MT454 / 5454 Combinatorics: Preliminary Problem Sheet

The purpose of this sheet is to remind you of the notation for sets, tuples and functions, and to give you some short and straightforward questions on which to practice the Basic Counting Principles.

Answers will be posted to Moodle on 11th October. Please do Question 2 before the lecture on 7th October.

You are very welcome to ask the lecturer about the problem sheets. You can do this either after lectures, or in office hours. You do not have to wait until the answers are posted to Moodle.

1. A menu has 3 starters, 4 main courses and 6 desserts.

(a) How many ways are there to order a starter, main course and dessert? [*Hint:* multiply choices using BCP1.]

(b) How many ways are there to order a two course meal, including exactly one main course? [*Hint:* use BCP1 and BCP2.]

2. Let

$$X = \{(a_1, a_2, a_3) : 1 \le a_1, a_2, a_3 \le 10,\ a_1, a_2, a_3 \text{ distinct}\},$$
$$Y = \{A \subseteq \{1, \ldots, 10\} : |A| = 3\}.$$

Define a function $f : X \to Y$ by $f\big((a_1, a_2, a_3)\big) = \{a_1, a_2, a_3\}$.
For example, $(3, 2, 5) \in X$ and $f\big((3, 2, 5)\big) = \{2, 3, 5\} \in Y$.

(a) Find $|X|$.

(b) Find the number of tuples $(a_1, a_2, a_3) \in X$ such that $f\big((a_1, a_2, a_3)\big) = \{2, 3, 5\}$.

(c) By generalizing the idea in (b), find $|Y|$.

[*Hint:* the point of this question is to show the ideas in one proof of the formula for binomial coefficients. So please do not assume this formula is true in (c).]

3. A deck consists of 52 cards. There are four Aces, four Kings, four Queens and four Jacks. How many hands of five cards are there that

(a) have at least one Ace, King, Queen and Jack? [*Hint:* first count hands of the form AKQJx, then hands of the form AAKQJ, and so on.]

(b) have at least one Ace, King and Queen?

Remark: In Section 5 of the course we will see the Principle of Inclusion and Exclusion: it gives a quick, unified, way to solve both problems.

4. Fix $n \in \mathbf{N}$. Let $X = \{(a,b) : 1 \le a \le b \le n\}$. Find a simple formula for $|X|$ in terms of n.

5. Let

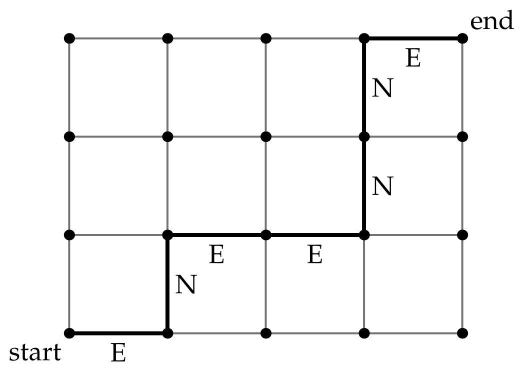
$$X = \left\{ \begin{array}{c} \text{placements of 4 indistinguishable balls into 7} \\ \text{numbered urns so that each ball is in a different urn} \end{array} \right\}$$

$$Y = \left\{ \begin{array}{c} \text{ways to walk 4 blocks East and 3 blocks North} \\ \text{on a New York grid, moving only East and North} \end{array} \right\}$$

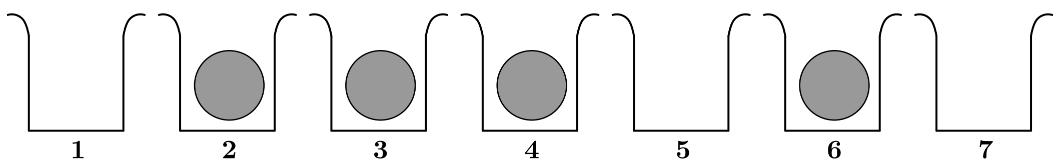
$$Z = \big\{ A \subseteq \{1,2,\ldots,7\} : |A| = 4 \big\}$$

(a) Define explicit bijective maps $f : X \to Z$ and $g : Y \to Z$. [*Hint:* it might help to first work out what you want the answers to (b) and (c) to be.]

(b) Which element of Z corresponds to the walking route ENEENNE $\in Y$?



(c) Which walking route corresponds to the ball-and-urn placement shown below?



(d) Using BCP3, find a binomial coefficient equal to $|X|$ and $|Y|$.

MT454 / 5454 Combinatorics: Sheet 1

Do questions 3, 4 and 5 and at least two other questions.

To be returned to McCrea 240 by noon on Tuesday 14th October or handed in at the Tuesday lecture.

Parts of questions marked ($\star$) are optional and harder than average.

1. Prove that
$$r\binom{n}{r} = n\binom{n-1}{r-1}$$
for n, $r \in \mathbf{N}$ in two ways:

(a) using the formula for a binomial coefficient;

(b) bijectively, by reasoning with subsets of $\{1, 2, \ldots, n\}$.

2. Prove that
$$\sum_{k=0}^{n} k\binom{m}{k}\binom{n}{k} = n\binom{m+n-1}{n}.$$
[*Hint: use Question 1 and then aim to apply Vandermonde's convolution.*]

3. Let n, $r \in \mathbf{N}$. Prove that
$$\binom{r}{r} + \binom{r+1}{r} + \binom{r+2}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1}$$
in two ways:

(a) by induction on n (where r is fixed in the inductive argument);

(b) bijectively, by reasoning with subsets of $\{1, 2, \ldots, n+1\}$.

4. Read from page 1 up to the end of Section 1.2 in *generatingfunctionology* and do parts (a), (b) and (c) of questions 1 and 3, and question 6(b) from the end of chapter exercises.

5. A lion tamer has n cages in a row. Let $g(n, k)$ be the number of ways is which she may accommodate k indistinguishable lions so that no cage contains more than one lion, and no two lions are housed in adjacent cages.

(a) Show that $g(n, k) = g(n-2, k-1) + g(n-1, k)$ if $n \geq 2$ and $k \geq 1$.

(b) Prove by induction that $g(n, k) = \binom{n-k+1}{k}$ for all $n \in \mathbf{N}$ and $k \in \mathbf{N}_0$ such that $k \leq n$.

($\star$) Find a bijective proof of the formula for $g(n, k)$.

6. Let n, $k \in \mathbf{N}$. How many solutions are there to the equation $x_1 + x_2 + \cdots + x_n = k$ if the x_i are *strictly* positive integers, i.e. $x_i \in \mathbf{N}$ for each i?

