$ *then*

$$(1+y)^\alpha = \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)\dots(\alpha-(n-1))}{n!} y^n$$

for all y *such that* $|y| < 1$*.*

Exercise: Let $\alpha \in \mathbf{Z}$.

(i) Show that if $\alpha \geq 0$ then Theorem 7.4 agrees with the Binomial Theorem for integer exponents, proved in Theorem 3.6.

(ii) Show that if $\alpha < 0$ then Theorem 7.4 agrees with Question 5 on Sheet 3. (Substitute $-x$ for y.)

§8: Recurrence Relations and Asymptotics

Three step programme for solving recurrences:

(a) Use the recurrence to write down an equation satisfied by the generating function $F(x) = \sum_{n=0}^{\infty} a_n x^n$;

(b) Solve the equation to get a closed form for the generating function;

(c) Use the closed form for the generating function to find a formula for the coefficients.

Example 8.1

Will solve (i) using generating functions and perform step (a) of the three-step programme (ii).

(i) $a_{n+2} = 5a_{n+1} - 6a_n$ for $n \in \mathbf{N}$

(ii) $b_r = 3b_{r-1} - 4b_{r-3}$ for $r \geq 3$.

Partial Fractions

Theorem 8.2

Let $f(x)$ *and* $g(x)$ *be polynomials with* $\deg f < \deg g$*.*

(i) *If* $g(x) = (x - \beta_1)(x - \beta_2) \dots (x - \beta_k)$ *where* $\beta_1, \beta_2, \dots, \beta_k$ *are distinct non-zero complex numbers, then there exist* $C_1, \dots, C_k \in \mathbf{C}$ *such that*

$$\frac{f(x)}{g(x)} = \frac{C_1}{1 - x/\beta_1} + \cdots + \frac{C_k}{1 - x/\beta_k}.$$

(ii) *If* $g(x) = \alpha(x - \beta_1)^{d_1}(x - \beta_2)^{d_2} \dots (x - \beta_k)^{d_k}$ *where* $\alpha, \beta_1, \beta_2, \dots, \beta_k$ *are distinct non-zero complex numbers and* $d_1, d_2, \dots, d_k \in \mathbf{N}$*, then there exist polynomials* $P_1, \dots, P_k$ *such that* $\deg P_i < d_i$ *and*

$$\frac{f(x)}{g(x)} = \frac{P_1(1 - x/\beta_1)}{(1 - x/\beta_1)^{d_1}} + \cdots + \frac{P_k(1 - x/\beta_k)}{(1 - x/\beta_k)^{d_k}}$$

where $P_i(1 - x/\beta_i)$ *is* P_i *evaluated at* $1 - x/\beta_i$*.*

More examples

Example 8.3

As an example of Theorem 8.2(ii), will finish steps (b) and (c) of the three-step programme on the recurrence in Example 8.1, $b_r = 3b_{r-1} - 4b_{r-3}$ for $r \geq 3$, with initial values $b_0 = 1$, $b_1 = 1$, $b_2 = 0$.

Derangements

Theorem 8.4

Let $p_n = d_n/n!$ be the probability that a permutation of $\{1, 2, \ldots, n\}$, chosen uniformly at random, is a derangement. Then

$$np_n = (n-1)p_{n-1} + p_{n-2}$$

for all $n \geq 2$ and

$$p_n = 1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{(-1)^n}{n!}.$$

Asymptotics

Definition 8.5

Given a sequence a_0, a_1, a_2, ... of real numbers and a function $t : \mathbf{R} \to \mathbf{R}$, we write $a_n = \mathrm{O}\big(t(n)\big)$ if there exists a constant $B \in \mathbf{R}$ such that $|a_n| < Bt(n)$ for all $n \in \mathbf{N}_0$.

Theorem 8.6

Let $F(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function for the sequence $a_0, a_1, a_2, \ldots$. Suppose that $F(x) = f(x)/g(x)$ where $f(x)$ and $g(x)$ are polynomials and $\deg f < \deg g$. If β is the root of $g(x)$ of minimum modulus then

$$a_n = \mathrm{O}\Big(\big(\frac{1}{|\beta|} + \epsilon\big)^n\Big)$$

for all $\epsilon > 0$.

Extra Non-Examinable Example: Dice Throwing

Problem 8.7

Let $d \in \mathbf{N}$. A d-sided die is rolled repeatedly until the total of all the throws is greater than or equal to 100. What is a good approximation to the expected value of the final total?

Extra Non-Examinable Example: Dice Throwing

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Let $d \in \mathbf{N}$. A d-sided die is rolled repeatedly until the total of all the throws is greater than or equal to 100. What is a good approximation to the expected value of the final total?

Here are some data gathered by solving the recurrence on a computer. (All figures are correct to 3 decimal places.)

d	1	2	3	4	5
$a_0 = b_{100+d-1}$	0.000	0.333	0.667	1.000	1.333

d	6	7	8	9	10
$a_0 = b_{100+d-1}$	1.667	2.000	2.333	2.667	3.000

Extra Non-Examinable Example: Dice Throwing

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d	6	7	8	9	10
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The generating function $F(x) = \sum_{n=0}^{\infty} b_n x^n$ will have an expression as a quotient $f(x)/g(x)$ where $f(x)$ and $g(x)$ are polynomials. What is $f(x)$?

Is there an intuitive explanation for the answer $(d-1)/3$?

Three-Step Programme for Dice Problem (Summary)

Step (a). Shifting by powers of x in the usual way gives

$$F(x) - \frac{1}{d}(x + x^2 + \cdots + x^d)F(x) = R(x)$$

where $R(x)$ is a polynomial. (This much can be seen by hand!) Computer algebra is helpful to show that

$$R(x) = \sum_{n=0}^{d-1}\Big(d - 1 - 2n - \frac{n(n+1)}{2}\Big)x^n.$$

Step (b). Hence $F(x) = \dfrac{dR(x)}{d - x - x^2 - \cdots - x^d}$.

Step (c). Let $S(x) = d - x - x^2 - \cdots - x^d$. The root of $S(x)$ of minimum modulus is 1, and all other roots have modulus > 1. By Theorem 8.2,

$$F(x) = \frac{C}{1-x} + Q(x)$$

where $Q(x)$ is the sum of the partial fraction summands from the other roots of $S(x)$. Asymptotically $Q(x)$ is unimportant. So the limiting value of b_n as $n \to \infty$ is C, which standard methods show is $(d-1)/3$.

