COMBINATORICS MT354/MT454/MT5454

MARK WILDON

These notes are intended to give the logical structure of the course; proofs and further remarks will be given in lectures. Further installments will be issued as they are ready. All handouts and problem sheets will be put on Moodle.

I would very much appreciate being told of any corrections or possible improvements to these notes.

You are warmly encouraged to ask questions in lectures, and to talk to me after lectures and in my office hours. I am also happy to answer questions about the lectures or problem sheets by email. My email address is `mark.wildon@rhul.ac.uk`.

Lectures: Tuesday 11am in C201, Thursday 9am in ABLT3 and Friday 1pm in C336.

Office hours in McCrea 240: Monday 4pm, Thursday 2pm and Friday 4pm or by appointment (email me).

Date: First term 2015/16.

1. INTRODUCTION

Combinatorial arguments may be found lurking in all branches of mathematics. Many people first become interested in mathematics by a combinatorial problem. But, strangely enough, at first many mathematicians tended to sneer at combinatorics. Thus one finds:

> *"Combinatorics is the slums of topology."*
>
> J. H. C. Whitehead (early 1900s, attr.)

Fortunately attitudes have changed, and the importance of combinatorial arguments is now widely recognised:

> *"The older I get, the more I believe that at the bottom of most deep mathematical problems there is a combinatorial problem."*
>
> I. M. Gelfand (1990)

Combinatorics is a very broad subject. It will often be useful to prove the same result in different ways, in order to see different combinatorial techniques at work. There is no shortage of interesting and easily understood motivating problems.

OVERVIEW. This course will give a straightforward introduction to four related areas of combinatorics. Each is the subject of current research, and taken together, they give a good idea of what the subject is about.

- **(A) Enumeration:** Binomial coefficients and their properties. Principle of Inclusion and Exclusion and applications. Rook polynomials.
- **(B) Generating Functions:** Ordinary generating functions and recurrence relations. Partitions and compositions. Catalan Numbers. Derangements.
- **(C) Ramsey Theory:** "Complete disorder is impossible". Pigeonhole Principle. Graph colouring.
- **(D) Probabilistic Methods:** Linearity of expectation. First moment method. Applications to counting permutations. Lovász Local Lemma.

RECOMMENDED READING.

[1] *A First Course in Combinatorial Mathematics*. Ian Anderson, OUP 1989, second edition.

[2] *Discrete Mathematics.* N. L. Biggs, OUP 1989, second edition.

[3] *Combinatorics: Topics, Techniques, Algorithms.* Peter J. Cameron, CUP 1994.

[4] *Concrete Mathematics.* Ron Graham, Donald Knuth and Oren Patashnik, Addison-Wesley 1994.

[5] *Invitation to Discrete Mathematics.* Jiri Matoušek and Jaroslav Nešetřil, OUP 2009, second edition.
[6] *Probability and Computing: Randomized Algorithms and Probabilistic Analysis.* Michael Mitzenmacher and Eli Upfal, CUP 2005.
[7] *generatingfunctionology*. Herbert S. Wilf, A K Peters 1994, second edition. Available from `http://www.math.upenn.edu/~wilf/DownldGF.html`.

In parallel with the first few weeks of lectures, you will be asked to do some reading from *generatingfunctionology*: the problem sheets will make clear what is expected.

PREREQUISITES.

- Basic definitions of graph theory: vertices, edges and complete graphs. (Required for Part C on Ramsey Theory.)
- Basic knowledge of discrete probability. This will be reviewed in lectures when we get to part D of the course. A handout with all the background results needed from probability theory will be issued later in term.

PRELIMINARY PROBLEM SHEET AND EXERCISES. The preliminary problem sheet is designed to get you thinking about the basic counting principles seen in the first three lectures. Exercises set in these notes are simple tests that you are following the material. Some will be done in lectures. **Doing the others will help you to review the lectures.**

PROBLEM SHEETS. There will be nine marked problem sheets; the first will be due in on Tuesday 14th October. **You are very welcome to discuss the problems with the lecturer.** You do not have to wait until answers appear on Moodle. I will give you full marks if you discuss the question with me but write up your answer on your own.

MOODLE. Provided you have a College computer account, you have access to the Moodle page for this course: `moodle.rhul.ac.uk/course/view.php?id=371`. If you are registered for the course then it will appear under 'My Courses' on Moodle.

MSC MINI-PROJECT. If you are an MSc student then you are doing MT5454 and must submit answers to the mini-project problem sheet by 12 noon, Friday 11th December. This problem sheet will be available from Moodle in week 3. It is on material on parts (A) and (B) of the course.

NOTE ON OPTIONAL QUESTIONS. Optional questions on problem sheets are included for interest and to give extra practice. Harder optional questions are marked $(\star)$.

> **Warning:** You will not pass this course by last minute cramming. You must attempt the compulsory questions on problem sheets and learn the techniques for yourself.

2. COUNTING PRINCIPLES AND DERANGEMENTS

In the first two lectures we will see the Derangements Problem and one way to solve it by *ad-hoc* methods. Later in the course we will develop techniques that can be used to solve this problem more easily.

On the way we will see three basic counting principles.

Definition 2.1. A *permutation* of a set X is a bijective function

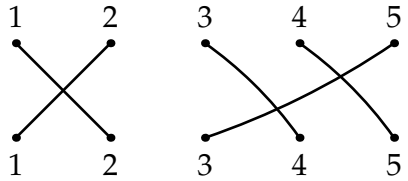
$$\sigma : X \to X.$$

A *fixed point* of a permutation σ of X is an element $x \in X$ such that $\sigma(x) = x$. A permutation is a *derangement* if it has no fixed points.

Usually we will consider permutations of $\{1, 2, \ldots, n\}$ for some natural number $n \in \mathbf{N}_0$. It is often useful to represent permutations by diagrams. For example, the diagram below shows the permutation $\sigma : \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4, 5\}$ defined by

$$\sigma(1) = 2,\ \sigma(2) = 1,\ \sigma(3) = 4,\ \sigma(4) = 5,\ \sigma(5) = 3.$$

Note that σ is a derangement.



Exercise: For $n \in \mathbf{N}_0$, how many permutations are there of $\{1, 2, \ldots, n\}$? How many of these permutations have 1 as a fixed point?

To solve this exercise we used the principle that when one choice is made after another, the number of choices should be multiplied. This will be used many times in this course.

> **BCP1: Multiplying Choices.** If an object can be specified uniquely by a sequence of r choices so that, when making the ith choice, we always have exactly c_i possibilities to choose from, then there are exactly $c_1 c_2 \ldots c_r$ objects.

In the special case where we make two choices, and one choice does not affect the next, so we first choose an element of a set A, then an element of a set B, this simply says that $|A \times B| = |A||B|$.

Problem 2.2 (Derangements)**.** *How many permutations of* $\{1, 2, \ldots, n\}$ *are derangements?*

Let d_n be the number of permutations of $\{1, 2, \ldots, n\}$ that are derangements. By definition, although you may regard this as a convention if you prefer, $d_0 = 1$.

Exercise: Check, by listing permutations, or some cleverer method, that $d_1 = 0$, $d_2 = 1$, $d_3 = 2$ and $d_4 = 9$.

Two further basic counting principles were used to get $d_4 = 9$.

> **BCP2: Adding choices.** If a finite set of objects can be partitioned into two disjoint sets A and B, then the total number of objects is $|A| + |B|$.

> **BCP0: Sets in bijection have the same size.** If there is a bijection between finite sets A and B then $|A| = |B|$.

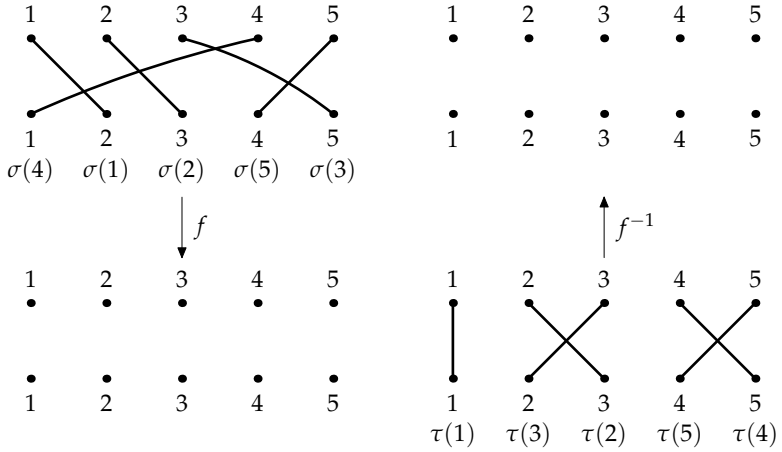
Both principles should seem obvious. For instance, all 'Adding choices' says is that if $X = A \cup B$ where $A \cap B = \varnothing$, then $|X| = |A| + |B|$.

Exercise: Suppose we try to construct a derangement of $\{1, 2, 3, 4, 5\}$ such that $\sigma(1) = 2$. Show that there are two derangements such that $\sigma(1) = 2, \sigma(2) = 1$, and three derangements such that $\sigma(1) = 2$, $\sigma(2) = 3$. How many choices are there for $\sigma(3)$ in each case?

The previous exercise shows that we can't hope to solve the derangements problem just by multiplying choices. Instead we shall find a recurrence for the numbers d_n.

Lemma 2.3. *If* $n \geq 2$ *then the number of derangements* σ *of* $\{1, 2, \ldots, n\}$ *such that* $\sigma(1) = 2$ *is* $d_{n-2} + d_{n-1}$.

Exercise: Let f be the 'swapping letters' bijection defined in the second lecture. Two permutations are shown overleaf. Apply f to the permutation σ on the left, and apply f^{-1} to the permutation τ on the right. Check in each case the image is in the expected set.



You can, of course, make up many similar examples yourself. Generally you should get into the habit of seeing how proofs work by **trying them out on particular examples**.

Theorem 2.4. *If $n \geq 2$ then $d_n = (n-1)(d_{n-2} + d_{n-1})$.*

Using this recurrence relation it is easy to find values of d_n for much larger n. Whenever one meets a new combinatorial sequence it is a good idea to look it up in N. J. A. Sloane's Online Encyclopedia of Integer Sequences: see `www.research.att.com/~njas/sequences/`. You will usually find it in there, along with references and often other combinatorial interpretations.

Corollary 2.5. *For all $n \in \mathbf{N}_0$,*

$$d_n = n!\Big(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\Big).$$

Exercise: Check directly that the right-hand side is an integer.

A more systematic way to derive Corollary 2.5 from Theorem 2.4 will be seen in Part B of the course. Question 9 on Sheet 1 gives an alternative proof that does not require knowing the answer in advance.

The proof of Corollary 2.5 and Question 9 show that it is helpful to consider the probability $d_n/n!$ that a permutation of $\{1, 2, \ldots, n\}$, chosen uniformly at random, is a derangement. Here 'uniformly at random' means that each of the $n!$ permutations of $\{1, 2, \ldots, n\}$ is equally likely to be chosen.