7. Define

$$b_n = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots$$

for $n \in \mathbf{N}_0$.

(a) Find the first few members of the sequence $b_0, b_1, b_2, b_3, \ldots$.

(b) State and prove a recurrence relating b_{n+2} to b_{n+1} and b_n.

8. (a) What is 11^4? Explain the connection to binomial coefficients.

(b) By considering a suitable binomial expansion prove that

$$\frac{4^n}{2n+1} \leq \binom{2n}{n} \leq 4^n.$$

9. Let $p_n = d_n/n!$ be the probability that a permutation of $\{1, 2, \ldots, n\}$, chosen uniformly at random, is a derangement. Using only the recurrence in Theorem 2.4, prove by induction that $p_n - p_{n-1} = (-1)^n/n!$; hence give an alternative proof of Corollary 2.5.

10. Some further results on derangements.

(a) Let $a_n(k)$ be the number of permutations of $\{1, 2, \ldots, n\}$ with exactly k fixed points. Note that $d_n = a_n(0)$. Use results from lectures to prove that

$$a_n(k) = \frac{n!}{k!}\Big(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{(-1)^{n-k}}{(n-k)!}\Big).$$

Hence, or otherwise, give a simple expression for $a_n(0) - a_n(1)$.

(b) Use part (a) to give an alternative proof of Theorem 2.6(ii), that the average number of fixed points of a permutation of $\{1, 2, \ldots, n\}$ is 1.

(c) ($\star$) Let e_n be the number of derangements of $\{1, 2, \ldots, n\}$ that are even permutations, and let o_n be the number that are odd permutations. By evaluating the determinant of the matrix

$$\begin{pmatrix} 0 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 1 & \ldots & 1 \\ 1 & 1 & 0 & \ldots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \ldots & 0 \end{pmatrix}$$

in two different ways, prove that $e_n - o_n = (-1)^{n-1}(n-1)$.

11. Assume that any two people are either friends or enemies. Show that in any room containing six people there are either three mutual friends, or three mutual enemies. (Generalizations of this problem will be solved in Part C of the course.)

MT454 / 5454 Combinatorics: Sheet 2

Do questions 2, 3 and 4 and at least one other question.

To be returned to McCrea 240 by 3pm on Thursday 23rd October or handed in at the Thursday lecture.

Parts of questions marked $(\star)$ are optional and harder than average.

1. How many numbers between 1 and 2011 are not divisible by either 2 or 3? How many are not divisible by either 2, 3 or 5? Illustrate your answers with Venn diagrams.

2. How many numbers in the interval $\{1, 2, \ldots, 100\}$ are not divisible by any of 2, 3, 5 or 7? Use the PIE, making it clear which sets you are using. Hence find the number of primes ≤ 100.

3. Do question 3(e)–(h) and Question 11 from the end of chapter exercises from Chapter 1 of Wilf *generatingfunctionology*. Please also read §1.3. [*Hint:* In Question 11, note the the notation $[n] = \{1, 2, \ldots, n\}$ is used.]

4. Euler's φ function is important in number theory. It is defined by

$$\varphi(N) = |\{a \in \mathbf{N} : 1 \leq a \leq N,\ a \text{ is coprime to } n\}|.$$

For example, when $N = 10$, the integers a such that $1 \leq a \leq 10$ and a is coprime to 10 are $1, 3, 7, 9$; note that these are precisely the numbers in $\{1, 2, \ldots, 10\}$ that are not divisible either by 2 or by 5.

(a) Show that $\varphi(p) = p - 1$ if p is prime.

(b) Let p, q, r denote distinct primes. Prove formulae for $\varphi(pq)$ and $\varphi(pqr)$ using the PIE. (Define the sets you use in the PIE precisely.)

(c) Recall that each integer N has a unique prime factorization $N = p_1^{e_1} \ldots p_r^{e_r}$ where $p_1 < p_2 < \cdots < p_r$ are primes and $e_1, e_2, \ldots, e_r \in \mathbf{N}$. Prove that

$$\varphi(N) = N\Big(1 - \frac{1}{p_1}\Big)\Big(1 - \frac{1}{p_2}\Big) \ldots \Big(1 - \frac{1}{p_r}\Big).$$

[*Hint:* you can use Theorem 5.6, or work directly from the PIE. Please do not assume the result in (d).]

(d) Deduce from (c) that if M and N are coprime then $\varphi(MN) = \varphi(M)\varphi(N)$.

5. How many non-decreasing sequences of length 3 can one make from the set $\{1,2,\ldots,8\}$? [*Hint: one approach is first to count the sequences with* 3 *distinct elements, then the sequences like* $(1, 1, 2)$ *with* 2 *distinct elements, and finally the sequences like* $(1, 1, 1)$ *with* 3 *equal elements. Or use Theorem 4.7.*]

6. Give a bijective proof of the identity $\displaystyle\sum_{k=0}^{n} \binom{n}{k} b^{n-k} = (1 + b)^n$ for $b \in \mathbf{N}_0$.

7. (a) Explain why there are

$$\binom{11}{4}\binom{7}{4}\binom{3}{2}$$

different ways to arrange the letters of the word 'mississippi'.

(b) How many ways are there to misspell 'abracadabra'?

8. Let a, $b \in \mathbf{N}_0$ and let $m \in \mathbf{N}_0$. By finding the coefficient of x^m in either side of

$$(1+x)^a(1+x)^b = (1+x)^{a+b}$$

give a generating function proof of Vandermonde's convolution,

$$\sum_{k=0}^{m} \binom{a}{k}\binom{b}{m-k} = \binom{a+b}{m}.$$

9. Let X denote the set of all functions $f : \{1, 2, \ldots, k\} \to \{1, 2, \ldots, n\}$. For each $i \in \{1, 2, \ldots, n\}$ define

$$A_i = \big\{f \in X : f(t) \neq i \text{ for any } t \in \{1, 2, \ldots, k\}\big\}.$$

(a) What is $|X|$? What is $|A_i|$?

(b) Let $I \subseteq \{1, 2, \ldots, n\}$ be a non-empty subset and let $A_I = \bigcap_{i \in I} A_i$. What condition must a function $f \in X$ satisfy to lie in A_I? Hence find $|A_I|$.

(c) Use the Principle of Inclusion and Exclusion to show that the number of surjective functions from $\{1, 2, \ldots, k\}$ to $\{1, 2, \ldots, n\}$ is

$$\sum_{r=0}^{n} \binom{n}{r}(-1)^r(n-r)^k.$$

(d) Show that the above expression is the number of ways to put k numbered balls into n numbered urns, so that each urn contains *at least* one ball.