§9: Convolutions and the Catalan Numbers

Definition 9.1

The convolution of the sequences a_0, a_1, a_2, $\ldots$ and b_0, b_1, b_2, $\ldots$ is the sequence c_0, c_1, c_2,... defined by

$$c_n = \sum_{m=0}^{n} a_m b_{n-m}.$$

Lemma 9.2

Let $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$ be sequences and let $c_0, c_1, c_2, \ldots$ be their convolution. Let $F(x) = \sum_{n=0}^{\infty} a_n x^n$, $G(x) = \sum_{n=0}^{\infty} b_n x^n$ and $H(x) = \sum_{n=0}^{\infty} c_n x^n$. Then

$$F(x)G(x) = H(x).$$

Example 9.3

A resident of Flatland (see *Flatland: A Romance of Many Dimensions*, Edwin A. Abbott 1884) is given an enormous number of indistinguishable 1×1 square bricks for his birthday. How many ways can he make a 'T' shape, using at least one brick for the vertical section and at least two bricks for the horizontal section?

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Exercise: suppose instead an 'C' shape is required, made up out of one vertical section of length ≥ 3, and two horizontal sections *of equal length* ≥ 2. Let c_n be the number of 'C's made using n bricks. Find a closed form for $C(x) = \sum_{n=0}^{\infty} c_n x^n$.

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A: $G(x) = \dfrac{x^7}{(1-x)^3}$

B: $\dfrac{x^5}{(1-x)^3}$

C: $G(x) = \dfrac{x^7}{(1-x)(1-x^2)}$

D: $\dfrac{x^5}{(1-x)(1-x^2)}$

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C: $G(x) = \dfrac{x^7}{(1-x)(1-x^2)}$

D: $\dfrac{x^5}{(1-x)(1-x^2)}$

How would your answers to both parts change if bricks came in two colours (but were otherwise still indistinguishable)?

Rooted binary trees

Definition 9.4

A rooted binary tree is either empty, or consists of a *root vertex* together with a pair of rooted binary trees: a *left subtree* and a *right subtree*. The *Catalan number* C_n is the number of rooted binary trees on n vertices.

Lemma 9.5

For each $n \in \mathbf{N}_0$ *we have*

$$C_{n+1} = C_0 C_n + C_1 C_{n-1} + \cdots + C_{n-1} C_1 + C_n C_0.$$

Theorem 9.6

If $n \in \mathbf{N}_0$ *then* $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Derangements by convolution

Lemma 9.7

If $n \in \mathbf{N}_0$ then

$$\sum_{k=0}^{n} \binom{n}{k} d_{n-k} = n!.$$

The sum in the lemma becomes a convolution after a small amount of rearranging.

Theorem 9.8

If $G(x) = \sum_{n=0}^{\infty} d_n x^n / n!$ then

$$G(x)\mathrm{e}^{x} = \frac{1}{1-x}.$$

§10: Partitions

Definition 10.1

A *partition* of a number $n \in \mathbf{N}_0$ is a sequence of natural numbers $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ such that

(i) $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1$.

(ii) $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$.

The entries in a partition λ are called the *parts* of λ. Let $p(n)$ be the number of partitions of n.

Example 10.2

Let a_n be the number of ways to pay for an item costing n pence using only 2p and 5p coins. Equivalently, a_n is the number of partitions of n into parts of size 2 and size 5. Will find the generating function for a_n.

Generating function

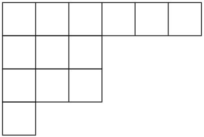
Theorem 10.3

The generating function for $p(n)$ is

$$\sum_{n=0}^{\infty} p(n)x^n = \frac{1}{(1-x)(1-x^2)(1-x^3)\dots}.$$

Young diagrams

It is often useful to represent partitions by *Young diagrams*. The Young diagram of $(\lambda_1, \ldots, \lambda_k)$ has k rows of boxes, with λ_i boxes in row i. For example, the Young diagram of $(6, 3, 3, 1)$ is



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Theorem 10.4

Let $n \in \mathbf{N}$ and let $k \leq n$. The number of partitions of n into parts of size $\leq k$ is equal to the number of partitions of n with at most k parts.

Two results from generating functions

While there are bijective proofs of the next theorem, it is much easier to prove it using generating functions.

Theorem 10.5

Let $n \in \mathbf{N}$. The number of partitions of n with at most one part of any given size is equal to the number of partitions of n into odd parts.

Theorem 10.6 (Van Lint's Upper Bound, Non-examinable)

If $n \in \mathbf{N}$ then $p(n) \leq \mathrm{e}^{c\sqrt{n}}$ where $c = 2\sqrt{\frac{\pi^2}{6}}$.

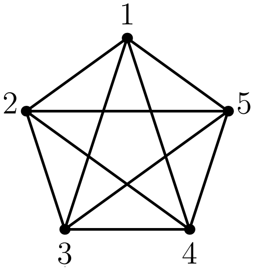
Part C: Ramsey Theory

§11: Introduction to Ramsey Theory

Definition 11.1

A *graph* consists of a set V of vertices together with a set E of 2-subsets of V called *edges*. The *complete graph* with vertex set V is the graph whose edge set is all 2-subsets of V.

The complete graph on $V = \{1, 2, 3, 4, 5\}$ is:



Today!

Oleg Pikhurko will give the Pure Mathematics Seminar at 2pm in C219 on

'On possible Turán densities'

Richard Pinch (GCHQ) will give the Coulter McDowell Lecture at 6.00pm (to start 6.15pm probably) in the Windsor Building on

'Modern Cryptography'.

There will be tea beforehand and drinks afterwards. To quote from the email: Dr Richard Pinch is a number theorist who worked at Cambridge University for many years before moving to GCHQ where he now applies his theoretical work.