Theorem 2.6. *Choose permutations of $\{1, 2, \ldots, n\}$ uniformly at random.*

(i) *The probability that a permutation of $\{1, 2, \ldots, n\}$ is a derangement tends to $1/\mathrm{e}$ as $n \to \infty$.*

(ii) *The mean number of fixed points of a permutation of $\{1, 2, \ldots, n\}$ is 1.*

We shall prove more results like this in Part D of the course.

Part A: Enumeration

3. BINOMIAL COEFFICIENTS

We have already used the notation $|X|$ for the size of a set X.

Notation 3.1. If Y is a set of size k then we say that Y is a *k-set*. To emphasise that Y is a subset of some other set X then we may say that Y is a *k-subset of* X.

We shall define binomial coefficients combinatorially.

Definition 3.2. Let $n, k \in \mathbf{N}_0$. Let $X = \{1, 2, \ldots, n\}$. The *binomial coefficient* $\binom{n}{k}$ is the number of k-subsets of X.

By this definition, if $k \notin \mathbf{N}_0$ then $\binom{n}{k} = 0$. Similarly if $k > n$ then $\binom{n}{k} = 0$. It should be clear that we could replace X with any other set of size n and we would define the same numbers $\binom{n}{k}$.

We should check that the combinatorial definition agrees with the usual definition.

Lemma 3.3. *If $n, k \in \mathbf{N}_0$ and $k \leq n$ then*

$$\binom{n}{k} = \frac{n(n-1)\ldots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}.$$

The technique of double counting used to prove Lemma 2.6(ii) and Lemma 3.3 is often useful in combinatorial counting problems.

Many of the basic properties of binomial coefficients can be given combinatorial proofs using explicit bijections and the three Basic Counting Principles. We say that such proofs are *bijective*.

Lemma 3.4. *If $n, k \in \mathbf{N}_0$ and $k \leq n$ then*

$$\binom{n}{k} = \binom{n}{n-k}.$$

Lemma 3.5 (Fundamental Recurrence)**.** *If $n, k \in \mathbf{N}$ then*

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Exercise: Prove bijectively that $(n-r)\binom{n}{r} = (r+1)\binom{n}{r+1}$ if $0 \leq r \leq n$.

Binomial coefficients are so-named because of the famous binomial theorem. (A binomial is a product of the form $x^r y^s$.)

Theorem 3.6 (Binomial Theorem). *Let $x, y \in \mathbf{C}$. If $n \in \mathbf{N}_0$ then*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Exercise: Write out an alternative proof of the Binomial Theorem by induction on n, using Lemma 3.5 in the inductive step. Which proof do you find more convincing?

4. Further binomial identities and balls and urns

This is a vast subject and we shall only cover a few aspects. Particularly recommended for optional further reading is Chapter 5 of *Concrete Mathematics*, [4] in the list on page 2.

PASCAL'S TRIANGLE. The entry in row n and column r of Pascal's Triangle is $\binom{n}{r}$. Pascal's Triangle can be computed by hand using $\binom{n}{0} = \binom{n}{n} = 1$ and the Fundamental Recurrence.

There are a number of nice identities that express row, column or diagonal sums in Pascal's Triangle.

Lemma 4.1 (Alternating row sums). *If $n \in \mathbf{N}$, $r \in \mathbf{N}_0$ and $r \leq n$ then*

$$\sum_{k=0}^{r} (-1)^k \binom{n}{k} = (-1)^r \binom{n-1}{r}.$$

Perhaps surprisingly, there is no simple formula for the unsigned row sums $\sum_{k=0}^{r} \binom{n}{k}$.

Lemma 4.2 (Diagonal sums, a.k.a. parallel summation). *If $n \in \mathbf{N}_0$, $r \in \mathbf{N}_0$ then*

$$\sum_{k=0}^{r} \binom{n+k}{k} = \binom{n+r+1}{r}.$$

For the column sums on Pascal's Triangle, see Sheet 1, Question 3. For the other diagonal sum, see Sheet 1, Question 7.

ARGUMENTS WITH SUBSETS. The two identities below are among the most useful in practice. Bijective proofs will be given in lectures.

Lemma 4.3 (Subset of a subset). *If $k, r, n \in \mathbf{N}_0$ and $k \leq r \leq n$ then*

$$\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}.$$

Lemma 4.4 (Vandermonde's convolution). *If $a, b \in \mathbf{N}_0$ and $m \in \mathbf{N}_0$ then*

$$\sum_{k=0}^{m} \binom{a}{k}\binom{b}{m-k} = \binom{a+b}{m}.$$

COROLLARIES OF THE BINOMIAL THEOREM. The following results can be obtained by making a strategic choice of x and y in the Binomial Theorem.

Corollary 4.5.

(i) *If* $n \in \mathbf{N}_0$ *then* $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

(ii) *If* $n \in \mathbf{N}$ *then* $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$.

Exercise: Find a bijective proof of (i) and a bijective proof of (ii) when n is odd. *Harder exercise:* Is there a bijective proof of (ii) when n is even?

Corollary 4.6. *For all* $n \in \mathbf{N}$ *there are equally many subsets of* $\{1, 2, \ldots, n\}$ *of even size as there are of odd size.*

BALLS AND URNS. We can now answer a basic combinatorial question: *How many ways are there to put k balls into n numbered urns?* The answer depends on whether the balls are distinguishable. We may consider urns of unlimited capacity, or urns that can only contain one ball.

	Numbered balls	Indistinguishable balls
≤ 1 ball per urn		
unlimited capacity		

Three of the entries can be found fairly easily. The entry in the bottom-right can be found in many different ways: two will be demonstrated in this lecture.

Theorem 4.7. *Let* $n \in \mathbf{N}$ *and let* $k \in \mathbf{N}_0$. *The number of ways to place k indistinguishable balls into n numbered urns of unlimited capacity is* $\binom{n+k-1}{k}$.

The following reinterpretation of Theorem 4.7 is often useful.

Corollary 4.8. *Let* $n \in \mathbf{N}$ *and let* $k \in \mathbf{N}_0$. *The number of n-tuples* $(t_1, \ldots, t_n)$ *such that* $t_1, t_2, \ldots, t_n \in \mathbf{N}_0$ *and*

$$t_1 + t_2 + \cdots + t_n = k$$

is $\binom{n+k-1}{k}$.

5. Principle of Inclusion and Exclusion

The Principle of Inclusion and Exclusion (PIE) is way to find the size of a union of a finite collection of subsets of a finite *universe set* X. The universe set we take will depend on the problem we are solving. If A is a subset of X, we denote by $\overline{A}$ the complement of A in X; i.e.,

$$\bar{A} = X \backslash A = \{x \in X : x \notin A\}.$$

We start with the two smallest non-trivial examples of the Principle of Inclusion and Exclusion.

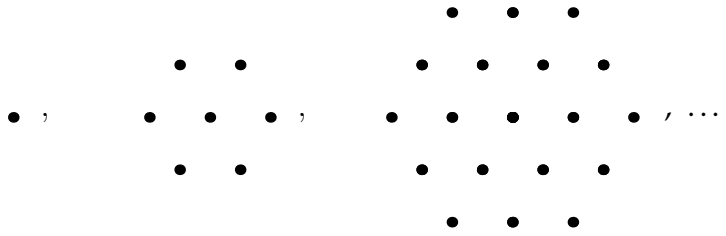
Example 5.1. If A, B, C are subsets of a finite set X then

$$|A \cup B| = |A| + |B| - |A \cap B|$$
$$|\overline{A \cup B}| = |X| - |A| - |B| + |A \cap B|$$

and

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| \\ &\quad - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C| \\ |\overline{A \cup B \cup C}| &= |X| - |A| - |B| - |C| \\ &\quad + |A \cap B| + |B \cap C| + |C \cap A| - |A \cap B \cap C| \end{aligned}$$

Example 5.2. The m-th (centred) hexagonal number is the number of dots in the m-th figure below. The Principle of Inclusion and Exclusion gives a nice way to find these numbers.



It is easier to find the sizes of the intersections of the three rhombi making up each hexagon than it is to find the sizes of their unions. Whenever intersections are easier to think about than unions, the PIE is likely to work well.

In the general setting we have a finite universe set X and subsets $A_1, A_2, \ldots, A_n \subseteq X$. For each non-empty subset $I \subseteq \{1, 2, \ldots, n\}$ we define

$$A_I = \bigcap_{i \in I} A_i.$$

Thus A_I is the set of elements which belong to all the sets A_i for $i \in I$. For example, if $i, j \in \{1, 2, \ldots, n\}$ then $A_{\{i\}} = A_i$ and $A_{\{i,j\}} = A_i \cap A_j$. By convention we set

$$A_\emptyset = X.$$

Theorem 5.3 (Principle of Inclusion and Exclusion)**.** *If $A_1, A_2, \ldots, A_n$ are subsets of a finite set X then*

$$|\overline{A_1 \cup A_2 \cup \cdots \cup A_n}| = \sum_{I \subseteq \{1,2,\ldots,n\}} (-1)^{|I|} |A_I|.$$

Exercise: Check that Theorem 5.3 holds when $n = 1$ and check that it agrees with Example 5.1 when $n = 2$ and $n = 3$.

Exercise: Deduce from Theorem 5.3 that

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{\substack{I \subseteq \{1,2,\ldots,n\} \\ I \neq \emptyset}} (-1)^{|I|-1} |A_I|.$$

PRIME NUMBERS AND EULER'S ϕ FUNCTION. Suppose we want to find the number of primes less than some number M. One approach, which is related to the Sieve of Eratosthenes, uses the Principle of Inclusion and Exclusion.

Example 5.4. Let $X = \{1, 2, \ldots, 48\}$. We define three subsets of X:

$$B(2) = \{m \in X : \ m \text{ is divisible by } 2\}$$
$$B(3) = \{m \in X : \ m \text{ is divisible by } 3\}$$
$$B(5) = \{m \in X : \ m \text{ is divisible by } 5\}.$$

Any composite number ≤ 48 is divisible by either 2, 3 or 5. So

$$\overline{B(2) \cup B(3) \cup B(5)} = \{1\} \cup \{p : 5 < p \leq 48, \ \ p \text{ is prime}\}.$$

We will find the size of the left-hand side using the PIE, and hence count the number of primes ≤ 48.

The example can be generalized to count numbers not divisible by any of a specified set of primes. Recall that if $x \in \mathbf{R}$ then $\lfloor x \rfloor$ denotes the largest natural number $\leq x$.

Lemma 5.5. *Let $r, M \in \mathbf{N}$. There are exactly $\lfloor M/r \rfloor$ numbers in $\{1, 2, \ldots, M\}$ that are divisible by r.*

Theorem 5.6. *Let $p_1, \ldots, p_n$ be distinct prime numbers and let $M \in \mathbf{N}$. The number of natural numbers $\leq M$ that are not divisible by any of primes $p_1, \ldots, p_n$ is*

$$\sum_{I \subseteq \{1,2,\ldots,n\}} (-1)^{|I|} \left\lfloor \frac{M}{\prod_{i \in I} p_i} \right\rfloor.$$

For $M \in \mathbf{N}$, let $\pi(M)$ be the number of prime numbers $\leq M$. It is possible to use Theorem 5.6 to show that there is a constant C such that

$$\pi(M) \leq \frac{CM}{\log\log M}$$

for all natural numbers $M \geq 2$. This is beyond the scope of this course, but I would be happy to go through the proof in an office-hour or supply a reference.