10. For k, $n \in \mathbf{N}_0$, the *Stirling number of the second kind* $\left\{ {k \atop n} \right\}$ is defined to be the number of set partitions of $\{1, 2, \ldots, k\}$ into n disjoint subsets. For example, $\left\{ {4 \atop 3} \right\} = 6$; one of the relevant set partitions is $\big\{\{1\}, \{2\}, \{3, 4\}\big\}$.

(a) Show that $\left\{ {k \atop 1} \right\} = 1$, $\left\{ {k \atop 2} \right\} = 2^{k-1} - 1$ and $\left\{ {k \atop k-1} \right\} = \binom{k}{2}$ for all $k \in \mathbf{N}$.

(b) Explain why $\left\{ {k \atop n} \right\}$ is the number of ways to put k numbered balls into n *indistinguishable* urns, so that each urn receives *at least* one ball.

(c) Find a formula for $\left\{ {k \atop n} \right\}$ using the previous question.

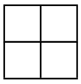
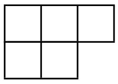
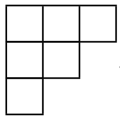
11. Recall that d_n is the number of derangements of $\{1, 2, ..., n\}$. Use the formula for d_n to prove that if $n > 0$ then d_n is the nearest integer to $n!/\mathrm{e}$.

MT454 / 5454 Combinatorics: Sheet 3

Do questions 1, 2 and 4 and at least one other question. Please write your answer to question 2 on a separate sheet: 2(d) is optional.

To be handed in at the lecture at 3pm on Thursday 30th October.

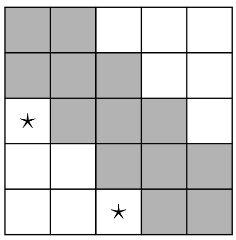
1. Find the rook polynomials of the boards below. (You may use any general lemmas proved in lectures.)

(i)  , (ii)  , (iii)  .

2. Let T be the set of all derangements σ of $\{1, 2, 3, 4, 5\}$ such that

- $\sigma(i) \neq i + 1$ if $1 \leq i \leq 4$,
- $\sigma(i) \neq i - 1$ if $2 \leq i \leq 5$.

(a) Explain why $|T|$ is the number of ways to place 5 non-attacking rooks on the board B formed by the unshaded squares below. (Give an explicit example of how a permutation corresponds to a rook placement.)



(b) Find the rook polynomial of B, and hence find $|T|$. [*Hint: consider the four possibilities for the starred squares. For example, if both are occupied, the contribution to the rook polynomial is* $x^2 f_1(x) f_2(x)$ *where* $f_n(x)$ *is the rook polynomial of the* $n \times n$ *square board.*]

(c) Use Theorem 6.10 to find the number of ways to place 5 non-attacking rooks on the shaded squares.

(d) ($\star$) By adapting the argument used to prove Theorem 6.10, find the number of ways to place 4 non-attacking rooks on the shaded squares.

3. Let B be the board in Example 6.3. Show that the complement of B in the 4×4 board has the same rook polynomial as B. [*Hint: for a calculation-free proof, argue that permuting the rows or columns of a board does not change its rook polynomial.*]

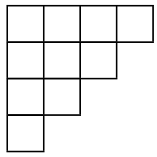
4. Prove by induction that if $n \in \mathbf{N}$ then

$$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k.$$

[*Hint: for the inductive step, try differentiating.*]

5. Find the number of permutations σ of $\{1,2,3,4,5,6\}$ such that $\sigma(m) \neq m$ for any even number m.

6. Let T_n denote the triangular board with k squares in row k for $k \in \{1,2,\ldots,n\}$. For example T_4 is shown in the margin.



(a) Prove that $\left\{ {n \atop s} \right\} = \left\{ {n-1 \atop s-1} \right\} + s\left\{ {n-1 \atop s} \right\}$ for all n, $s \in \mathbf{N}$. (Stirling numbers of the second kind were defined on Question 10 of Sheet 2.)

(b) Prove that $r_k(T_n) = r_k(T_{n-1}) + \big(n - (k-1)\big) r_{k-1}(T_{n-1})$ for all n, $k \in \mathbf{N}$.

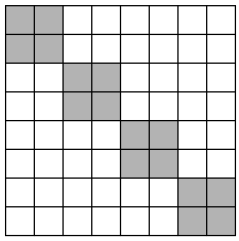
(c) Hence show that $f_{T_n}(x) = \sum_{k=0}^{n} \left\{ {n+1 \atop n+1-k} \right\} x^k$.

(d) ($\star$) Give a bijective proof of (c).

7. How many numbers between 100 and 300 can be formed from the digits 1, 2, 3, 4 if (i) repetition of digits is not allowed, (ii) repetition of digits is allowed?

8. Use Theorem 6.10 to find the number of ways that eight non-attacking rooks can be placed on the unshaded part of the board shown below. It may be helpful to note that

$$(1+4x+2x^2)^4 = 1+16x+104x^2+352x^3+664x^4+704x^5+416x^6+128x^7+16x^8.$$



9. Let X be a finite set and let A_1, A_2, $\ldots$, A_n be subsets of X.

(a) Set $C = A_1 \cup \cdots \cup A_{n-1}$. Show that

$$|\overline{A_1 \cup A_2 \cup \cdots \cup A_n}| = |X| - |C| - |A_n| + |C \cap A_n|.$$

(b) Use (a) to prove the Principle of Inclusion and Exclusion by induction on n. [*Hint: in the inductive step let $A'_i = A_i \cap A_n$ and apply the PIE to the sets $A'_1, \ldots, A'_{n-1}$ inside the universe set A_n.*]

10. (For those who know about group homomorphisms.) Let G denote the set of all permutations of $\{1,2,...,n\}$, thought of as the symmetric group of degree n. Given $\sigma \in G$, define an $n \times n$ matrix $A(\sigma)$ by

$$\begin{cases} A_{ij} = 1 & \text{if } \sigma(j) = i \\ A_{ij} = 0 & \text{otherwise.} \end{cases}$$

Show that the map $\sigma \mapsto A(\sigma)$ is an injective group homomorphism from G into the group of all invertible $n \times n$ real matrices.

MT454 / 5454 Combinatorics: Sheet 4

Do questions 1, 2 and 3 at least one other question.

To be returned to McCrea 240 by 11am on Tuesday 11th November or handed in at the Tuesday lecture.