Colourings

Definition 11.2

Let c, $n \in \mathbf{N}$. A *c-colouring* of the complete graph K_n is a function from the edge set of K_n to $\{1, 2, \ldots, c\}$. If S is an s-subset of the vertices of K_n such that all the edges between vertices in X have the same colour, then we say that S is a *monochromatic* K_s

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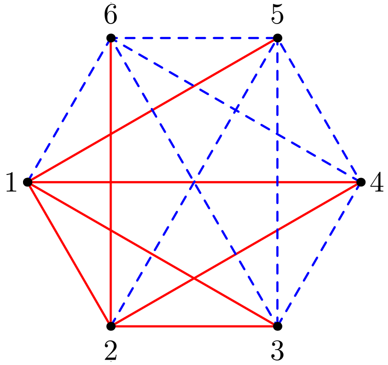
Exercise: find all red K_3s and blue K_4s in this colouring of K_6:



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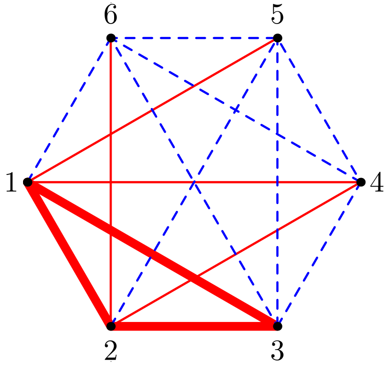

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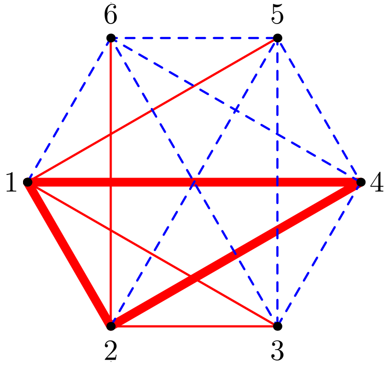

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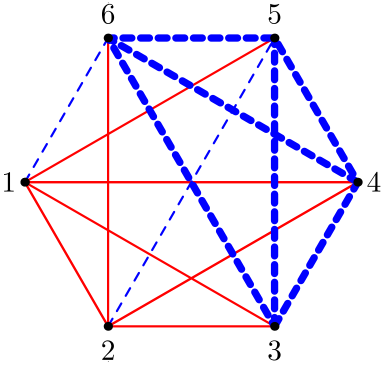

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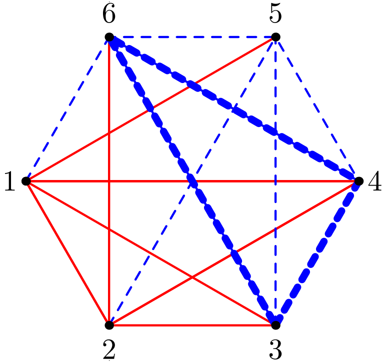

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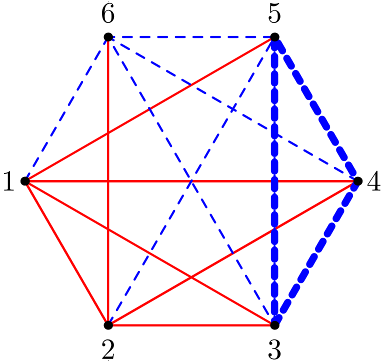

In any room with six people . . .

Example 11.3

In any red-blue colouring of the edges of K_6 there is either a red triangle or a blue triangle.

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In any red-blue colouring of the edges of K_6 there is either a red triangle or a blue triangle.

Definition 11.4

Given s, $t \in \mathbf{N}$, with $s, t \geq 2$, we define the Ramsey number $R(s, t)$ to be the smallest n (if one exists) such that in any red-blue colouring of the complete graph on n vertices, there is either a red K_s or a blue K_t.

Exercise: Let $s, t \geq 2$ and suppose that $R(s, t) = n$. Show that if $N \geq n$ then in any red-blue colouring of K_N there is either a red K_s or a blue K_t.

In any room with six people ...

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Given s, $t \in \mathbf{N}$, with $s, t \geq 2$, we define the Ramsey number $R(s, t)$ to be the smallest n (if one exists) such that in any red-blue colouring of the complete graph on n vertices, there is either a red K_s or a blue K_t.

Exercise: Let $s, t \geq 2$ and suppose that $R(s, t) = n$. Show that if $N \geq n$ then in any red-blue colouring of K_N there is either a red K_s or a blue K_t.

So if $N \geq R(s, t)$ we *always have order*. I.e, any colouring of K_N has either a red K_s or a blue K_t. If $N < R(s, t)$ then there are colourings of K_N not having either configuration.

Sheet 6: please hand in on Thursday if you'd like more time.

$R(3,4) \leq 10$

Lemma 11.5

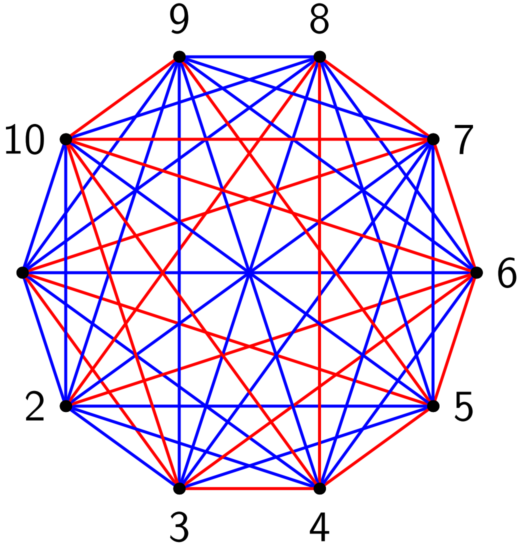
For any $s \in \mathbf{N}$ *we have* $R(s,2) = R(2,s) = s$.

The main idea need to prove the existence of all the Ramsey Numbers $R(s,t)$ appears in the next example.

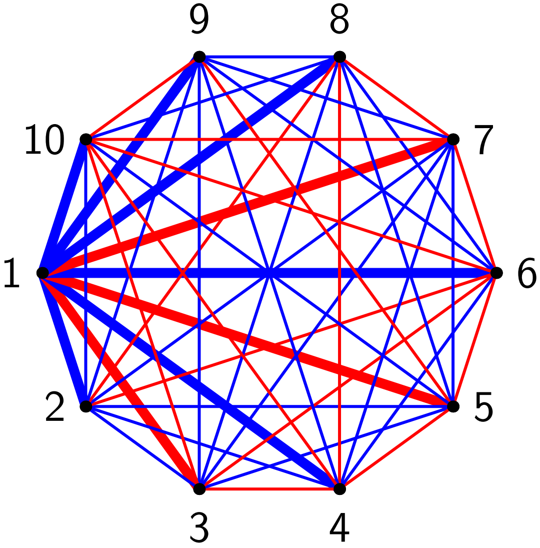
Example 11.6

In any two-colouring of K_{10} there is either a red K_3 or a blue K_4. Hence $R(3,4) \leq 10$.