The next example will be helpful for the questions on Sheet 2. In it, we say that numbers n, M are *coprime* if n and M have no common prime divisors. For example, 12 and 35 are coprime, but 7 and 14 are not.

Example 5.7. Let $M = pq$ where p and q are distinct prime numbers. The numbers of natural numbers less than or equal to M that are coprime to M is

$$M\Big(1 - \frac{1}{p}\Big)\Big(1 - \frac{1}{q}\Big).$$

APPLICATION TO DERANGEMENTS. The Principle of Inclusion and Exclusion gives a particularly elegant proof of the formula for the derangement numbers d_n first proved in Corollary 2.5:

$$d_n = n!\Big(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{(-1)^n}{n!}\Big).$$

Recall from Definition 2.1 that a permutation

$$\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$$

is a derangement if and only if it has no fixed points. Let X be the set of all permutations of $\{1, 2, \ldots, n\}$ and let

$$A_i = \{\sigma \in X : \sigma(i) = i\}$$

be the set of permutations which have i as a fixed point. To apply the PIE we need the results in the following lemma.

Lemma 5.8. (i) *A permutation $\sigma \in X$ is a derangement if and only if*

$$\sigma \in \overline{A_1 \cup A_2 \cup \cdots \cup A_n}.$$

(ii) *If $I \subseteq \{1, 2, \ldots, n\}$ then A_I consists of all permutations of $\{1, 2, \ldots, n\}$ which fix the elements of I. If $|I| = k$ then*

$$|A_I| = (n - k)!.$$

It is often helpful to think of each A_i as the set of all objects in X satisfying a property P_i. Then the Principle of Inclusion and Exclusion counts all the objects in X that satisfy *none* of the properties $P_1, \ldots, P_n$. In the derangements example

$$P_i(\sigma) = \text{`}\sigma \text{ has } i \text{ as a fixed point}'$$

and we count the permutations σ such that $P_i(\sigma)$ is false for all $i \in \{1, 2, \ldots, n\}$.

OTHER APPLICATIONS. There are many other applications of the Principle of Inclusion and Exclusion. Here are three.

- The PIE can be used to count the number of irreducible polynomials of a given degree over a finite field. Such polynomials are important in coding theory and cryptography.
- See Question 9 on Sheet 2 for an application of the Principle of Inclusion and Exclusion to counting the number of surjective functions from $\{1, \ldots, k\}$ to $\{1, \ldots, n\}$.
- Theorem 5.6 is a basic result of sieve theory: deep generalizations of this result were used in Yitang Zhang's breakthrough proof in 2013 that there are infinitely many prime numbers p and q such that $q > p$ and $q - p \leq 70\,000\,000$.

6. ROOK POLYNOMIALS

Many enumerative problems can be expressed as problems about counting permutations with some restriction on their structure. The derangements problem is a typical example. In this section we shall see a unified way to solve this sort of problem.

Recommended optional reading: Ian Anderson, *A First Course in Combinatorial Mathematics*, §5.2 ([1] on the list on page 2) and Victor Bryant, *Aspects of Combinatorics*, Chapter 12 (Cambridge University Press). Examples 6.3 and 6.4 below are based on those in Bryant's book.

Definition 6.1. A *board* is a subset of the squares of an $n \times n$ grid. Given a board B, we let $r_k(B)$ denote the number of ways to place k indistinguishable rooks on B, so that no two rooks are in the same row or column. Such rooks are said to be *non-attacking*. The *rook polynomial* of B is defined to be

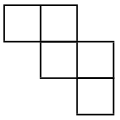
$$f_B(x) = r_0(B) + r_1(B)x + r_2(B)x^2 + \cdots + r_n(B)x^n.$$

Note that $f_B(x)$ is the generating function of the sequence

$$r_0(B), r_1(B), r_2(B), \ldots$$

Since $r_k(B) = 0$ if $k > n$, the power series $\sum_{k=0}^{\infty} r_k(B)x^k$ is a polynomial.

Example 6.2. Let B be the board shown below.



The rook polynomial of B is $1 + 5x + 6x^2 + x^3$.

Exercise: Let B be a board. Show that $r_0(B) = 1$ and that $r_1(B)$ is the number of squares in B.

Example 6.3. After the recent spate of cutbacks, only four professors remain at the University of Erewhon. Prof. W can lecture courses 1 or 4; Prof. X is an all-rounder and can lecture 2, 3 or 4; Prof. Y refuses to lecture anything except 3; Prof. Z can lecture 1 or 2. If each professor must lecture exactly one course, how many ways are there to assign professors to courses?

Example 6.4. How many derangements σ of $\{1, 2, 3, 4, 5\}$ have the property that $\sigma(i) \neq i + 1$ for $1 \leq i \leq 4$?

Lemma 6.5. *The rook polynomial of the $n \times n$ board is*

$$\sum_{k=0}^{n} k! \binom{n}{k}^2 x^k.$$

The two following lemmas are very useful when calculating rook polynomials. Lemma 6.5 will be illustrated with an example in lectures, and proved later in Section 9 using convolutions of generating functions.

Lemma 6.6. *Let C be a board. Suppose that the squares in C can be partitioned into sets A and B so that no square in A lies in the same row or column as a square of B. Then*

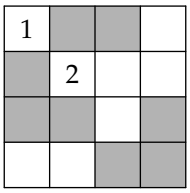
$$f_C(x) = f_A(x) f_B(x).$$

This is the first of many times that multiplying generating functions will help us to solve combinatorial problems.

Lemma 6.7. *Let B be a board and let s be a square in B. Let D be the board obtained from B by deleting s and let E be the board obtained from B by deleting the entire row and column containing s. Then*

$$f_B(x) = f_D(x) + x f_E(x).$$

Example 6.8. The rook-polynomial of the boards in Examples 6.3 and 6.4 can be found using Lemma 6.7. For the board in Example 6.3 it works well to apply the lemma first to the square marked 1, then to the square marked 2 (in the new boards).



Our final result on rook polynomials is often the most useful in practice. The proof uses the Principle of Inclusion and Exclusion. The following lemma isolates the key idea. Its proof needs the same idea we used in Lemma 5.8(ii) to count permutations with a specified set of fixed points.

Lemma 6.9. *Let B be a board contained in an $n \times n$ grid and let $0 \leq k \leq n$. The number of ways to place k red rooks on B and $n-k$ blue rooks anywhere on the grid, so that the n rooks are non-attacking, is $r_k(B)(n-k)!$.*

Theorem 6.10. *Let B be a board contained in an $n \times n$ grid. Let $\bar{B}$ denote the board formed by all the squares in the grid that are not in B. The number of ways to place n non-attacking rooks on $\bar{B}$ is*

$$n! - (n-1)!r_1(B) + (n-2)!r_2(B) - \cdots + (-1)^n r_n(B).$$

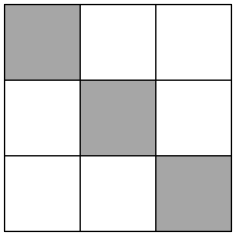
As an easy corollary we get our third proof of the derangements formula (Corollary 2.5), that

$$d_n = n!\Big(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{(-1)^n}{n!}\Big).$$

See Problem Sheet 3 for some other applications of Theorem 6.10.

Theorem 6.10 is one of the harder results in the course. If you find the proof difficult, you may find the following exercise helpful.

Exercise: Let $n = 3$ and let B be the board formed by the *shaded* squares below.



Draw the rook placements lying in each of the sets $A_\varnothing$, $A_{\{1\}}$, $A_{\{2\}}$, $A_{\{3\}}$, $A_{\{1,2\}}$, $A_{\{1,3\}}$, $A_{\{2,3\}}$, $A_{\{1,2,3\}}$ defined in the proof of Theorem 6.10, and check the main claim in the proof for $k = 0, 1, 2, 3$. For instance, for $k = 1$, you should find that $|A_{\{1\}}| + |A_{\{2\}}| + |A_{\{3\}}|$ is the number of non-attacking placements with one red rook on B and two blue rooks anywhere on the grid; according to Lemma 6.9 there are $r_1(B)(3-1)!$ such placements.

Part B: Generating Functions

7. Introduction to generating functions

Generating functions can be used to solve the sort of recurrence relations that often arise in combinatorial problems. But better still, they can help us to think about combinatorial problems in new ways and suggest new results.

Definition 7.1. The *ordinary generating function* associated to the sequence $a_0, a_1, a_2, \ldots$ is the power series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots .$$

To indicate that $F(x)$ is the ordinary generating function of the sequence $a_0, a_1, a_2, \ldots$ we may use the notation in §2.2 of Wilf *generatingfunctionology* and write

$$(a_n) \stackrel{ogf}{\longleftrightarrow} F(x).$$

Usually we shall drop the word 'ordinary' and just write 'generating function'.

If there exists $N \in \mathbf{N}$ such that $a_n = 0$ if $n > N$, then the generating function of the sequence $a_0, a_1, a_2, \ldots$ is a polynomial. Rook polynomials (see Definition 6.1) are therefore generating functions.

Operations on generating functions. Let $F(x) = \sum_{n=0}^{\infty} a_n x^n$ and $G(x) = \sum_{n=0}^{\infty} b_n x^n$ be generating functions. From

$$F(x) + G(x) = \sum_{n=0}^{\infty} (a_n + b_n) x^n$$

and

$$F(x)G(x) = \sum_{n=0}^{\infty} c_n x^n$$

where $c_n = \sum_{m=0}^{n} a_m b_{n-m}$. The derivative of $F(x)$ is

$$F'(x) = \sum_{n=0}^{\infty} n x^{n-1}.$$

Note that if $(a_n) \stackrel{ogf}{\longleftrightarrow} F(x)$ and $(b_n) \stackrel{ogf}{\longleftrightarrow} G(x)$ then

$$(a_n + b_n) \stackrel{ogf}{\longleftrightarrow} F(x) + G(x).$$

The sequence (c_n) such that $(c_n) \stackrel{ogf}{\longleftrightarrow} F(x)G(x)$ often arises in combinatorial problems. This was seen for rook polynomials in Lemma 6.6, and will be studied in §9.

It is also possible to define $1/F(x)$ whenever $a_0 \neq 0$. By far the most important case is the case $F(x) = 1 - x$, when

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

is the usual formula for the sum of a geometric progression.

ANALYTIC AND FORMAL INTERPRETATIONS. There are at least two ways to think of a generating function $\sum_{n=0}^{\infty} a_n x^n$. Either:

- As a formal power series with x acting as a place-holder. This is the 'clothes-line' interpretation (see the first page of Wilf *generatingfunctionology*), in which we regard the power-series as a convenient way to display the terms in our sequence.
- As a function of a real or complex variable x convergent when $|x| < r$, where r is the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$.