1. (a) Suppose that $2a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$. Use generating functions to find a formula for a_n in terms of a_0 and a_1.

(b) Let $A \in \mathbf{N}$. Solve the recurrence $a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3}$ for $n \geq 3$ subject to the initial conditions $a_0 = 0$, $a_1 = 1$, $a_2 = A$.

2. Write out a complete proof of Theorem 8.4 following the three-step programme.

3. Let $n \in \mathbf{N}$ be given. Let b_k be the number of n-tuples $(t_1, \ldots, t_n)$ such that $t_i \in \mathbf{N}$ for each i and $t_1 + \cdots + t_n = k$. (Note that this differs from Example 7.3 since now $t_i \in \mathbf{N}$, rather than $t_i \in \mathbf{N}_0$.)

(a) Show that $b_k = 0$ if $k < n$ and give formulae for b_n and b_{n+1}.

(b) Let $F(x) = \sum_{k=0}^{\infty} b_k x^k$. By adapting the argument in Example 7.3 show that

$$F(x) = \left(\frac{x}{1-x}\right)^n.$$

(c) Deduce from Theorem 7.4 (or Question 4 on Sheet 3) that

$$F(x) = \sum_{r=0}^{\infty} \binom{n+r-1}{r} x^{r+n}.$$

Find the coefficient of x^k in the right-hand side and show that $b_k = \binom{k-1}{n-1}$.

4. Let a, $b \in \mathbf{N}_0$ and let $m \in \mathbf{N}_0$. By finding the coefficient of x^{2m} in either side of $(1-x)^a(1+x)^a = (1-x^2)^a$ prove that

$$\sum_{k=0}^{2m} (-1)^k \binom{a}{k}\binom{a}{2m-k} = (-1)^m \binom{a}{m}.$$

5. A *Latin square* is an $n \times n$ square in which every row and column contains each of the numbers 1, 2, ..., n exactly once. Let L be the incomplete Latin square shown below

1	2	3	4	5
2	3	1	5	4

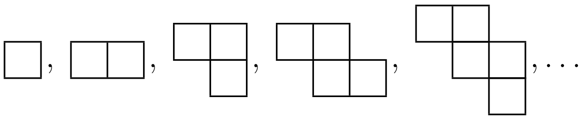
.

Let B be the board with a square in position (i, j) if and only if the number i can be put in row 3 and column j of L. Find the rook polynomial of $\overline{B}$ and hence find the number of ways to complete the third row of L.

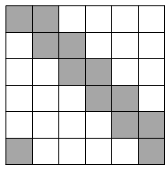
6. Do part (ii) of the exercise below Theorem 7.5. Show that if $c, m \in \mathbf{N}$ then

$$\sum_{k=0}^{m} (-1)^k \binom{c+k-1}{k} \binom{c}{m-k} = 0.$$

7. (Problème des Ménages.) Let B_m denote the board with exactly m squares in the sequence shown below.



(a) Prove that the rook polynomial of B_m is $\sum_k \binom{m-k+1}{k} x^k$. [*Hint: there is a very short proof using Question 5 on Sheet 1.*]



(b) Find the number of ways to place 6 non-attacking rooks on the unshaded squares of the board shown in the margin.

(c) At a dinner party six married couples are to be seated around a circular table. Men and women must sit in alternate places, and no-one may sit next to their spouse. In how many ways can this be done? [*Hint: first seat the women, then use* (b) *to count the number of ways to seat the men.*]

8. This question gives an alternative proof of the Principle of Inclusion and Exclusion. Fix a set X. For each $A \subseteq X$, define a function $1_A : X \to \{0, 1\}$ by

$$1_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

We say that 1_A is the *indicator function* of A.

(a) Show that if B, $C \subseteq X$ then $1_{B \cap C}(x) = 1_B(x) 1_C(x)$ for all $x \in X$.

(b) Let A_1, A_2, $\ldots$, A_n be subsets of X. Show that

$$1_{\overline{A_1 \cup A_2 \cup \cdots \cup A_n}} = (1_X - 1_{A_1})(1_X - 1_{A_2}) \ldots (1_X - 1_{A_n}).$$

(c) By multiplying out the right-hand side and using (a) show that

$$1_{\overline{A_1 \cup A_2 \cup \cdots \cup A_n}} = \sum_{I \subseteq \{1,2,\ldots,n\}} (-1)^{|I|} 1_{A_I}$$

where A_I is as defined just before Theorem 5.3.

(d) Prove Theorem 5.3 by summing the previous equation over all $x \in X$.

9. A deck consists of 52 cards. There are four Aces, four Kings, four Queens and four Jacks. Use the Principle of Inclusion and Exclusion (rather than direct enumeration, as in the Preliminary Problem Sheet) to find the number of hands of five cards that

(a) have at least one Ace, King, Queen and Jack;

(b) have at least one Ace, King and Queen.

MT454 / 5454 Combinatorics: Sheet 5

Do questions 1, 2 and 3 and at least one other question.

To be handed in at the lecture on Tuesday 18th November.

1. Complete the proof of Theorem 9.5 by showing that

$$xF(x) = \frac{1 - \sqrt{1-4x}}{2}.$$

Then use Theorem 7.5 to show that the coefficient of x^{n+1} in $xF(x)$ is $\dfrac{1}{n+1}\dbinom{2n}{n}$.

2. Let $a_0, a_1, a_2, \ldots$ be a sequence of real numbers and let $F(x) = \sum_{n=0}^{\infty} a_n x^n$ be the associated generating function. Let $c_0, c_1, c_2, \ldots$ be the convolution of $a_0, a_1, a_2, \ldots$ with the constant sequence $1, 1, 1, \ldots$.

(a) Write down a formula for c_n.

(b) Express the generating function $\sum_{n=0}^{\infty} c_n x^n$ in terms of F.

3. Let $u_0, u_1, u_2, \ldots$ denote the sequence of Fibonacci numbers, as defined by $u_0 = 0$, $u_1 = 1$ and $u_n = u_{n-1} + u_{n-2}$ for $n \geq 2$. Let $F(x) = \sum_{n=0}^{\infty} u_n x^n$ be the associated generating function. You may assume that $F(x) = x/(1 - x - x^2)$.

(a) Let $v_n = u_{n+2} - 1$. Find the generating function of $v_0, v_1, v_2, \ldots$ in terms of F.

(b) Let $c_n = \sum_{k=0}^{n} u_k$. Find the generating function of $c_0, c_1, c_2, \ldots$ in terms of F.