Example $R(3, 4) \leq 10$



Example $R(3, 4) \leq 10$



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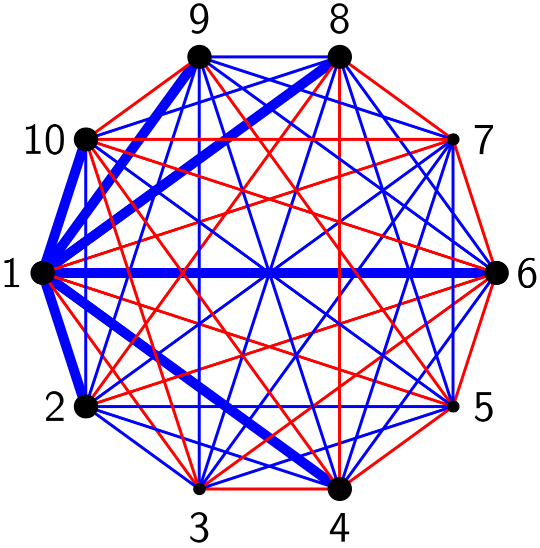

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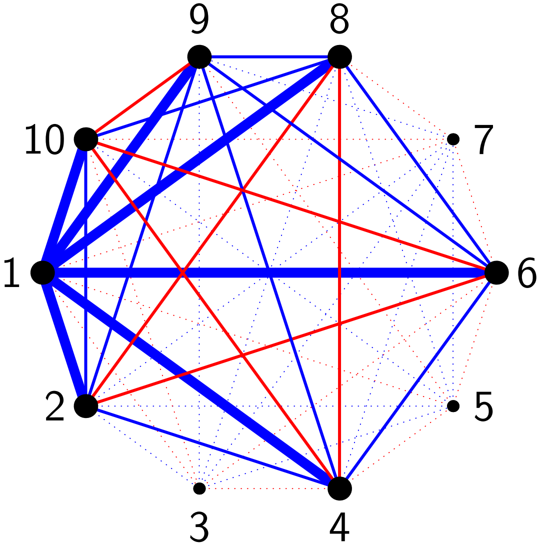

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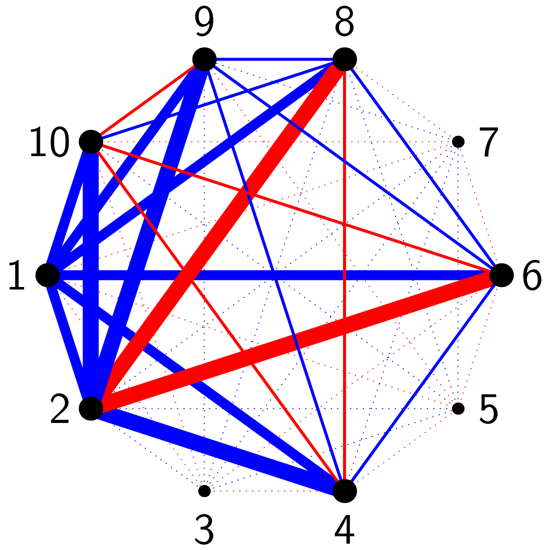

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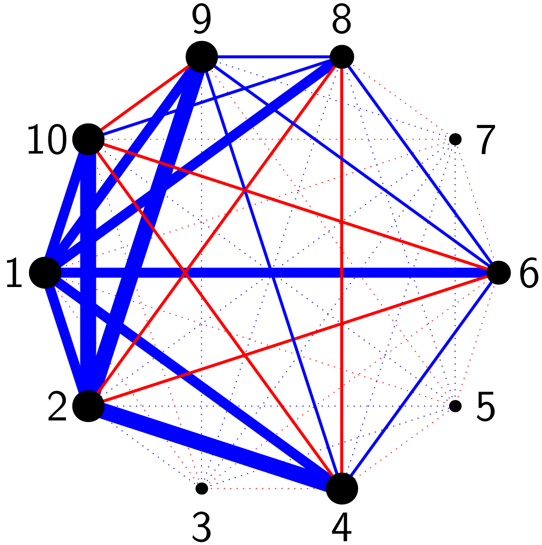

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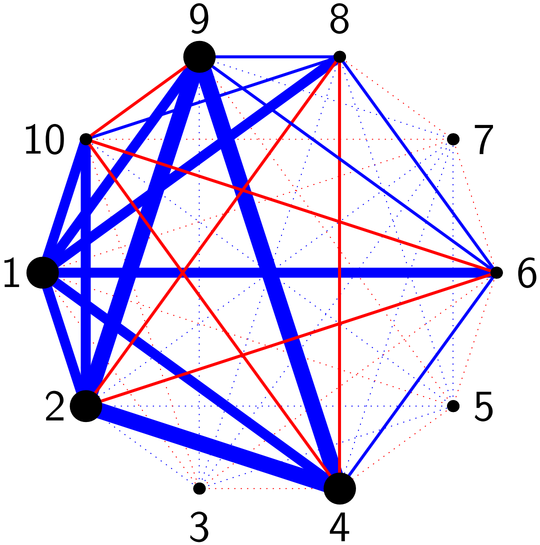
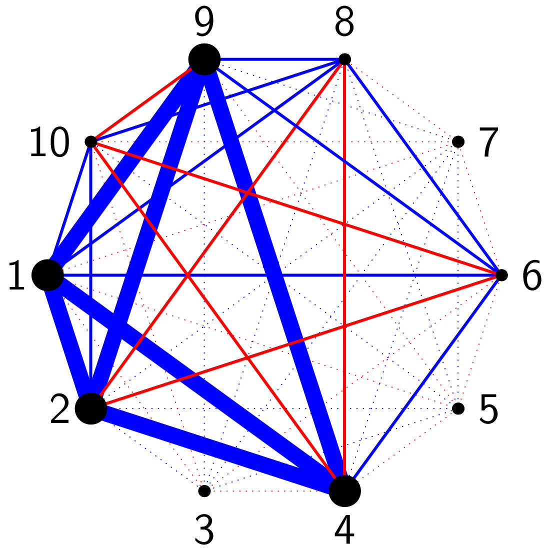