The formal point of view is often the most convenient because it allows us to define and manipulate power series by the operations on the previous page without worrying about convergence. From this point of view,

$$0! + 1!x + 2!x^2 + 3!x^3 + \cdots$$

is a perfectly respectable formal power series, even though it only converges when $x = 0$. The analytic point of view is useful for proving asymptotic results.[1]

All the generating functions one normally encounters have positive radius of convergence, so in practice, the two approaches are equivalent. For a more careful discussion of these issues and the general definition of $1/F(x)$, see §2.1 of Wilf *generatingfunctionology*.

TWO EXAMPLES OF GENERATING FUNCTIONS.

Example 7.2. What is the generating function for the number of ways to tile a $2 \times n$ path with bricks that are either 1×2 () or 2×1 ()?

[1] From the analytic perspective, the formula for the derivative $F'(x)$ on the previous page expresses a non-trivial theorem, namely that a power series is a differentiable functions (within its radius of convergence) with derivative given by term-by-term differentiation. A similar remark applies to the formulae for the sum $F(x) + G(x)$ and product $F(x)G(x)$.

See the exercise on page 19 for how to extract a formula for the number of tilings from the generating function.

In the second example we shall use products of power series to give a proof of Corollary 4.8 that is logically independent of Part A. (We assume $n = 3$, to make the notation simpler, but once you understand this case, you should see that the general case is no harder.)

Example 7.3. Let $k \in \mathbf{N}_0$. Let b_k be the number of 3-tuples (t_1, t_2, t_3) such that $t_1, t_2, t_3 \in \mathbf{N}_0$ and $t_1 + t_2 + t_3 = k$. Then

$$\sum_{k=0}^{\infty} b_k x^k = \frac{1}{(1-x)^3}$$

and so $b_k = \binom{k+2}{2}$.

USEFUL POWER SERIES. To complete Example 7.3 we needed a special case of the result below, which was proved on Question 4 of Sheet 3.

Theorem 7.4. *If $n \in \mathbf{N}$ then*

$$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k$$

A more general result is stated below.

Theorem 7.5 (Binomial Theorem for general exponent)**.** *If $\alpha \in \mathbf{R}$ then*

$$(1+y)^\alpha = \sum_{k=0}^{\infty} \frac{\alpha(\alpha-1)\dots(\alpha-(k-1))}{k!} y^k$$

for all y such that $|y| < 1$.

Exercise: Let $\alpha \in \mathbf{Z}$.

(i) Show that if $\alpha \geq 0$ then Theorem 7.4 agrees with the Binomial Theorem for integer exponents, proved in Theorem 3.6, and with Theorem 7.5.

(ii) Show that if $\alpha < 0$ then Theorem 7.4 agrees with Question 4 on Sheet 3. (Substitute $-x$ for y.)

We shall need the case $\alpha = 1/2$ of the general Binomial Theorem to find the Catalan Numbers in §9.

As we saw in Example 7.3, geometric series often arise in generating functions problem. So you need to get used to spotting either side of the identity $1/(1-rx) = \sum_{n=0}^{\infty} r^n x^n$. The exponential series, $\exp x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, is also often useful.

8. Recurrence relations and asymptotics

We have seen that combinatorial problems often lead to recurrence relations. For example, in §2 we found the derangement numbers d_n by solving the recurrence relation in Theorem 2.4. See also Questions 5 and 7 on Sheet 1 for other examples.

Generating functions are very useful for solving recurrence relations. The method is clearly explained at the end of §1.2 of Wilf *generatingfunctionology*. Given a recurrence satisfied by the sequence $a_0, a_1, a_2, \ldots$ proceed as follows:

(a) Use the recurrence to write down an equation satisfied by the generating function $F(x) = \sum_{n=0}^{\infty} a_n x^n$;

(b) Solve the equation to get a closed form for the generating function;

(c) Use the closed form for the generating function to find a formula for the coefficients.

Step (a) will become routine with practice. To obtain terms like na_{n-1}, try differentiating $F(x)$. Powers of x will usually be needed to get everything to match up correctly. In Step (c) it is often necessary to use partial fractions.

Example 8.1. Solve $a_{n+2} = 5a_{n+1} - 6a_n$ for $n \in \mathbf{N}_0$ subject to the initial conditions $a_0 = A$, $a_1 = B$.

Another way to proceed is to first rewrite the recurrence as $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$; then the shifts are done by multiplication by x and x^2 rather than division.

The next theorem gives a general form for the partial fraction expressions needed to solve these recurrences, Recall that if

$$f(x) = c_d x^d + c_{d-1} x^{d-1} + \cdots + c_0$$

and $c_d \neq 0$ then f is said to have *degree* d; we write this as $\deg f = d$. This theorem will not be proved in lectures: see instead Chapter 25 of Biggs *Discrete Mathematics* ([2] in the list of recommended reading).

Theorem 8.2. *Let $f(x)$ and $g(x)$ be polynomials with* $\deg f < \deg g$. *If*

$$g(x) = \alpha(x - 1/\beta_1)^{d_1} \ldots (x - 1/\beta_k)^{d_k}$$

where $\alpha, \beta_1, \beta_2, \ldots, \beta_k$ are distinct non-zero complex numbers and $d_1, d_2, \ldots, d_k \in \mathbf{N}$, then there exist polynomials $P_1, \ldots, P_k$ such that $\deg P_i < d_i$ *and*

$$\frac{f(x)}{g(x)} = \frac{P_1(x)}{(1 - \beta_1 x)^{d_1}} + \cdots + \frac{P_k(x)}{(1 - \beta_k x)^{d_k}}.$$

Theorem 7.4 can then be used to find the coefficient of x^n in $f(x)/g(x)$. If $d_i = 1$ for all i then each polynomial P_i is just a constant $B_i \in \mathbf{C}$ and Theorem 8.2 states that

$$\frac{f(x)}{g(x)} = \frac{B_1}{1-\beta_1 x} + \cdots + \frac{B_k}{1-\beta_k x}.$$

In this case the coefficient of x^n in $f(x)/g(x)$ is $B_1\beta_1^n + \cdots + B_k\beta_k^n$. Moreover, we have $B_i = \lim_{x\to 1/\beta_i}(1-\beta_i x)f(x)/g(x)$.

When $f(x)/g(x)$ is a generating function for a sequence $a_0, a_1, a_2, \ldots$ it is usually easiest to use values of the sequence to determine any unknown constants.

Example 8.3. Will solve $b_n = 3b_{n-1} - 4b_{n-3}$ for $n \geq 3$.

The next exercise completes the solution to Example 7.2.

Exercise: In Example 7.2 we saw that if a_n is the number of ways to tile a $2 \times n$ path with bricks that are either 1×2 () or 2×1 (), then $a_n = a_{n-1} + a_{n-2}$, and that the generating function

$$F(x) = \sum_{n=0}^{\infty} a_n x^n$$

satisfies $(1 - x - x^2)F(x) = 1$. Show that $x^2 + x - 1 = (x - \phi)(x - \psi)$ where $\phi = \frac{-1+\sqrt{5}}{2}$ and $\psi = \frac{-1-\sqrt{5}}{2}$. Show that $1/\phi = -\psi$ and $1/\psi = -\phi$ and deduce from Theorem 8.2 that

$$a_n = C\Big(\frac{1+\sqrt{5}}{2}\Big)^n + D\Big(\frac{1-\sqrt{5}}{2}\Big)^n$$

for some $C, D \in \mathbf{C}$. Find C and D by using the values $a_0 = a_1 = 1$ and solving a pair of simultaneous equations. (Or by some other method for finding partial fractions, if you prefer.) You should get

$$C = \frac{1}{2} + \frac{1}{2\sqrt{5}} \quad \text{and} \quad D = \frac{1}{2} - \frac{1}{2\sqrt{5}}.$$

In §2 we used the recurrence $d_n = (n-1)(d_{n-1} + d_{n-2})$ for the derangement numbers to prove Theorem 2.5 by induction on n. This required us to already know the formula for d_n. Generating functions give a more systematic approach. (You are asked to fill in the details in this proof in Question 2 on Sheet 4.)

Theorem 8.4. *Let $p_n = d_n/n!$ be the probability that a permutation of the set $\{1, 2, \ldots, n\}$, chosen uniformly at random, is a derangement. Then*

$$np_n = (n-1)p_{n-1} + p_{n-2}$$

for all $n \geq 2$ *and*

$$p_n = 1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{(-1)^n}{n!}.$$

The steps needed in this proof can readily be performed using computer algebra packages. Indeed, MATHEMATICA implements a more refined version of our three step programme for solving recurrences in its `RSolve` command. (See the discussion in Appendix A of Wilf *generatingfunctionology*.)

It is usually possible to get some information about the asymptotic growth of a sequence from its generating function. For this, it is essential to use the analytic interpretation, and think of the generating function as a function defined on the complex numbers.

Let G be a function taking values in $\mathbf{C}$. A *singularity* of G is a point where G is undefined. For example, if $G(z) = 1/(1 - z^2)$ then G has singularities at $z = 1$ and $z = -1$ and G has no singularities w such that $|w| < 1$.

Theorem 8.5. *Let* $F(z) = \sum_{n=0}^{\infty} a_n z^n$ *be the generating function for the sequence* $a_0, a_1, a_2, \ldots$. *Fix* $R \in \mathbf{R}$. *Suppose that* F *has no singularities* w *such that* $|w| < R$. *Then for any* $\epsilon > 0$ *we have*

$$|a_n| \leq \left(\frac{1}{R} + \epsilon\right)^n$$

for all sufficiently large $n \in \mathbf{N}$. *Moreover, if* F *has a singularity* w *such that* $|w| = R$ *then there exist infinitely many* n *such that*

$$|a_n| \geq \left(\frac{1}{R} - \epsilon\right)^n.$$

See §2.4 in Wilf *generatingfunctionology* for a proof of Theorem 8.4. The proofs of this theorem, and Theorem 8.2, are non-examinable, but you might be asked to apply these results in simple cases.

The following example shows how a combination of the three-step programme, Theorem 8.5, and simple computer algebra can give useful asymptotic results on recurrences that would be very unpleasant to solve explicitly by hand.

Example 8.6. Consider the recurrence relation $a_{n+3} = a_n + a_{n+1} + a_{n+2}$. Step (a) of the three-step programme shows that the generating function for a_n is $F(z) = P(z)/(1 - z - z^2 - z^3)$ for some polynomial $P(z)$. The roots of $1 - z - z^2 - z^3 = 0$ are, to five decimal places,

$$0.543689, \quad 0.7718445 + 1.115143i, \quad 0.7718445 - 1.115143i.$$

So the singularity of $F(z)$ of smallest modulus is at $0.543689\ldots$. By Theorem 8.5, $a_n \leq \frac{1}{0.543689^n} \leq 2^n$ for all sufficiently large n. (Note that the initial values a_0, a_1 and a_2 were not needed.)

If $F(z) = \sum_{n=0}^{\infty} a_n z^n$ has no singularities then Theorem 8.5 implies that for any $S > 0$,

$$|a_n| \leq \frac{1}{S^n}$$

for all sufficiently large n.