(c) Hence prove that $\sum_{k=0}^{n} u_k = u_{n+2} - 1$ for all $n \geq 0$.

4. (a) Arguing directly from Definition 9.4 show that the Catalan numbers satisfy the recurrence

$$C_n = C_0 C_{n-1} + C_1 C_{n-2} + \cdots + C_{n-2} C_1 + C_{n-1} C_0$$

for all $n \in \mathbf{N}$.

(b) Hence show that if $F(x) = \sum_{n=0}^{\infty} C_n x^n$ then $xF(x)^2 = F(x) - 1$.

5. The grocer sells apples, bananas, cantaloupe melons and dates. Find, in as simple form as possible, the generating function for the number of ways to buy n pieces of fruit, such that all of the following hold:

(i) the number of apples purchased is a multiple of 5;

(ii) at most 4 bananas are bought;

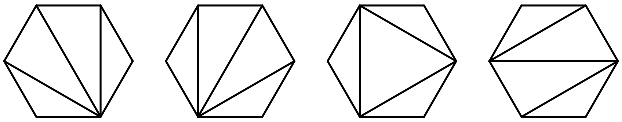
(iii) at most 1 melon is bought;

(iv) the number of dates purchased is odd.

Hence find the number of possible purchases of n pieces of fruit.

6. Define the sequence of Fibonacci numbers as in Question 3. Let $G(x) = \sum_{n=0}^{\infty} u_n x^n / n!$. Show that $G''(x) = G'(x) + G(x)$ and hence find a formula for u_n without making any use of partial fractions.

7. For each $n \geq 3$ let T_n denote the number of ways in which a regular n-gon can be divided into triangles. For example, four of the 14 possible divisions of a hexagon are shown below. (Note that the n-gon sits in a fixed position in the plane: rotations and reflections should *not* be considered in this question.)



(a) Find T_3, T_4 and T_5.

(b) Prove that

$$T_{n+1} = T_n + T_{n-1}T_3 + T_{n-2}T_4 + \cdots + T_3T_{n-1} + T_n$$

for all $n \geq 3$. Hence prove that $T_n = C_{n-2}$. [*Hint: use the recurrence proved in Question 4.*]

8. Let $r \in \mathbf{N}$ and let $\zeta = \exp(2\pi \mathrm{i}/r)$. Show that if $F(x) = \sum_{n=0}^{\infty} a_n x^n$ then

$$F(x) + F(\zeta x) + F(\zeta^2 x) + \cdots + F(\zeta^{r-1} x) = r \sum_{n=0}^{\infty} a_{nr} x^{nr}.$$

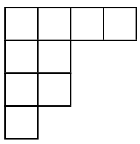
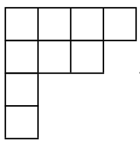
9. Prove that

$$\frac{1}{\sqrt{1-4x}} = \sum_{n=0}^{\infty} \binom{2n}{n} x^n.$$

By squaring both sides deduce the identity

$$\sum_{m=0}^{n} \binom{2m}{m} \binom{2n-2m}{n-m} = 4^n.$$

10. The *conjugate* of a partition is obtained by reflecting its Young diagram in its major diagonal. For example $(4,2,2,1)$ has conjugate $(4,3,1,1)$ since

 reflects to  .

We write λ' for the conjugate of a partition λ.

(a) Show that λ has exactly k parts if and only if k is the largest part of λ'.

(b) Show that the number of partition λ of n such that $\lambda = \lambda'$ is equal to the number of partitions of n into odd distinct parts.

(c) Hence find the generating function for the number of partitions of n that are equal to their conjugate partition.

MT454 / 5454 Combinatorics: Sheet 6

Do questions 1, 2 and 5 and at least one other.

To be returned to McCrea 240 by 3pm on Tuesday 25th November or handed in at the Tuesday lecture.

1. Let a_n be the number of partitions of $n \in \mathbf{N}$ into parts of size 3 and 5.

(a) Show that $a_{15} = 2$ and find a_{14} and a_{16}.

(b) Explain why
$$\sum_{n=0}^{\infty} a_n x^n = \frac{1}{(1-x^3)(1-x^5)}.$$

(c) Let c_n be the number of partitions of with parts of sizes 3 and 5 whose sum of parts is *at most* n. (For example, $c_6 = 5$, the relevant partitions are $\varnothing$, (3), (5), $(3,3)$.) Find the generating function of c_n.

2. Let b_n be the number of partitions of n that have at most one part of each odd size. For example, $b_6 = 5$: the relevant partitions are $(6), (5,1), (4,2), (3,2,1), (2,2,2)$. Express the generating function $\sum_{n=0}^{\infty} b_n x^n$ as an infinite product.

3. Show that there is a red-blue colouring of K_5 with no monochromatic triangle.

4. Let G be a graph with vertex set $\{1, 2, ..., n\}$ and edge set $E(G)$. Let G' be the graph on $\{1, 2, ..., n\}$ with edge set $E(G')$ defined by $\{i, j\} \in E(G')$ if and only if $\{i, j\} \notin E(G)$.

(a) Show that at least one of G and G' is connected.

(b) Can both G and G' be connected?

(c) Show that in red-blue colouring of K_n either the red edges or the blue edges form a connected graph.

5. Let $s, t \geq 2$. By constructing a suitable red-blue colouring of $K_{(s-1)(t-1)}$ prove that $R(s,t) > (s-1)(t-1)$. [*Hint: start by partitioning the vertices into* $s-1$ *blocks each of size* $t-1$. *Colour edges within each block with one colour* ...]

6. Let $s, t \geq 2$.

(a) Prove that if $R(s,t)$ exists then $R(t,s)$ exists and $R(s,t) = R(t,s)$.

(b) Prove that if $s' \geq s$, $t' \geq t$ and $R(s',t')$ exists, then $R(s,t)$ exists and $R(s,t) \leq R(s',t')$.

7. Given a non-empty partition λ, let $r(\lambda)$ denote the greatest $r \in \mathbf{N}$ such that $\lambda_r \geq r$. For example, if $\lambda = (7,5,3,3,2)$ then $r(\lambda) = 3$. The *Durfee square* of λ consists of all the boxes in the Young diagram of λ that are in both its first $r(\lambda)$ rows and its first $r(\lambda)$ columns. Use Durfee squares to prove the identity
$$\prod_{j=1}^{\infty} \frac{1}{1-q^j} = 1 + \sum_{r=1}^{\infty} \frac{q^{r^2}}{(1-q)^2(1-q^2)^2 \dots (1-q^r)^2}.$$

8. Let $\ell \geq 2$. A partition is said to be *ℓ-regular* if it has at most $\ell - 1$ parts of any given size.

(a) Show that the generating function for ℓ-regular partitions is

$$\prod_{j=1}^{\infty} \left(1 + x^j + x^{2j} + \cdots + x^{(\ell-1)j}\right).$$

(b) Show that for each $n \in \mathbf{N}$, the number of ℓ-regular partitions of n is equal to the number of partitions of n into parts not divisible by ℓ.