Example $R(3,4) \le 10$



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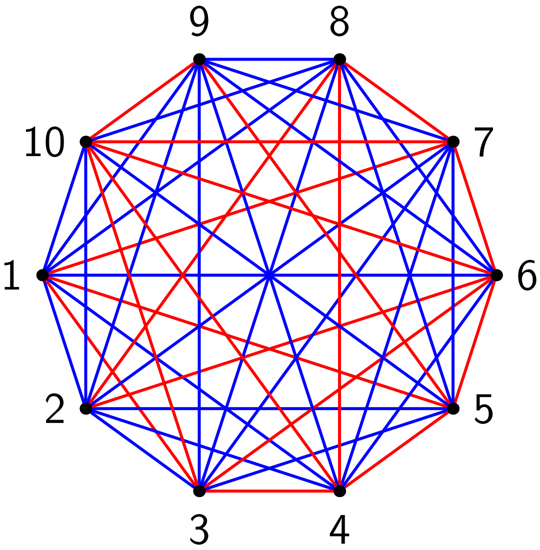
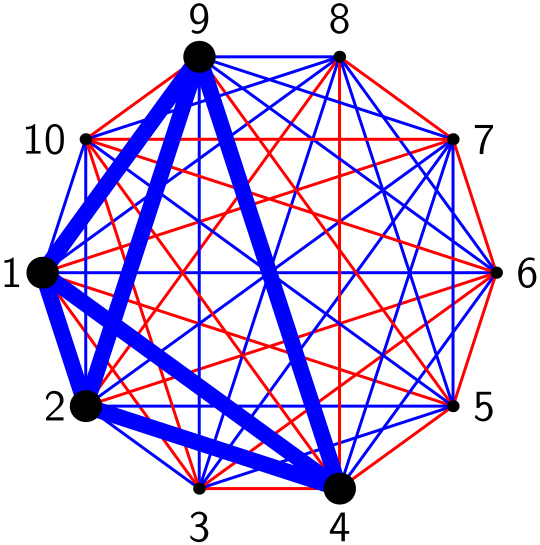


Example $R(3,4) \leq 10$



$R(3,4) = 9$

Lemma 11.7 (Hand-Shaking Lemma)

Let G be a graph with vertex set $\{1, 2, ..., n\}$ and exactly e edges. If d_i is the degree of vertex i then

$$2e = d_1 + d_2 + \cdots + d_n.$$

In particular, the number of vertices of odd degree is even.

Theorem 11.8

$R(3,4) = 9$.

The red-blue colouring of K_8 used to show that $R(3,4) > 8$ is a special case of a more general construction: see Question 2 on Sheet 7.

Theorem 11.9

$R(4,4) \leq 18$.

§12: Ramsey's Theorem

We shall prove that $R(s,t)$ exists, and get an upper bound for it, by induction on $s+t$.

Lemma 12.1

Let $s, t \in \mathbf{N}$ with $s, t \geq 3$. If $R(s-1, t)$ and $R(s, t-1)$ exist then $R(s,t)$ exists and

$$R(s,t) \leq R(s-1,t) + R(s,t-1).$$

Theorem 12.2

For any $s, t \in \mathbf{N}$ with $s, t \geq 2$, the Ramsey number $R(s,t)$ exists and

$$R(s,t) \leq \binom{s+t-2}{s-1}.$$

Inductive Proof of Ramsey's Theorem

$s \backslash t$	2	3	4	5	6	$\ldots$
2						
3						
4						
5						
6						
$\vdots$						

Inductive Proof of Ramsey's Theorem

$s \backslash t$	2	3	4	5	6	...
2	2	3	4	5	6	...
3	3					
4	4					
5	5					
6	6					
$\vdots$	$\vdots$					

Base case: $R(2, s) = R(s, 2) = s$ for all $s \geq 2$.

Inductive Proof of Ramsey's Theorem

$s \backslash t$	2	3	4	5	6	...
2	2	3	4	5	6	...
3	3	6				
4	4					
5	5					
6	6					
$\vdots$	$\vdots$					

Base case: $R(2, s) = R(s, 2) = s$ for all $s \geq 2$.
Inductive step by Lemma 13.1

Inductive Proof of Ramsey's Theorem

$s \backslash t$	2	3	4	5	6	$\dots$
2	2	3	4	5	6	$\dots$
3	3	6	10			
4	4	10				
5	5					
6	6					
$\vdots$	$\vdots$					

Base case: $R(2, s) = R(s, 2) = s$ for all $s \geq 2$.
Inductive step by Lemma 13.1

Inductive Proof of Ramsey's Theorem

$s \backslash t$	2	3	4	5	6	...
2	2	3	4	5	6	...
3	3	6	10	15		
4	4	10	20			
5	5	15				
6	6					
$\vdots$	$\vdots$					

Base case: $R(2, s) = R(s, 2) = s$ for all $s \geq 2$.
Inductive step by Lemma 13.1

Inductive Proof of Ramsey's Theorem

$s \backslash t$	2	3	4	5	6	...
2	2	3	4	5	6	...
3	3	6	10	15	21	
4	4	10	20	35		
5	5	15	35			
6	6	21				
$\vdots$	$\vdots$					

Base case: $R(2, s) = R(s, 2) = s$ for all $s \geq 2$.
Inductive step by Lemma 13.1

Inductive Proof of Ramsey's Theorem

$s \backslash t$	2	3	4	5	6	$\ldots$
2	2	3	4	5	6	$\ldots$
3	3	6	10	15	21	
4	4	10	20	35	56	
5	5	15	35	70		
6	6	21	56			
$\vdots$	$\vdots$					

Base case: $R(2, s) = R(s, 2) = s$ for all $s \geq 2$.
Inductive step by Lemma 13.1

Inductive Proof of Ramsey's Theorem

$s \backslash t$	2	3	4	5	6	...
2	2	3	4	5	6	...
3	3	6	10	15	21	
4	4	10	20	35	56	
5	5	15	35	70	126	
6	6	21	56	126		
$\vdots$	$\vdots$					

Base case: $R(2, s) = R(s, 2) = s$ for all $s \geq 2$.
Inductive step by Lemma 13.1

Inductive Proof of Ramsey's Theorem

$s \backslash t$	2	3	4	5	6	...
2	2	3	4	5	6	...
3	3	6	10	15	21	
4	4	10	20	35	56	
5	5	15	35	70	126	
6	6	21	56	126	252	
⋮	⋮					

Base case: $R(2, s) = R(s, 2) = s$ for all $s \geq 2$.
Inductive step by Lemma 13.1