Example 8.7. Let $G(z) = \sum_{n=0}^{\infty} p_n z^n$ be the generating function for the proportion of permutations of $\{1, 2, \ldots, n\}$ that are derangements. We saw that

$$G(z) = \frac{\exp(-z)}{1-z}.$$

Since G has a singularity at 1, a direct application of Theorem 8.5 gives only that $p_n \leq (1+\epsilon)^n$ for all sufficiently large n. (Why is this uninteresting?) In such cases, it is a good idea to take out the part of the function that causes $G(z)$ to blow up. Define $g(z)$ by

$$G(z) = \frac{\mathrm{e}^{-1}}{1-z} + g(z).$$

Then we can extend g to a function defined on all of $\mathbf{C}$. Using the result just mentioned, it follows that $|p_n - \mathrm{e}^{-1}| < 1/10^n$ for all sufficient large n. (Here 10 is just one possible choice of S.)

Note that we got the results in Example 8.7 without using the formula for the p_n. This is important because in trickier problems we might know the generating function, but not have an exact formula for its coefficients.

9. Convolutions and the Catalan numbers

The problems in this section fit into the following pattern: suppose that $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are classes of combinatorial objects and that each object has a *size* in $\mathbf{N}_0$. Write $\mathrm{size}(X)$ for the size of the object X. Suppose that there are finitely many objects of any given size.

Let a_n, b_n and c_n denote the number of objects of size n in $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, respectively.

Theorem 9.1. *Suppose that for all $n \in \mathbf{N}_0$ there is a bijection*

$$\Big\{ Z \in \mathcal{C} : \mathrm{size}(Z) = n \Big\} \leftrightarrow \Big\{ (X, Y) : \begin{array}{c} X \in \mathcal{A}, Y \in \mathcal{B} \\ \mathrm{size}(X) + \mathrm{size}(Y) = n \end{array} \Big\}$$

Then

$$\sum_{n=0}^{\infty} c_n x^n = \Big(\sum_{n=0}^{\infty} a_n x^n \Big) \Big(\sum_{n=0}^{\infty} b_n x^n \Big)$$

The critical step in the proof is to show that

$$c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_{n-1} b_1 + a_n b_0 = \sum_{m=0}^{n} a_m b_{n-m}.$$

If sequences (a_n), (b_n) and (c_n) satisfy this relation then we say that (c_n) is the *convolution* of (a_n) and (b_n).

Example 9.2. The grocer sells indistinguishable apples and bananas in unlimited quantities. Bananas are only sold in bunches of three.

(a) What is the generating function for the number of ways to buy n pieces of fruit?
(b) How would your answer to (a) change if dates are also sold?

You could either do (b) by two applications of Theorem 9.1, or by using the more general version where the objects are decomposed into three (or more) subobjects.

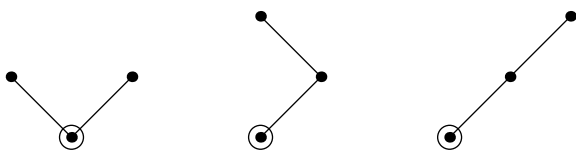
Example 9.3. Lemma 6.6 on rook placements states that if C is a board that decomposes as a disjoint union of boards A and B where no square in A lies in the same row or column as a square in B then the rook polynomials $f_A(x)$, $f_B(x)$, $f_C(x)$ satisfy $f_C(x) = f_A(x) f_B(x)$. It has a very short proof using Theorem 9.1.

Exercise: Show that splitting a non-attacking placement of rooks on C into the placements on the sub-boards A and B gives a bijection satisfying the hypotheses of Theorem 9.1. (Define the size of a rook placement and the sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$.) Hence prove Lemma 6.6.

The canonical application of convolutions is to the Catalan numbers. These numbers have many different combinatorial interpretations; we shall define them using rooted binary trees drawn in the plane.

Definition 9.4. A rooted binary tree is either empty, or consists of a *root vertex* together with a pair of rooted binary trees: a *left subtree* and a *right subtree*. The *Catalan number* C_n is the number of rooted binary trees on n vertices.

For example, there are five rooted binary trees with three vertices, so $C_3 = 5$. Three of them are shown below, with the root vertex circled. The other two can be obtained by reflecting the two asymmetric diagrams.



Theorem 9.5. *If $n \in \mathbf{N}_0$ then $C_n = \frac{1}{n+1}\binom{2n}{n}$.*

We shall prove Theorem 9.5 using our usual three-step programme. Let $F(x) = \sum_{n=0}^{\infty} C_n x^n$ be the generating function for the Catalan numbers. In outline the steps are:

(a) Use Theorem 9.1 to show that $F(x)$ satisfies the quadratic equation
$$xF(x)^2 = F(x) - 1.$$
(b) Solve the quadratic equation to get the closed form
$$xF(x) = \frac{1 - \sqrt{1-4x}}{2}.$$
(c) Use the general version of the Binomial Theorem in Theorem 7.5 to deduce the formula for C_n.

The Catalan Numbers have a vast number of combinatorial interpretations. See Question 4 on Sheet 6 for one more. A further 64 (and counting) are given in Exercise 6.19 in Stanley *Enumerative Combinatorics II*, CUP 2001.

Exercise: Explain the unusual structure of the decimal expansion
$$\tfrac{1}{2} - \sqrt{\tfrac{1}{4} - \tfrac{1}{1000}} = 0.001\,001\,002\,005\,014\,042\ldots.$$

As a further application of convolutions we will give yet another proof (probably the shortest yet!) of the formula for the derangement numbers d_n.

Lemma 9.6. *If $n \in \mathbf{N}_0$ then*
$$\sum_{k=0}^{n} \binom{n}{k} d_{n-k} = n!.$$

The sum in the lemma becomes a convolution after a small amount of rearranging.

Theorem 9.7. *If $G(x) = \sum_{n=0}^{\infty} d_m x^m/m!$ then*
$$G(x)\exp(x) = \frac{1}{1-x}.$$

It is now easy to deduce the formula for d_n; the argument needed is the same as the final step in the proof of Theorem 8.4. The generating function G used above is an example of an *exponential generating function*.

10. PARTITIONS

Definition 10.1. A *partition* of a number $n \in \mathbf{N}_0$ is a sequence of natural numbers $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ such that

(i) $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1$.
(ii) $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$.

The entries in a partition λ are called the *parts* of λ. Let $p(n)$ be the number of partitions of n.

By this definition the unique partition of 0 is the empty partition $\emptyset$, and so $p(0) = 1$. The sequence of partition numbers begins

$$1, 1, 2, 3, 5, 7, 11, 15, \ldots.$$

Example 10.2. Let a_n be the number of ways to pay for an item costing n pence using only 2p and 5p coins. Equivalently, a_n is the number of partitions of n into parts of size 2 and size 5. Will find the generating function for a_n.

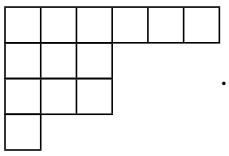
The next theorem can be proved using a generalized version of Theorem 9.1 in which a partition of n decomposes into subobjects consisting of its parts of size 1, its parts of size 2, and so on.

Instead we will give a direct proof that repeats the main idea in Theorem 9.1.

Theorem 10.3. *The generating function for $p(n)$ is*

$$\sum_{n=0}^{\infty} p(n)x^n = \frac{1}{(1-x)(1-x^2)(1-x^3)\ldots}.$$

It is often useful to represent partitions by *Young diagrams*. The Young diagram of $(\lambda_1, \ldots, \lambda_k)$ has k rows of boxes, with λ_i boxes in row i. For example, the Young diagram of $(6, 3, 3, 1)$ is



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The next theorem has a very simple proof using Young diagrams. (See also Question 9 on Sheet 5.)

Theorem 10.4. *Let $n \in \mathbf{N}$ and let $k \leq n$. The number of partitions of n into parts of size $\leq k$ is equal to the number of partitions of n with at most k parts.*

While there are bijective proofs of the next theorem using Young diagrams, it is much easier to prove it using generating functions. Note how we adapt the proof of Theorem 10.3 to get the generating functions for two special types of partitions.

Theorem 10.5. *Let $n \in \mathbf{N}$. The number of partitions of n with at most one part of any given size is equal to the number of partitions of n into odd parts.*

See Question 8 on Sheet 6 for a generalization of Theorem 10.5.

EXTRAS ON PARTITIONS. The following material is included for interest only and is non-examinable.

There are many deep combinatorial and number-theoretic properties of the partition numbers. For example, in 1919 Ramanujan used analytic arguments with generating functions to prove that

$$p(4), p(9), p(14), p(19), \ldots, p(5m+4), \ldots$$

are all divisible by 5. In 1944 Freeman Dyson found a bijective proof of this result while still an undergraduate. A number of deep generalizations of Ramanujan's congruences have since been proved, most recently by Mahlburg in 2005. See Chapter 10 of G. E. Andrews, *The theory of partitions*, Cambridge University Press 1984 for an introduction.

Many easily stated problems remain open: for example, is $p(n)$ even about half the time?

The problem of finding an estimate for the size of the partition number $p(n)$ was solved in 1919 by Hardy and Ramanujan as the original application of the circle method. The crudest version of their result is

$$p(n) \sim \frac{\mathrm{e}^{c\sqrt{n}}}{4n\sqrt{3}}$$

where $c = 2\sqrt{\frac{\pi^2}{6}}$, and $\sim$ means that the ratio of the two sides tends to 1 as $n \to \infty$.

A much more elementary result helps to explain the constant c in the Hardy–Ramanujan theorem.

Theorem 10.6 (Van Lint's upper bound)**.** *If $n \in \mathbf{N}$ then $p(n) \leq \mathrm{e}^{c\sqrt{n}}$ where $c = 2\sqrt{\frac{\pi^2}{6}}$.*

Outline proof. Let $P(x) = \prod_{m=1}^{\infty} 1/(1-x^m)$ be the generating function for the partition numbers found in Theorem 10.3. Taking logs we get

$$\log P(x) = -\sum_{m=1}^{\infty} \log(1-x^m).$$

Using the power series $-\log(1-y) = \sum_{r=1}^{\infty} y^r/r$ we get

$$\log P(x) = \sum_{r=1}^{\infty} \frac{x^r}{r(1-x^r)}.$$

Substitute $x = \mathrm{e}^{-y}$ where $y > 0$ to get

$$\log P(\mathrm{e}^{-y}) \leq \sum_{r=1}^{\infty} \frac{\mathrm{e}^{-ry}}{r(1-\mathrm{e}^{-ry})}.$$

Now rewrite each summand as $1/r(\mathrm{e}^{ry}-1)$ and use the inequality $\mathrm{e}^{ry} - 1 = 1 + (ry) + (ry)^2/2! + \cdots - 1 \geq ry$ to get

$$\log P(\mathrm{e}^{-y}) \leq \sum_{r=1}^{\infty} \frac{1}{r^2 y}.$$

Since $\sum_{r=1}^{\infty} 1/r^2 = \pi^2/6$, we have $\log P(\mathrm{e}^{-y}) \leq \pi^2/6y$. The result now follows by making a strategic choice of y: see Question 9 on Sheet 6 for the remaining steps. □

A different proof that $p(n) \leq \mathrm{e}^{c\sqrt{n}}$, still using only real analysis, was given by Erdős in a beautiful paper, *On an elementary proof of some asymptotic formulas in the theory of partitions*, Ann. of Math. **43** (1942) 437–450. It is an open problem to find an entirely combinatorial proof that $p(n) \leq A^{\sqrt{n}}$ for some constant A.