(c) ($\star$) Find a bijective proof of (b), taking $\ell = 2$ if you wish.

9. Three applications of the Pigeonhole Principle.

(a) Making any reasonable assumptions, prove that there are two students at British universities whose bank balances agree to the nearest penny.

(b) Prove that if five points are chosen inside an equilateral triangle of size 1 then there are two points whose distance is $\leq 1/2$.

(c) Show that in any sequence of n integers, there is a consecutive subsequence whose sum is divisible by n. (For example, in $1, 4, 5, 1, 2, 2, 1$, the sum of $4, 5, 1, 2, 2$ is divisible by 7.)

10. (For hints, see page 26 of printed notes.) Let $P(x) = \sum_{n=0}^{\infty} p(n)x^n$.

(a) Use Theorem 10.3 to prove that

$$\log P(x) = \sum_{r=1}^{\infty} \frac{x^r}{r(1 - x^r)}.$$

(b) Hence show that if $y \geq 1$ then $\log P(\mathrm{e}^{-y}) \leq \pi^2/6y$.

(c) Using the inequality $p(n)\mathrm{e}^{-yn} \leq P(y)$ and taking logs, show that $\log p(n) \leq ny + \pi^2/6y$.

(d) By making a strategic choice of y, prove that $p(n) \leq \mathrm{e}^{c\sqrt{n}}$ where $c = 2\sqrt{\frac{\pi^2}{6}}$.

11. Suppose that the edges of the complete graph K_{2m} are coloured red and blue. Define a *fork* to be a pair $(x, \{y, z\})$ where $x, y, z \in \{1, 2, \ldots, 2m\}$ are distinct and the edges $\{x, y\}$ and $\{x, z\}$ have the same colour. Let f be the number of forks. Let t be the number of monochromatic triangles.

(a) Show that $f = \binom{2m}{3} + 2t$.

(b) Show that $f \geq 2m\left(\binom{m}{2} + \binom{m-1}{2}\right)$.

(c) Hence show that $t \geq 2\binom{m}{3}$.

Taking $m = 3$ in (c) implies that in any red-blue colouring of K_6 there are at least two monochromatic triangles. Show that this result is best possible.

MT454 / 5454 Combinatorics: Sheet 7

Do at least questions 1, 5, 7 and 11.

To be returned to McCrea 240 by 11am on Tuesday 2nd December or handed in at the Tuesday lecture. Question 4(d) is optional.

1. Prove that $R(4,4) \leq 18$. You may assume Theorem 11.9.

2. Suppose that the edges of K_{17} are coloured red, blue and green. By adapting the argument used in Examples 11.3, 11.6 and Lemma 12.1, show that there is a monochromatic triangle. [*Hint: to get started, show that there are* 6 *edges of the same colour meeting vertex* 1.]

3. Given $t \in \mathbf{N}$, let G_t denote the complete graph on $\{1, 2, \ldots, 3(t-1)-1\}$, coloured so that the edge $\{x, y\}$ with $x < y$ is red if $y - x \equiv 1 \bmod 3$, and blue if $y - x \equiv 0$ or 2 mod 3.

(a) Draw G_2 and G_3.

(b) Prove that G_t has no red K_3.

(c) Suppose that $S \subseteq \{1, 2, \ldots, n\}$ is a blue K_t in $G(t)$. Let $S = \{x_1, x_2, \ldots, x_t\}$ where $x_1 < x_2 < \cdots < x_t$. By considering the differences $x_j - x_i$ for $1 \leq i < j \leq t$, get a contradiction.

(d) Deduce that $R(3,t) \geq 3(t-1)$.

4. (a) Use Lemma 12.1 to prove that $R(3,t) \leq t(t+1)/2$ for all $s \in \mathbf{N}$. (Please do not use Theorem 12.2.)

(b) Give a self-contained proof that if $t \leq t'$ then $R(3,t) \leq R(3,t')$.

(c) Use parts (a) and (b) together with the result of Question 3 to give upper and lower bounds for $R(3,6)$ and $R(3, R(3,6))$.

(d) ($\star$) Prove the stronger result that if $t < t'$ then $R(3,t) < R(3,t')$.

5. Find an explicit n such that if the edges of K_n are coloured red, blue, green and yellow, then there exists a monochromatic K_4. (You may use any known bounds on the two-colour Ramsey Numbers.)

6. Given $s, t \in \mathbf{N}$, let $D(s,t)$ denote the smallest n (if one exists) such that whenever the 3-subsets of $\{1, 2, \ldots, n\}$ are coloured red and blue then either there is an s-subset $S \subseteq \{1, 2, \ldots, n\}$ such that all the 3-subsets of S are red; or there is a t-subset $T \subseteq \{1, 2, \ldots, n\}$ such that all the 3-subsets of T are blue.

(a) Prove that $D(3,s) = D(s,3) = s$ for all $s \in \mathbf{N}$.

(b) Prove that $D(4,4) \leq R(4,4) + 1 = 19$. [*Hint: consider the colouring on the* 2*-subsets of* $\{2, 3, \ldots, 19\}$ *induced by giving* $\{x, y\}$ *the colour of* $\{1, x, y\}$.]

(c) Give an explicit upper bound for $D(5,5)$.

7. Let x_1, x_2, ..., x_N be a sequence of distinct integers. Prove that, provided N is sufficiently large, there is either an increasing subsequence of length 2014 or a decreasing subsequence of length 2014. [*Hint: given i and j such that $1 \le i < j \le N$, colour the edge $\{i, j\}$ of K_N red if $x_i < x_j$ and blue if $x_i > x_j$.*]

8. Let $V = \{0, 1, 2, \ldots, 16\}$ and let G be the complete graph on V. Given x, $y \in V$ with $x < y$, colour the edge $\{x, y\}$ red if $y - x$ is a square number modulo 17, and blue otherwise. For example $\{2, 10\}$ is red because $10 - 2 \equiv 5^2 \bmod 17$.

(a) Show if $x, y, u \in V$ and $u \neq 0$ then $\{x + u, y + u\}$ and $\{xu^2, yu^2\}$ have the same colour as $\{x, y\}$. (Here $x + u$ etc. should be taken modulo 17.)