Inductive Proof of Ramsey's Theorem

$s \backslash t$	2	3	4	5	6	...
2	2	3	4	5	6	...
3	3	6	9	14	18	
4	4	9	18	25	41	
5	5	14	25	49	87	
6	6	18	41	87	143	
⋮	⋮					

Base case: $R(2, s) = R(s, 2) = s$ for all $s \geq 2$.
Inductive step by Lemma 13.1
Best known upper bounds and lower bounds (black if Ramsey number known exactly)

Inductive Proof of Ramsey's Theorem

$s \backslash t$	2	3	4	5	6	...
2	2	3	4	5	6	...
3	3	6	9	14	18	
4	4	9	18	25	35	
5	5	14	25	43	58	
6	6	18	35	58	102	
⋮	⋮					

Base case: $R(2, s) = R(s, 2) = s$ for all $s \geq 2$.
Inductive step by Lemma 13.1
Best known upper bounds and lower bounds (black if Ramsey number known exactly)

Diagonal Ramsey Numbers

Corollary 12.3

If $s \in \mathbf{N}$ *and* $s \geq 2$ *then*

$$R(s,s) \leq \binom{2s-2}{s-1} \leq 4^{s-1}.$$

Games and Multiple Colours

Red and Blue play a game. Red starts by drawing a red line between two corners of a hexagon, then Blue draws a blue line and so on. A player *loses* if they makes a triangle of their colour.

Exercise: can the game end in a draw?

Games and Multiple Colours

Red and Blue play a game. Red starts by drawing a red line between two corners of a hexagon, then Blue draws a blue line and so on. A player *loses* if they makes a triangle of their colour.

Exercise: can the game end in a draw?

Theorem 12.4

There exists $n \in \mathbf{N}$ such that if the edges of K_n are coloured red, blue and yellow then there exists a monochromatic triangle.

Sheet 6

1. Let a_n be the number of partitions of $n \in \mathbf{N}$ into parts of size 3 and 5.

(a) Show that $a_{15} = 2$ and find a_{14} and a_{16}.

(b) Explain why

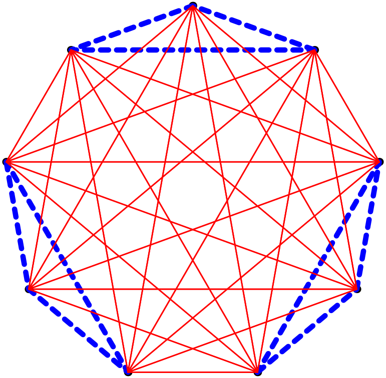
$$\sum_{n=0}^{\infty} a_n x^n = \frac{1}{(1-x^3)(1-x^5)}.$$

(c) Let c_n be the number of partitions with parts of sizes 3 and 5 whose sum of parts is *at most* n. Find the generating function of c_n.

6. Let $s, t \geq 2$. By constructing a suitable red-blue colouring of $K_{(s-1)(t-1)}$ prove that $R(s,t) > (s-1)(t-1)$. [*Hint: start by partitioning the vertices into* $s-1$ *blocks each of size* $t-1$*. Colour edges within each block with one colour . . .*]

6. Let $s, t \geq 2$. By constructing a suitable red-blue colouring of $K_{(s-1)(t-1)}$ prove that $R(s,t) > (s-1)(t-1)$. [*Hint: start by partitioning the vertices into* $s-1$ *blocks each of size* $t-1$. *Colour edges within each block with one colour* ...]

Example for $s = t = 4$.



Part D: Probabilistic Methods

§13: Revision of Discrete Probability

Definition 13.1

- A *probability measure* p on a finite set Ω assigns a real number p_ω to each $\omega \in \Omega$ so that $0 \leq p_\omega \leq 1$ for each ω and

$$\sum_{\omega \in \Omega} p_\omega = 1.$$

  We say that p_ω is the *probability of* ω.
- A *probability space* is a finite set Ω equipped with a probability measure. The elements of a probability space are sometimes called *outcomes*.
- An *event* is a subset of Ω.
- The *probability* of an event $A \subseteq \Omega$, denoted $\mathbf{P}[A]$ is the sum of the probability of the outcomes in A; that is $\mathbf{P}[A] = \sum_{\omega \in A} p_\omega$.

Example 13.2: Probability Spaces

(1) To model a throw of a single unbiased die, we take

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

and put $p_\omega = 1/6$ for each outcome $\omega \in \Omega$. The event that we throw an even number is $A = \{2, 4, 6\}$ and as expected, $\mathbf{P}[A] = p_2 + p_4 + p_6 = 1/6 + 1/6 + 1/6 = 1/2$.

(2) To model a throw of a pair of dice we could take

$$\Omega = \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\}$$

and give each element of Ω probability $1/36$, so $p_{(i,j)} = 1/36$ for all $(i, j) \in \Omega$. Alternatively, if we know we only care about the sum of the two dice, we could take $\Omega = \{2, 3, \ldots, 12\}$ with

n	2	3	...	6	7	8	...	12
p_n	$1/36$	$2/36$	...	$5/36$	$6/36$	$5/36$	...	$1/36$

The former is natural and more flexible.

Example 13.2: Probability Spaces

(3) A suitable probability space for three flips of a coin is

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

where H stands for heads and T for tails, and each outcome has probability $1/8$. To allow for a biased coin we fix $0 \le q \le 1$ and instead give an outcome with exactly k heads probability $q^k(1-q)^{3-k}$.

Exercise: Let A be the event that there is at least one head, and let B the the event that there is at least one tail. Find $\mathbf{P}[A]$, $\mathbf{P}[B]$, $\mathbf{P}[A \cap B]$, $\mathbf{P}[A \cup B]$.

(4) Let $n \in \mathbf{N}$ and let Ω be the set of all permutations of $\{1, 2, \ldots, n\}$. Set $p_\sigma = 1/n!$ for each permutation $\sigma \in \Omega$. This gives a suitable setup for Theorem 2.6.

Conditional Probability

Definition 13.3

Let Ω be a probability space, and let A, $B \subseteq \Omega$ be events.