Part C: Ramsey Theory

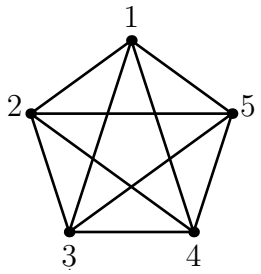
11. Introduction to Ramsey Theory

A typical result in Ramsey Theory says that any sufficiently large combinatorial structure always contain a substructure with some regular pattern. For example, any infinite sequence of real numbers contains either an increasing or a decreasing subsequence (the Bolzano–Weierstrass theorem). A finite version of this result is given in Question 7 on Problem Sheet 7.

Most of the results in Ramsey Theory concern graphs. In this course we will concentrate on the finite case.

Definition 11.1. A *graph* consists of a set V of vertices together with a set E of 2-subsets of V called *edges*. The *complete graph* with vertex set V is the graph whose edge set is all 2-subsets of V.

For example, the complete graph on $V = \{1,2,3,4,5\}$ is drawn below. Its edge set is $\big\{\{1,2\},\{1,3\},\ldots,\{4,5\}\big\}$.



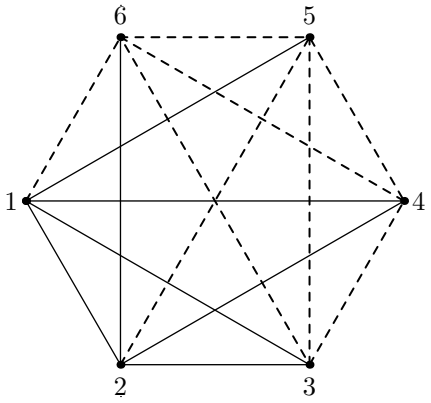
We denote the complete graph on $\{1,2,\ldots,n\}$ by K_n.

Definition 11.2. Let c, $n \in \mathbf{N}$. A *c-colouring* of the complete graph K_n is a function from the edge set of K_n to $\{1,2,\ldots,c\}$. If S is an s-subset of the vertices of K_n such that all the edges between vertices in S have the same colour, then we say that S is a *monochromatic K_s*

A monochromatic K_3 is usually said to be a monochromatic triangle. Note that it is the edges of the complete graph K_n that are coloured, *not the vertices*.

In practice we shall specify graphs and colours rather less formally, following the convention that colour 1 is red and colour 2 is blue. In these notes, red will be indicated by solid lines and blue by dashed lines.

Exercise: Show that in the colouring of K_6 below there is a unique blue (dashed) K_4 and exactly two red (solid) triangles. Find all the blue triangles.



Example 11.3. In any red-blue colouring of the edges of K_6 there is either a red triangle or a blue triangle.

Definition 11.4. Given $s, t \in \mathbf{N}$, with $s, t \geq 2$, we define the Ramsey number $R(s,t)$ to be the smallest n (if one exists) such that in any red-blue colouring of the complete graph on n vertices, there is either a red K_s or a blue K_t.

For example, we know from Example 11.3 that $R(3,3) \leq 6$.

Lemma 11.5. *Let $s, t \in \mathbf{N}$ with $s, t \geq 2$. Let $N \in \mathbf{N}$. Assume that $R(s,t)$ exists.*

(i) *If $N < R(s,t)$ there exist colourings of K_N with no red K_s or blue K_t.*

(ii) *If $N \geq R(s,t)$ then in any red-blue colouring of K_N there is either a red K_s or a blue K_t.*

By Question 3 on Sheet 6 there is a red-blue colouring of K_5 with no monochromatic triangle. Hence, by Example 11.3 and Lemma 11.5(ii), $R(3,3) = 6$.

Exercise: Let $s, t \in \mathbf{N}$ with $s, t \geq 2$. Show that $R(s,t) = R(t,s)$.

We will prove in Theorem 12.3 that all the two-colour Ramsey numbers $R(s,t)$ exist, and that $R(s,t) \leq \binom{s+t-2}{s-1}$. (Please *do not* assume this result when doing Sheet 6.)

One family of Ramsey numbers is easily found.

Lemma 11.6. *If $s \geq 2$ then $R(s,2) = R(2,s) = s$.*

The main idea need to prove Theorem 12.3 appears in the next example.

Example 11.7. In any two-colouring of K_{10} there is either a red K_3 or a blue K_4. Hence $R(3,4) \leq 10$.

This bound can be improved using a result from graph theory. Recall that if v is a vertex of a graph G then the *degree* of v is the number of edges of G that meet v.

Lemma 11.8 (Hand-Shaking Lemma)**.** *Let G be a graph with vertex set $\{1, 2, ..., n\}$ and exactly e edges. If d_x is the degree of vertex x then*

$$2e = d_1 + d_2 + \cdots + d_n.$$

Theorem 11.9. $R(3,4) = 9$.

The proof of the final theorem is left to you: see Question 1 on Sheet 7.

Theorem 11.10. $R(4,4) \leq 18$.

There is a red-blue colouring of K_{17} with no red K_4 or blue K_4 so $R(4,4) = 18$. A construction is given in Question 8 of Sheet 7.

It is a very hard problem to find the exact values of Ramsey numbers for larger s and t. For a survey of other known results on $R(s,t)$ for small s and t, see Stanislaw Radziszowski, *Small Ramsey Numbers*, Electronic Journal of Combinatorics, available at `www.combinatorics.org/Surveys`. For example, it was shown in 1965 that $R(4,5) = 25$, but all that is known about $R(5,5)$ is that it lies between 43 and 49. It is probable that no-one will ever know the exact value of $R(6,6)$.

12. Ramsey's Theorem

Since finding the Ramsey numbers $R(s,t)$ exactly is so difficult, we settle for proving that they exist, by proving an upper bound for $R(s,t)$. We work by induction on $s+t$. The following lemma gives the critical inductive step.

Lemma 12.1. *Let $s, t \in \mathbf{N}$ with $s, t \geq 3$. If $R(s-1,t)$ and $R(s,t-1)$ exist then $R(s,t)$ exists and*

$$R(s,t) \leq R(s-1,t) + R(s,t-1).$$

Theorem 12.2. *For any $s, t \in \mathbf{N}$ with $s, t \geq 2$, the Ramsey number $R(s,t)$ exists and*

$$R(s,t) \leq \binom{s+t-2}{s-1}.$$

We now get a bound on the diagonal Ramsey numbers $R(s,s)$. Note that because of the use of induction on $s+t$, we could not have obtained this result without first bounding all the Ramsey numbers $R(s,t)$.

Corollary 12.3. *If $s \in \mathbf{N}$ and $s \geq 2$ then*

$$R(s,s) \leq \binom{2s-2}{s-1} \leq 4^{s-1}.$$

One version of Stirling's Formula states that if $m \in \mathbf{N}$ then

$$\sqrt{2\pi m}\left(\frac{m}{\mathrm{e}}\right)^m \leq m! \leq \sqrt{2\pi m}\left(\frac{m}{\mathrm{e}}\right)^m \mathrm{e}^{1/12m}.$$

These bounds lead to the asymptotically stronger result that

$$R(s,s) \leq \frac{4^s}{\sqrt{s}} \quad \text{for all } s \in \mathbf{N}.$$

Corollary 12.3 was proved by Erdös and Szekeres in 1935. We have followed their proof above. The strongest improvement known to date is due to David Conlon, who showed in 2004 that, up to a rather technical error term, $R(s,s) \leq 4^s/s$. In 1947 Erdös proved the lower bound $R(s,s) \geq 2^{(s-1)/2}$. His argument becomes clearest when stated in probabilistic language: we will see it in Part D of the course.

To end this introduction to Ramsey Theory we give some related results.

PIGEONHOLE PRINCIPLE. The Pigeonhole Principle states that if n pigeons are put into $n-1$ holes, then some hole must contain two or more pigeons. See Question 8 on Sheet 6 for some applications of the Pigeonhole Principle.

In Examples 11.3 and 11.6, and Lemma 12.1, we used a similar result: if $r+s-1$ objects (in these cases, edges) are coloured red and blue, then either there are r red objects, or s blue objects. This is probably the simplest result that has some of the general flavour of Ramsey theory.

MULTIPLE COLOURS. It is possible to generalize all the results proved so far to three or more colours.

Theorem 12.4. *There exists $n \in \mathbf{N}$ such that if the edges of K_n are coloured red, blue and yellow then there exists a monochromatic triangle.*

There are (at least) two ways to prove Theorem 12.4. The first adapts our usual argument, looking at the edges coming out of vertex 1 and concentrating on those vertices joined by edges of the majority colour. The second uses a neat trick to reduce to the two-colour case.

14. Introduction to probabilistic methods

In this section we shall solve some problems involving permutations (including, yet again, the derangements problem) using probabilistic arguments. We shall use the language of probability spaces and random variables recalled in §13. It will be particularly important for you to ask questions if the use of anything from this section is unclear.

Throughout this section we fix $n \in \mathbf{N}$ and let Ω be the set of all permutations of the set $\{1, 2, \ldots, n\}$. We define a probability measure $q : \Omega \to \mathbf{R}$ by $q_\sigma = 1/n!$ for each permutation σ of $\{1, 2, \ldots, n\}$. This makes Ω into a probability space in which all the permutations have equal probability. We say that the permutations are *chosen uniformly at random*.

Recall that, in probabilistic language, *events* are subsets of Ω.

Exercise: Let $x \in \{1, 2, \ldots, n\}$ and let $A_x = \{\sigma \in \Omega : \sigma(x) = x\}$. Then A_x is the event that a permutation fixes x. What is the probability of A_x?

Building on this we can give a better proof of Theorem 2.6(ii).

Theorem 14.1. *Let $F : \Omega \to \mathbf{N}_0$ be defined so that $F(\sigma)$ is the number of fixed points of the permutation $\sigma \in \Omega$. Then $\mathbf{E}[F] = 1$.*

To give a more general result we need cycles and the cycle decomposition of a permutation.

Definition 14.2. A permutation σ of $\{1, 2, \ldots, n\}$ *acts as a k-cycle* on a k-subset $S \subseteq \{1, 2, \ldots, n\}$ if S has distinct elements $x_1, x_2, \ldots, x_k$ such that

$$\sigma(x_1) = x_2,\ \sigma(x_2) = x_3,\ \ldots,\ \sigma(x_k) = x_1.$$

If $\sigma(y) = y$ for all $y \in \{1, 2, \ldots, n\}$ such that $y \notin S$ then we say that σ is a *k-cycle*, and write

$$\sigma = (x_1, x_2, \ldots, x_k).$$

Note that there are k different ways to write a k-cycle. For example, the 3-cycle $(1, 2, 3)$ can also be written as $(2, 3, 1)$ and $(3, 1, 2)$.