(b) Prove that G has no monochromatic set of size 4. [*Hint: use symmetry and* (a) *to reduce the number of cases that have to be considered.*]

(c) Hence prove that $R(4, 4) = 18$. You may assume Theorem 11.10.

9. By comparing $\int_1^n \log x \, \mathrm{d}x$ with $\log n!$ prove that

$$\left(\frac{n}{\mathrm{e}}\right)^n \le n! \le \left(\frac{n}{\mathrm{e}}\right)^n \mathrm{e}n$$

for all $n \in \mathbf{N}$. (These bounds are crude, but often useful in practice.)

10. At the University of Erewhon, whenever any of its n employees has a birthday, the university closes and everyone takes the day off. Apart from this there are no holidays whatsoever. Local laws require that people are appointed without regard to their date of birth (and there are no leap years).

(a) Show that the probability that the university is open on 25th December is $\left(1 - \frac{1}{365}\right)^n$.

(b) Prove, using linearity of expectation, that the expected number of days of the year when the university is open is $365\left(1 - \frac{1}{365}\right)^n$.

(c) The Pro-Vice Chancellor for Administrative Affairs wishes to maximize the number of person-days worked over the year. Advise him on an optimal choice for n.

11. Let $0 \le p \le 1$ and let $n \in \mathbf{N}$. Suppose that a coin biased to land heads with probability p is tossed n times. Let X be the number of times the coin lands heads.

(a) Describe a suitable probability space Ω and probability measure $p : \Omega \to \mathbf{R}$ and define X as a random variable $\Omega \to \mathbf{R}$.

(b) Find $\mathbf{E}[X]$ and $\mathbf{Var}[X]$. [*Hint: write X as a sum of n independent random variables and use linearity of expectation and* Lemma 13.14(ii).]

(c) Find a simple closed form for the generating function $\sum_{k=0}^{\infty} \mathbf{P}[X = k] x^k$. (Such power series are called *probability generating functions*.)

MT454 / 5454 Combinatorics: Sheet 8

Do questions 2, 3 and 6 and at least one other.

The questions marked ($\star$) are a little harder than average. To be returned to McCrea 240 by 11am on Tuesday 9th December or handed in at the Tuesday lecture.

1. (a) Show, by counting permutations, that the probability 1 and 2 lie in the same cycle of a permutation of $\{1, 2, 3, 4\}$, chosen uniformly at random, is $1/2$.

(b) Let $\sigma = (1, 2, 3, 4, 5, 6)$ and let $\tau = (3, 5)$. Write $\tau \circ \sigma$ and $\tau \circ \sigma \circ \tau$ as compositions of disjoint cycles.

2. Let $n \geq 2$ and let $1 \leq x < y \leq n$. Let τ be the transposition (x, y).

(a) Show that if σ is a permutation of $\{1, 2, \ldots, n\}$ then x and y lie in the same cycle of σ if and only if x and y lie in different cycles of $\tau \circ \sigma$.

(b) Hence find the probability that x and y lie in the same cycle of a permutation of $\{1, 2, \ldots, n\}$ chosen uniformly at random.

3. A lion-tamer has n numbered cages, arranged in a line, and k lions. Each cage can accommodate at most one lion.

(a) Let $1 \leq r < n$. If the lion-tamer puts the lions into the cages at random, what is the probability that both cages r and $r + 1$ are occupied?

(b) On average, how many pairs of adjacent cages will both contain lions? [*Hint: use linearity of expectation.*]

4. Let Ω be the probability space of all permutations of $\{1, 2, 3, 4, 5, 6\}$ in which each permutation has probability $1/6!$. Define

$$\begin{aligned} A &= \{\sigma \in \Omega : \sigma(2) < \sigma(1) < \sigma(4)\} \\ B &= \{\sigma \in \Omega : \sigma(6) < \sigma(1) < \sigma(2)\} \\ C &= \{\sigma \in \Omega : \sigma(6) < \sigma(1) < \sigma(4)\}. \end{aligned}$$

(a) Show that $\mathbf{P}[A] = \mathbf{P}[B] = \mathbf{P}[C] = 1/3!$. [*Hint: in a permutation of $\{1, 2, \ldots, 6\}$, there are $3!$ possible relative orders for $\sigma(2)$, $\sigma(1)$, $\sigma(4)$. Each relative order is equally likely, so $1/6$ of all permutations have $\sigma(2) < \sigma(1) < \sigma(4)$, a further $1/6$ have $\sigma(2) < \sigma(4) < \sigma(1)$, and so on.*]

(b) Show that $\mathbf{P}[A \cap B] = 0$ and that $\mathbf{P}[A \cap C] = \mathbf{P}[B \cap C] = 2/4!$.

(c) Using the Principle of Inclusion and Exclusion, find the number of ways in which the letters A, B, C, D, E, F may be arranged so that none of the words BAD, FAB, FAD can be obtained by crossing out some of the letters.

5. Let F be the number of fixed points of a permutation of $\{1, 2, \ldots, n\}$, chosen uniformly at random. By adapting the argument used to prove Theorem 14.1, find $\mathbf{E}[F^2]$. Hence find $\mathbf{Var}[F]$.

6. Describe each of the proofs you have seen that the number of derangements of $\{1, 2, ..., n\}$ is

$$n! - \frac{n!}{1!} + \frac{n!}{2!} - \cdots + \frac{(-1)^n}{n!}.$$

(One or two lines per proof is ample.) Which proof is your favourite?

7. Let Ω be a probability space and let $X : \Omega \to \mathbf{N}_0$ be a random variable. Prove, using the formula after Definition 13.10, that

$$\mathbf{E}[X] = \sum_{k=1}^{\infty} \mathbf{P}[X \geq k].$$

Deduce Markov's inequality, that $\mathbf{P}[X \geq k] \leq \mathbf{E}[X]/k$ for each $k \in \mathbf{N}$.

8. ($\star$) In a room there are 100 numbered lockers. Each locker contains a piece of paper numbered between 1 and 100 so that each number is used exactly once. A team of 100 numbered people are let into the room, one at a time in numerical order. Each person is allowed to open up to 50 lockers before leaving the room. If every team member finds the piece of paper with his or her number on it, the team succeeds, otherwise they fail. (After each visit the room is returned to its original state, and once someone has visited the room, they cannot communicate with their colleagues.)

Find a strategy that gives the team a probability of success $\geq 1/10$.