- If $\mathbf{P}[B] \neq 0$ then we define the *conditional probability of A given B* by
$$\mathbf{P}[A|B] = \frac{\mathbf{P}[A \cap B]}{\mathbf{P}[B]}.$$
- The events A, B are said to be *independent* if
$$\mathbf{P}[A \cap B] = \mathbf{P}[A]\mathbf{P}[B].$$

Exercise: Let $\Omega = \{HH, HT, TH, TT\}$ be the probability space for two flips of a fair coin. Let A be the event that both flips are heads, and let B be the event that at least one flip is a head. Write A and B as subsets of Ω and show that $\mathbf{P}[A|B] = 1/3$.

The Most Misunderstood Problem Ever?

Example 13.4 (The Monty Hall Problem)

On a game show you are offered the choice of three doors. Behind one door is a car, and behind the other two are goats. You pick a door and then the host, *who knows where the car is*, opens another door to reveal a goat. You may then either open your original door, or change to the remaining unopened door. Assuming you want the car, should you change?

More Examples of Conditional Probability

Example 13.5 (Sleeping Beauty)

Beauty is told that if a coin lands heads she will be woken on Monday and Tuesday mornings, but after being woken on Monday she will be given an amnesia inducing drug, so that she will have no memory of what happened that day. If the coin lands tails she will only be woken on Tuesday morning. At no point in the experiment will Beauty be told what day it is. Imagine that you are Beauty and are awoken as part of the experiment and asked for your credence that the coin landed heads. What is your answer?

Example 13.6

Suppose that one in every 1000 people has disease X. There is a new test for X that will always identify the disease in anyone who has it. There is, unfortunately, a tiny probability of $1/250$ that the test will falsely report that a healthy person has the disease. What is the probability that a person who tests positive for X actually has the disease?

Random Variables

Definition 13.7

Let Ω be a probability space. A *random variable* on Ω is a function $X : \Omega \to \mathbf{R}$.

Definition 13.8

If $X, Y : \Omega \to \mathbf{R}$ are random variables then we say that X and Y are *independent* if for all $x, y \in \mathbf{R}$ the events

$$A = \{\omega \in \Omega : X(\omega) = x\} \quad \text{and}$$
$$B = \{\omega \in \Omega : Y(\omega) = y\}$$

are independent.

If $X : \Omega \to \mathbf{R}$ is a random variable, then '$X = x$' is the event $\{\omega \in \Omega : X(\omega) = x\}$. We mainly use this shorthand in probabilities, so for instance

$$\mathbf{P}[X = x] = \mathbf{P}\big[\{\omega \in \Omega : X(\omega) = x\}\big].$$

Example of Independence of Random Variables

Example 13.9

Let $\Omega = \{HH, HT, TH, TT\}$ be the probability space for two flips of a fair coin. Define $X : \Omega \to \mathbf{R}$ to be 1 if the first coin is heads, and zero otherwise. So

$$X(HH) = X(HT) = 1 \quad \text{and} \quad X(TH) = X(TT) = 0.$$

Define $Y : \Omega \to \mathbf{R}$ similarly for the second coin.

(i) The random variables X and Y are independent.

(ii) Let Z be 1 if exactly one flip is heads, and zero otherwise. Then X and Z are independent, and Y and Z are independent.

(iii) There exist $x, y, z \in \{0, 1\}$ such that

$$\mathbf{P}[X = x, Y = y, Z = z] \neq \mathbf{P}[X = x]\mathbf{P}[Y = y]\mathbf{P}[Z = z].$$

Expectation

Definition 13.10

Let Ω be a probability space with probability measure p. The *expectation* $\mathbf{E}[X]$ of a random variable $X : \Omega \to \mathbf{R}$ is defined to be

$$\mathbf{E}[X] = \sum_{\omega \in \Omega} X(\omega) p_w.$$

Lemma 13.11

Let Ω be a probability space. If $X_1, X_2, \ldots, X_k : \Omega \to \mathbf{R}$ are random variables then

$$\mathbf{E}[a_1 X_1 + a_2 X_2 + \cdots + a_k X_k] = a_1 \mathbf{E}[X_1] + a_2 \mathbf{E}[X_2] + \cdots + a_k \mathbf{E}[X_k]$$

for any $a_1, a_2, \ldots, a_k \in \mathbf{R}$.

Lemma 13.12

If $X, Y : \Omega \to \mathbf{R}$ are independent random variables then $\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y]$.

Example of Linearity of Expectation (Question 10, Sheet 7)

10. Let $0 \leq p \leq 1$ and let $n \in \mathbf{N}$. Suppose that a coin biased to land heads with probability p is tossed n times. Let X be the number of times the coin lands heads.

(a) Describe a suitable probability space Ω and probability measure $p : \Omega \to \mathbf{R}$ and define X as a random variable $\Omega \to \mathbf{R}$.

(b) Find $\mathbf{E}[X]$ and $\mathbf{Var}[X]$. [*Hint: write X as a sum of n independent random variables and use linearity of expectation and Lemma 13.14(ii).*]

(c) Find a simple closed form for the generating function $\sum_{k=0}^{\infty} \mathbf{P}[X = k]x^k$. (Such power series are called *probability generating functions*.)

Variance

Definition 13.13

Let Ω be a probability space. The *variance* $\mathbf{Var}[X]$ of a random variable $X : \Omega \to \mathbf{R}$ is defined to be

$$\mathbf{Var}[X] = \mathbf{E}\big[(X - \mathbf{E}[X])^2\big].$$

Lemma 13.14

Let Ω be a probability space.

(i) *If $X : \Omega \to \mathbf{R}$ is a random variable then*

$$\mathbf{Var}[X] = \mathbf{E}[X^2] - (\mathbf{E}[X])^2.$$

(ii) *If $X, Y : \Omega \to \mathbf{R}$ are independent random variables then*

$$\mathbf{Var}[X + Y] = \mathbf{Var}[X] + \mathbf{Var}[Y].$$

Questionnaires

The batch number is 965017.

If you are a 3rd year or 4th year undergraduate you are doing MT4540. If you are an MSc student you are doing MT5454.

The additional questions are:

17. For this course, Library study space met my needs.
18. The course books in the Library met my needs for this course.
19. The online Library resources met my needs for this course.
20. I was satisfied with the Moodle elements of this course.
21. I received feedback on my work within the 4 week norm specified by College.

Please write any further comments on the back of the form. (In particular, please answer the old version of Q17: whether you found the speed too fast, too slow, or about right.)

§14: Introduction to Probabilistic Methods

Throughout this section we fix $n \in \mathbf{N}$ and let Ω be the set of all permutations of the set $\{1, 2, \ldots, n\}$. Define a probability measure so that permutations are chosen uniformly at random.