Definition 14.3. We say that cycles $(x_1, x_2, \ldots, x_k)$ and $(y_1, y_2, \ldots, y_\ell)$ are *disjoint* if

$$\{x_1, x_2, \ldots, x_k\} \cap \{y_1, y_2, \ldots, y_\ell\} = \varnothing.$$

Lemma 14.4. *A permutation σ of $\{1, 2, \ldots, n\}$ can be written as a composition of disjoint cycles. The cycles in this composition are uniquely determined by σ.*

The proof of Lemma 14.4 is non-examinable and will not be given in full in lectures. What is more important is that you can apply the result. We shall use it below in Theorem 14.5

Exercise: Write the permutation of $\{1,2,3,4,5,6\}$ defined by $\sigma(1) = 3$, $\sigma(2) = 6$, $\sigma(3) = 1$, $\sigma(4) = 2$, $\sigma(5) = 5$, $\sigma(6) = 4$ as a composition of disjoint cycles.

Given a permutation σ of $\{1,2,\ldots,n\}$ and $k \in \mathbf{N}$, we can ask: what is the probability that a given $x \in \{1,2,\ldots,n\}$ lies in a k-cycle of σ? The first exercise in this section shows that the probability that x lies in a 1-cycle is $1/n$.

Exercise: Check directly that the probability that 1 lies in a 2-cycle of a permutation of $\{1,2,3,4\}$ selected uniformly at random is $1/4$.

Theorem 14.5. *Let $1 \leq k \leq n$ and let $x \in \{1,2,\ldots,n\}$. The probability that x lies in a k-cycle of a permutation of $\{1,2,\ldots,n\}$ chosen uniformly at random is $1/n$.*

Theorem 14.6. *Let p_n be the probability that a permutation of $\{1,2,\ldots,n\}$ chosen uniformly at random is a derangement. If $n \in \mathbf{N}$ then*

$$p_n = \frac{p_{n-2}}{n} + \frac{p_{n-3}}{n} + \cdots + \frac{p_1}{n} + \frac{p_0}{n}.$$

It may be helpful to compare this result with Lemma 9.6: there we get a recurrence by considering fixed points; here we get a recurrence by considering cycles.

We now use generating functions to recover the usual formula for p_n.

Corollary 14.7. *For all $n \in \mathbf{N}_0$,*

$$p_n = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}.$$

We can also generalize Theorem 14.1.

Theorem 14.8. *Let $C_k : \Omega \to \mathbf{R}$ be the random variable defined so that $C_k(\sigma)$ is the number of k-cycles in the permutation σ of $\{1,2,\ldots,n\}$. Then $\mathbf{E}[C_k] = 1/k$ for all k such that $1 \leq k \leq n$.*

Note that if $k > n/2$ then a permutation can have at most one k-cycle. So in these cases, $\mathbf{E}[C_k]$ is the probability that a permutation of $\{1,2,\ldots,n\}$, chosen uniformly at random, has a k-cycle.

15. Ramsey Numbers and the First Moment Method

The grandly named 'First Moment Method' is nothing more than the following simple observation.

Lemma 15.1 (First Moment Method)**.** *Let Ω be a probability space and let $M : \Omega \to \mathbf{N}_0$ be a random variable taking values in $\mathbf{N}_0$. If $\mathbf{E}[M] = x$ then*

(i) $\mathbf{P}[M \geq x] > 0$, *so there exists $\omega \in \Omega$ such that $M(\omega) \geq x$.*
(ii) $\mathbf{P}[M \leq x] > 0$, *so there exists $\omega' \in \Omega$ such that $M(\omega') \leq x$.*

Exercise: Check that the lemma holds in the case when

$$\Omega = \{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$$

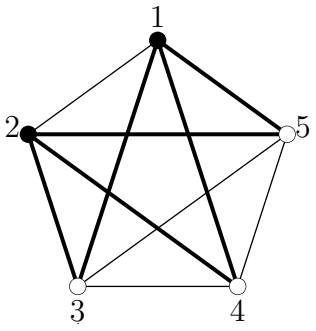
models the throw of two fair dice (see Example 13.2(2)) and if $(\alpha, \beta) \in \Omega$ then $M(\alpha, \beta) = \alpha + \beta$.

The kth *moment* of a random variable X is defined to be $\mathbf{E}[X^k]$. Sometimes stronger results can be obtained by considering higher moments. We shall concentrate on first moments, where the power of the method is closely related to the linearity property of expectation (see Lemma 13.11).

Our applications will come from graph theory.

Definition 15.2. Let G be a graph with vertex set V. A *cut* (S, T) of G is a partition of V into subsets A and B. The *capacity* of a cut (S, T) is the number of edges of G that meet both S and T.

Note that $T = V \backslash S$ and $S = V \backslash T$, so a cut can be specified by giving either of the sets making up the partition. The diagram below shows the cut in the complete graph on $\{1,2,3,4,5\}$ where $S = \{1,2,3\}$ and $T = \{4,5\}$. The capacity of the cut is 6, corresponding to the 6 edges $\{x, y\}$ with $x \in S$ and $y \in T$ shown with thicker lines.



Theorem 15.3. *Let G be a graph with vertex set $\{1,2,\ldots,n\}$ and exactly m edges. There is a cut of G with capacity $\geq m/2$.*

In 1947 Erdös proved a lower bound on the Ramsey Numbers $R(s,s)$ that is still almost the best known result in this direction. Our version of his proof will use the First Moment Method in the following probability space.

Lemma 15.4. *Let $N \in \mathbf{N}$ and let Ω be the set of all red-blue colourings of the complete graph K_N. Let $p_\omega = 1/|\Omega|$ for each $\omega \in \Omega$. Then*

(i) *each colouring in Ω has probability $1/2^{\binom{N}{2}}$;*

(ii) *given any m edges in G, the probability that all m of these edges have the same colour is 2^{1-m}.*

Theorem 15.5. *Let $N \in \mathbf{N}$ and let $s \in \mathbf{N}$ with $s \geq 2$. If*

$$\binom{N}{s} 2^{1-\binom{s}{2}} < 1$$

then there is a red-blue colouring of the complete graph on $\{1, 2, \ldots, N\}$ with no red K_s or blue K_s.

Corollary 15.6. *For any $s \in \mathbf{N}$ with $s \geq 2$ we have*

$$R(s,s) \geq 2^{(s-1)/2}.$$

For example, since

$$\binom{42}{8} 2^{1-\binom{8}{2}} \approx 0.879 < 1,$$

if we repeatedly colour the complete graph on $\{1, 2, ..., 42\}$ at random, then we will fairly soon get a colouring with no monochromatic K_8. However, to check that we have found such a colouring, we will have to look at all $\binom{42}{8} \approx 1.18 \times 10^8$ subsets of $\{1, 2, \ldots, 42\}$. Thus Theorem 15.5 does not give an effective construction.

It is a major open problem to find, for each $s \geq 2$, an explicit colouring of the complete graph on 1.01^s vertices with no monochromatic K_s. (Here 1.01 could be replaced with $1 + \epsilon$ for any $\epsilon > 0$.

The bound in Corollary 15.6 can be slightly improved by the Lovász Local Lemma: see the final section.

16. LOVÁSZ LOCAL LEMMA

The section is non-examinable, and is included for interest only.

MOTIVATION. In the proof of Theorem 15.5, we considered a random colouring of the complete graph on $\{1, 2, \ldots, n\}$ and used Lemma 15.1 to show that, provided $\binom{N}{s}2^{1-\binom{s}{2}} < 1$ there was a positive probability that this colouring had no monochromatic K_s. As motivation for the Lovász Local Lemma, consider the following alternative argument, which avoids the use of Lemma 15.1.

Alternative proof of Theorem 15.5. As before, let Ω be the probability space of all colourings of the complete graph on $\{1, 2, \ldots, N\}$, where each colouring gets the same probability. For each s-subset

$$S \subseteq \{1, 2, \ldots, N\},$$

let A_S be the event that S is a monochromatic K_s. The event that no K_s is monochromatic is then $\bigcap_S \bar{A}_S$, where the intersection is taken over all s-subsets $S \subseteq \{1, 2, \ldots, N\}$ and $\bar{A}_S = \Omega \backslash A_S$. So it will suffice to show that $\mathbf{P}[\bigcap \bar{A}_S] > 0$, or equivalently, that $\mathbf{P}[\bigcup A_S] < 1$.

In lectures we used Lemma 15.4 to show that if S is any s-subset of $\{1, 2, \ldots, N\}$ then

$$\mathbf{P}[A_S] = 2^{1-\binom{n}{s}}.$$

By the exercise on page 32, the probability of a union of events is at most the sum of their probabilities, so

$$\mathbf{P}[\bigcup_S A_S] \leq \binom{n}{s} 2^{1-\binom{n}{s}}.$$

Hence the hypothesis implies that $\mathbf{P}\left[\bigcup_S E_S\right] < 1$, as required. □

If the events A_S were independent, we would have

$$\mathbf{P}[\bigcap_S \bar{A}_S] = \prod_S \mathbf{P}[\bar{A}_S].$$

Since each event A_S has non-zero probability, it would follow that their intersection has non-zero probability, giving another way to finish the proof. However, the events are not independent, so this is not an admissible strategy. The Lovász Local Lemma gives a way to get around this obstacle.

STATEMENT OF THE LOVÁSZ LOCAL LEMMA. Let Ω be a probability space, and let $A_1, A_2, \ldots, A_n \subseteq \Omega$ be events.

Definition 16.1. Let $J \subseteq \{1, 2, \ldots, n\}$. We say that A_i is *mutually independent* of the events $\{A_k : k \in K\}$ if for all $L, L' \subseteq K$ such that $L \cap L' = \varnothing$ we have

$$\mathbf{P}\Big[A_i \,\Big|\, \Big(\bigcap_{\ell \in L} A_\ell\Big) \cap \Big(\bigcap_{\ell' \in L'} \bar{A}_{\ell'}\Big)\Big] = \mathbf{P}[A_i],$$

provided the event conditioned on has non-zero probability.

For example, if the events A_S are as defined above (so events are now indexed by subsets of $\{1, 2, \ldots, N\}$ rather than the numbers $\{1, 2, \ldots, n\}$), then A_S is independent of the events $\{A_T : |S \cap T| \leq 1\}$. This can be checked quite easily: informally the reason is that since each $S \cap T$ has at most one vertex, no edge is common to both S and T, and so knowing whether or not T is monochromatic gives no information about S.

This information about dependencies can be recorded by a digraph.

Definition 16.2. Let G be a digraph with edge set

$$E \subseteq \{(i, j) : 1 \leq i, j \leq n, i \neq j\}.$$

If A_i is mutually independent of $\{A_j : (i, j) \notin E\}$ for all $i \in \{1, 2, \ldots, n\}$ then we say that G is a *dependency digraph* for the events $A_1, A_2, \ldots, A_n$.

It is easy to prove that $\mathbf{P}[A_i|A_j] \neq \mathbf{P}[A_i]$ if and only if $\mathbf{P}[A_j|A_i] \neq A_j$. So if (i, j) is an 'essential' edge of a dependency digraph (i.e. $\mathbf{P}[A_i|A_j] \neq \mathbf{P}[A_i]$) then (j, i) is also an edge.