9. In an election there are two candidates A and B, each of whom gets exactly n votes. Let c_n be the number of ways in which the votes may be counted so that candidate A is never behind candidate B. (For example, $c_3 = 5$; the corresponding ballot sequences are $AAABBB$, $AABABB$, $AABBAB$, $ABAABB$, $ABABAB$.)

(a) Show that $c_n = \sum_{j=1}^{n} c_{j-1}c_{n-j}$ for each $n \in \mathbf{N}$.

(b) Hence show that c_n is equal to the nth Catalan Number C_n.

(c) Find the probability that when the votes are counted, A is never behind B.

10. Let $m, n \in \mathbf{N}$. A platoon of mn soldiers is arranged in m rows of n soldiers. The sergeant orders the soldiers in each row to rearrange themselves in decreasing order of height and then issues the same order for the columns.

(a) Show that the tallest soldier is now in the first row and the first column.

(b) Show that the rows are still arranged in decreasing order of height. [*Hint: there is an argument using the pigeonhole principle.*]

11. ($\star$) Let $n \in \mathbf{N}$. Let $f \in \mathbf{N}$ be such that $f \leq n$. Show that the number of permutations of $\{1, 2, \ldots, n\}$ with *at least* f fixed points is

$$\frac{n!}{(f-1)!} \sum_{r=f}^{n} \frac{(-1)^{r-f}}{r(r-f)!}.$$

MT454 / 5454 Combinatorics: Sheet 9

Do at least questions 2, 4 and 5.

The questions marked ($\star$) are harder than average. To be returned to McCrea 240 by 5pm Tuesday on the first week of next term.

1. Let σ be a permutation of $\{1, 2, ..., n\}$, chosen uniformly at random. Find the average length of the cycle of σ containing 1.

2. Let e_n be the expected number of cycles in a permutation of $\{1, 2, \ldots, n\}$ chosen uniformly at random. Show, using linearity of expectation, that $e_n = \sum_{k=1}^{n} 1/k$. (You may use Theorem 14.8.)

3. Let t_n be the probability that a permutation of $\{1, 2, \ldots, n\}$, chosen uniformly at random, has a cycle of length $> n/2$.

(a) Use Theorem 14.8 to show that $t_n = \sum_{n/2<k\leq n} 1/k$.

(b) Hence show that $t_n \to \log 2$ as $n \to \infty$.

4. Suppose that the edges of the complete graph on $\{1, 2, ..., n\}$ are coloured red, blue and green. Adapt the proof of Theorem 15.5 to show that if

$$3^{1-\binom{s}{2}} \binom{n}{s} < 1$$

then there is a colouring with no monochromatic K_s. What is the resulting bound on the three-colour Ramsey number for $s = 10$?

5. Let $n \in N$ and let G be the complete graph on $\{1, 2, \ldots, 9\}$. Suppose that a subset A of $\{1, 2, \ldots, 9\}$ is chosen uniformly at random. Let $B = \{1, 2, \ldots, 9\}\backslash A$. What is the probability that the cut (A, B) has capacity $\geq m/2$, where m is the number of edges of G?

6. Let K denote the complete graph on $\mathbf{N}$, so $\{x, y\}$ is an edge of K for all distinct $x, y \in \mathbf{N}$. Show that if the edges of K are coloured red and blue then there is an infinite subset S of $\mathbf{N}$ such that all the edges $\{x, y\}$ for $x, y \in S$ have the same colour.

7. ($\star$) Let A_k be the set of permutations of $\{1, 2, \ldots, n\}$ in which 1 lies in a k cycle. Find a bijective proof that $|A_k| = |A_{k+1}|$ for all k such that $1 \leq k < n$.

8. An aircraft has exactly 100 seats. The 100 people due to travel on it are lined up, in a random order. The first person in the queue has forgotten his seat number, and so sits in one of the seats at random. The remaining 99 people all know their seat numbers and so if their seat is not taken, they sit in it. If their seat is taken, they are too shy to complain and so they sit in a free seat which they choose at random.

Find the probability that the last person in the queue sits in his or her own seat.

9. This question gives an alternative proof of Theorem 6.10 using ideas from generating functions. Let B be a board contained in an $n \times n$ grid. Let $c_m(B)$ be the number of ways to place n non-attacking rooks on the $n \times n$ grid so that *exactly* m rooks are on B.

(a) Show that the number of ways to place k red rooks on B and $n - k$ blue rooks anywhere on the grid, so that all n rooks are non-attacking, is $\sum_{m=k}^{n} \binom{m}{k} c_m(B)$.

(b) Deduce from Lemma 6.9 that $\sum_{m=k}^{n} \binom{m}{k} c_m(B) = r_k(B)(n-k)!$.

(c) Hence show that if $N(x) = \sum_{m=0}^{n} c_m(B)x^m$ then

$$N(x+1) = \sum_{k=0}^{n} r_k(B)(n-k)!x^k.$$

(d) By substituting $x = -1$ in the above equation, prove Theorem 6.10.

10. Prove that if $n, r \in \mathbf{N}$ then

$$r(r-1)\binom{n}{r} = 2\binom{n}{2}\binom{n-2}{r-2}$$

by interpreting each side as the number of ways to choose a committee of r people, one of whose members is the secretary and another is the chairperson.

11. Use generating functions to find formulae for the nth term of the sequences defined by the recurrence relations: (a) $a_n = 6a_{n-2} - a_{n-1}$; (b) $mb_m = (m+2)b_{m-1}$, $b_0 = 1$.

12. By adapting the argument used in Wilf *generatingfunctionology*, Example 4, page 37, find a formula for $\sum_{k=1}^{n} k^3$.

13. There are 10 pirates who have recently acquired a bag containing 100 coins. The leader, number 1, must propose a way to divide up the loot. For instance he might say 'I'll take 91 coins and the rest of you can have one each'. A vote is then taken. If the leader gets half or more of the votes (the leader getting one vote himself), the loot is so divided. Otherwise he is made to walk the plank by his dissatisfied subordinates, and number 2 takes over, with the same responsibility to propose an acceptable division.

Assuming that the pirates are all greedy, untrustworthy, and capable mathematicians, what happens? [*Hint: try thinking about a smaller 2 or 3 pirate problem to get started.*]

14. ($\star$) Let $c_e(n)$ and $c_o(n)$ be the number of derangements of $\{1, 2, \ldots, n\}$ be the number of derangements of $\{1, 2, \ldots, n\}$ with, respectively, an even number of cycles and an odd number of cycles, in their disjoint cycle decomposition. Prove that $c_o(n) - c_e(n) = n - 1$ for all $n \in \mathbf{N}_0$.