Exercise: Let $x \in \{1, 2, \ldots, n\}$ and let $A_x = \{\sigma \in \Omega : \sigma(x) = x\}$. Then A_x is the event that a permutation fixes x. What is the probability of A_x?

Theorem 14.1
Let $F : \Omega \to \mathbf{N}_0$ be defined so that $F(\sigma)$ is the number of fixed points of the permutation $\sigma \in \Omega$. Then $\mathbf{E}[F] = 1$.

Cycles

Definition 14.2

A permutation σ of $\{1, 2, \ldots, n\}$ *acts as a k-cycle* on a k-subset $S \subseteq \{1, 2, \ldots, n\}$ if S has distinct elements $x_1, x_2, \ldots, x_k$ such that

$$\sigma(x_1) = x_2,\ \sigma(x_2) = x_3,\ \ldots,\ \sigma(x_k) = x_1.$$

If $\sigma(y) = y$ for all $y \in \{1, 2, \ldots, n\}$ such that $y \notin S$ then we say that σ is a *k-cycle*, and write

$$\sigma = (x_1, x_2, \ldots, x_k).$$

Definition 14.3

We say that cycles $(x_1, x_2, \ldots, x_k)$ and $(y_1, y_2, \ldots, y_\ell)$ are *disjoint* if

$$\{x_1, x_2, \ldots, x_k\} \cap \{y_1, y_2, \ldots, y_\ell\} = \varnothing.$$

Cycle decomposition of a permutation

Lemma 14.4
A permutation σ of $\{1, 2, \ldots, n\}$ can be written as a composition of disjoint cycles. The cycles in this composition are uniquely determined by σ.

Exercise: Write the permutation of $\{1, 2, 3, 4, 5, 6\}$ defined by $\sigma(1) = 3$, $\sigma(2) = 4$, $\sigma(3) = 1$, $\sigma(4) = 6$, $\sigma(5) = 5$, $\sigma(6) = 2$ as a composition of disjoint cycles.

Theorem 14.5
Let $1 \leq k \leq n$ and let $x \in \{1, 2, \ldots, n\}$. The probability that x lies in a k-cycle of a permutation of $\{1, 2, \ldots, n\}$ chosen uniformly at random is $1/n$.

Application to derangements

Theorem 14.6

Let p_n be the probability that a permutation of $\{1, 2, \ldots, n\}$ chosen uniformly at random is a derangement. Then

$$p_n = \frac{p_{n-2}}{n} + \frac{p_{n-3}}{n} + \cdots + \frac{p_1}{n} + \frac{p_0}{n}.$$

Corollary 14.7

For all $n \in \mathbf{N}$,

$$p_n = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}.$$

Administration

- As discussed on Tuesday, the final Combinatorics lecture on Friday 14th December will be moved to 1pm on Tuesday 11th December.

  The extra lecture will be held in ABLT2 and will follow on from the usual lecture at 12 noon in HLT1.

- Please take Problem Sheet 8 if you don't already have it. I will mark work handed it on Tuesday and leave outside office by Thursday lunchtime.

- Also there are copies of the Problems published in the October issue of the American Mathematical Monthly. Problem 11668 can be solved in several different ways by the techniques developed in this course (and in still further ways using Chapter 3 of *generatingfunctionology*).

  The deadline for submission of solutions is February 28, 2013. Send to `monthlyproblems@math.tamu.edu`, mentioning that you are a student at Royal Holloway.

Counting cycles

We can also generalize Theorem 14.1.

Theorem 14.8

Let $C_k : \Omega \to \mathbf{R}$ *be the random variable defined so that* $C_k(\sigma)$ *is the number of* k*-cycles in the permutation* σ *of* $\{1, 2, \ldots, n\}$*. Then* $\mathbf{E}[C_k] = 1/k$ *for all* k *such that* $1 \leq k \leq n$*.*

§15: Ramsey Numbers and the First Moment Method

Lemma 15.1 (First Moment Method)

Let Ω be a probability space and let $M : \Omega \to \mathbf{N}_0$ be a random variable taking values in $\mathbf{N}_0$. If $\mathbf{E}[M] = x$ then

(i) $\mathbf{P}[M \geq x] > 0$, *so there exists $\omega \in \Omega$ such that $M(\omega) \geq x$.*

(ii) $\mathbf{P}[M \leq x] > 0$, *so there exists $\omega' \in \Omega$ such that $M(\omega') \leq x$.*

Exercise: Check that the lemma holds in the case when

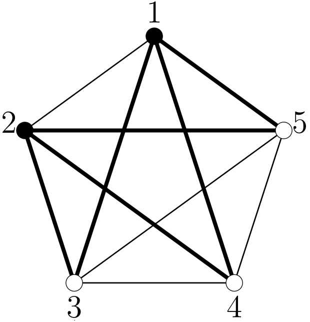
$$\Omega = \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\}$$

models the throw of two fair dice and $M(x, y) = x + y$.

Cut sets in graphs

Definition 15.2

Let G be a graph with vertex set V. A *cut* (A, B) of G is a partition of V into two subsets A and B. The *capacity* of a cut (A, B) is the number of edges of G that meet both A and B.



({1,2}, {3,4,5})

Theorem 15.3

Let G be a graph with vertex set $\{1, 2, \ldots, n\}$ and m edges. There is a cut of G with capacity $\geq m/2$.

Application to Ramsey Theory

Lemma 15.4

Let $n \in \mathbf{N}$ and let Ω be the set of all red-blue colourings of the complete graph K_n. Let $p_\omega = 1/|\Omega|$ for each $\omega \in \Omega$. Then

(i) *each colouring in Ω has probability $1/2^{\binom{n}{2}}$;*

(ii) *given any m edges in G, the probability that all m of these edges have the same colour is 2^{1-m}.*

Theorem 15.5

Let $n, s \in \mathbf{N}$. If

$$\binom{n}{s} 2^{1-\binom{s}{2}} < 1$$

then there is a red-blue colouring of the complete graph on $\{1, 2, \ldots, n\}$ with no red K_s or blue K_s.

Lower bound on $R(s, s)$

Corollary 15.6

For any $s \in \mathbf{N}$ we have

$$R(s, s) \geq 2^{(s-1)/2}.$$

This result can be strengthened slightly using the Lovász Local Lemma. See the printed lecture notes for an outline. (The contents of §16 are non-examinable.)