This might suggest using graphs instead. However digraphs are usual in the literature, and it is convenient in the proof to be able to read off from (i, j) that we are working with the event A_i conditioned on A_j (and maybe some other events).

Lemma 16.3 (Asymmetric Lovász Local Lemma)**.** *Let G be a dependency digraph with edge set E for the events $A_1, \ldots, A_n$. Suppose there exist $x_i \in \mathbf{R}$ such that $0 \leq x_i < 1$ and*

$$P[A_i] \leq x_i \prod_{j:(i,j) \in E} (1 - x_j)$$

for all i. Then

$$\mathbf{P}\Big[\bigcap_{i=1}^{n} \bar{A}_i\Big] \geq \prod_{i=1}^{n} (1 - x_i).$$

Some remarks that may clarify the statement of the lemma.

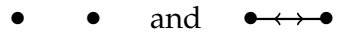
(1) The events A_i should be thought of as unlikely; the conclusion of the lemma is that none of these unlikely events occur. For example, provided s is not too small compared to N, the event that a particular s-subset of $\{1, 2, \ldots, N\}$ is a red K_s is unlikely.

(2) A key step in the proof will be to show that $J \subseteq \{1, 2, \ldots, n\}$ then

$$P\Big[A_i \Big| \bigcap_{j \in J} \bar{A}_j \Big] \leq x_i.$$

Informally, it seems good to think of x_i as the 'worst case conditional probability' for the unlikely event A_i, given that some of the likely events $\bar{A}_j$ occur. Conditioning on likely events is a reasonable thing to do, and can be expected not to skew probabilities too much.

(3) It is a good exercise to prove the lemma for the two possible dependency graphs on 2 vertices shown below.



A solution is given below.

(4) We will use many times in the proof that if B is an event with $\mathbf{P}[B] \neq 0$ and $i \in \{1, 2, \ldots, n\}$ then $\mathbf{P}[A_i|B] = 1 - \mathbf{P}[\bar{A}_i|B]$, and so

$$\mathbf{P}[A_i|B] \leq x_i \iff 1 - \mathbf{P}[\bar{A}_i|B] \geq 1 - x_i.$$

Solution to exercise. For the first graph we have $\mathbf{P}[\bar{A}_1 \cap \bar{A}_2] = \mathbf{P}[\bar{A}_1]\mathbf{P}[\bar{A}_2] = (1 - \mathbf{P}[A_1])(1 - \mathbf{P}[A_2]) = (1 - x_1)(1 - x_2)$, since A_1 and A_2 are independent, and by hypothesis $\mathbf{P}[A_1] = x_1$, $\mathbf{P}[A_2] = x_2$, there being no edges to consider. For the second graph we have

$$\begin{aligned} \mathbf{P}[\bar{A}_1 \cap \bar{A}_2] &= \mathbf{P}[\bar{A}_1|\bar{A}_2]\mathbf{P}[\bar{A}_2] \\ &= (1 - \mathbf{P}[A_1|\bar{A}_2])(1 - \mathbf{P}[A_2]) \end{aligned}$$

Now $\mathbf{P}[A_2] \leq x_2(1 - x_1)$ by hypothesis, so $\mathbf{P}[A_2] \leq x_2$ and $1 - \mathbf{P}[A_2] \geq 1 - x_2$. We now use this to get

$$\begin{aligned} \mathbf{P}[A_1|\bar{A}_2] &= \mathbf{P}[A_1 \cap \bar{A}_2]/\mathbf{P}[\bar{A}_2] \\ &\leq \mathbf{P}[A_1]/\mathbf{P}[\bar{A}_2] \\ &\leq x_1(1 - x_2)/(1 - x_2) \\ &= x_1. \end{aligned}$$

Hence, $\mathbf{P}[\bar{A}_1 \cap \bar{A}_2] = (1 - \mathbf{P}[A_1|\bar{A}_2])(1 - \mathbf{P}[A_2]) \geq (1 - x_1)(1 - x_2)$ as required. The cancellation above reappears in the general proof.

PROOF OF ASYMMETRIC LOVÁSZ LOCAL LEMMA. The following proof is based on Chapter 5 of Noga Alon and Joel H. Spencer *The Probabilistic Method*, 3rd edition, but with some extra details to make it obvious that the events we condition on have non-zero probability. The lemma follows from the case $J = \{1, 2, \ldots, n\}$ of the following claim.

Claim 16.4. *Let $J \subseteq \{1, 2, \ldots, n\}$. Then $\mathbf{P}\left[\bigcap_{j\in J} \bar{A}_j\right] \geq \prod_{j\in J}(1 - x_j)$ and*

$$P\Big[A_i \Big| \bigcap_{j\in J} \bar{A}_j \Big] \leq x_i$$

for all $i \in \{1, 2, \ldots, n\}$.

Proof. We work by induction on $|J|$. When $J = \varnothing$ the empty intersection is Ω and we have $\mathbf{P}[\Omega] = 1$ and

$$\mathbf{P}[A_i|\Omega] = \mathbf{P}[A_i] \leq x_i \prod_{j:(i,j)\in E} (1 - x_j) \leq x_i,$$

so $\mathbf{P}[\bar{A}_i] \geq 1 - x_i$, as required. Suppose inductively that the claim holds when $|J| < m$. Let $|J| = m$ where $m \geq 1$.

We first show that $\mathbf{P}\left[\bigcap_{j\in J} \bar{A}_j\right] \geq \prod_{j\in J}(1 - x_j)$. Let $j^\star \in J$. Then

$$\begin{aligned}
\mathbf{P}\Big[\bigcap_{j\in J} \bar{A}_j\Big] &= \mathbf{P}\Big[\bar{A}_{j^\star} \Big| \bigcap_{\substack{j\in J\\ j\neq j^\star}} \bar{A}_j\Big] \mathbf{P}\Big[\bigcap_{\substack{j\in J\\ j\neq j^\star}} \bar{A}_j\Big] \\
&= \Big(1 - \mathbf{P}\Big[A_{j^\star} \Big| \bigcap_{\substack{j\in J\\ j\neq j^\star}} \bar{A}_j\Big]\Big) \mathbf{P}\Big[\bigcap_{\substack{j\in J\\ j\neq j^\star}} \bar{A}_j\Big] \\
&\geq (1 - x_{j^\star}) \prod_{\substack{j\in J\\ j\neq j^\star}} (1 - x_j) \\
&= \prod_{j\in J} (1 - x_j)
\end{aligned}$$

where we used both parts of the inductive hypothesis to bound the first and second terms when going from line 2 to line 3.

We now prove the second part of the claim. Let $i \in \{1, 2, \ldots, n\}$. Let

$$\{j_1, \ldots, j_r\} = \{j : j \in J, (i, j) \in E\}$$

and let $K = J \backslash \{j_1, \ldots, j_r\}$. Let $B = \bigcap_{k\in K} \bar{A}_k$. Note that A_i is independent of B and B contains $\bigcap_{j\in J} \bar{A}_j$, so has non-zero probability. We have

$$\mathbf{P}\Big[A_i \Big| \bigcap_{j\in J} \bar{A}_j\Big] = \frac{\mathbf{P}\Big[A_i \cap \bar{A}_{j_1} \cap \cdots \cap \bar{A}_{j_r} | B\Big]}{\mathbf{P}[\bar{A}_{j_1} \cap \cdots \cap \bar{A}_{j_r} | B]}.$$

For the numerator we make the estimate

$$\mathbf{P}[A_i \cap \bar{A}_{j_1} \cap \cdots \cap \bar{A}_{j_r} | B] \leq \mathbf{P}[A_i|B] = \mathbf{P}[A_i] \leq x_i \prod_{j:(i,j)\in E} (1 - x_j).$$

For the denominator we use an extension of the argument used for the first part of the claim, as follows:

$$\begin{aligned}\mathbf{P}[\bar{A}_{j_1} \cap \cdots \cap \bar{A}_{j_r}|B] &= \mathbf{P}[\bar{A}_{j_1}|\bar{A}_{j_2} \cap \cdots \cap \bar{A}_{j_r} \cap B] \\ &\qquad \mathbf{P}[\bar{A}_{j_2}|\bar{A}_{j_3} \cap \cdots \cap \bar{A}_{j_r} \cap B] \ldots \mathbf{P}[\bar{A}_{j_r}|B] \\ &\geq (1-x_{j_1})(1-x_{j_2})\ldots(1-x_{j_r}) \\ &\geq \prod_{j:(i,j)\in E}(1-x_j),\end{aligned}$$

where the second line follows by r applications of the second part of the claim, taking $i = j_1, i = j_2, \ldots, i = j_r$. Combining these results we get

$$\mathbf{P}\Big[A_i \Big| \bigcap_{j\in J} \bar{A}_j\Big] \leq \frac{x_i \prod_{j:(i,j)\in E}(1-x_j)}{\prod_{j:(i,j)\in E}(1-x_j)} = x_i$$

as required. □

SYMMETRIC LOVÁSZ LOCAL LEMMA. As a fairly easy corollary we get the symmetric form of the Lovász Local Lemma, which suffices for many applications.

Corollary 16.5 (Symmetric Lovász Local Lemma)**.** *Suppose that the maximum degree in a dependency digraph for the events $A_1, \ldots, A_n$ is d. If $\mathbf{P}[A_i] \leq p$ for all i and $\mathrm{e}p(d+1) \leq 1$ then*

$$\mathbf{P}\Big[\bigcap_{i=1}^{n} \bar{A}_i] \geq 0.$$

Proof. We aim to apply the Asymmetric Lovász Local Lemma by taking $x_i = x$ for some $x \in \mathbf{R}$ with $0 \leq x < 1$. By the hypothesis on the maximum degree in the dependency graph, it suffices to find x such that $p \leq x(1-x)^d$. By the first part of the exercise below, the function $x \mapsto x(1-x)^d$ is maximized when $x = 1/(d+1)$. By the second part we have

$$x(1-x)^d = \frac{1}{d+1}\Big(\frac{d}{d+1}\Big)^d = \frac{1}{d+1}\Big(1-\frac{1}{d+1}\Big)^d \geq \frac{1}{\mathrm{e}(d+1)}.$$

Hence

$$p \leq \frac{1}{\mathrm{e}(d+1)} \leq x(1-x)^d,$$

as required. □

Exercise: Show that $x \mapsto x(1-x)^d$ is maximized when $x = 1/(d+1)$. Let $g(y) = (1-1/y)^{y-1}$ for $y > 1$. Show that

$$g'(y) = 1/y + \log(1-1/y) = -\sum_{m=2}^{\infty} \frac{1}{my^m}$$

and deduce that g is decreasing for $y > 1$. Show that $\lim_{y\to\infty} g(y) = 1/\mathrm{e}$ and hence that $(1 - \frac{1}{d+1})^d \geq 1/\mathrm{e}$ for all $d \in \mathbf{N}$. [For comparison, $(1 - \frac{1}{d})^d \leq 1/\mathrm{e}$ follows easily from the inequality $1 - x \leq \mathrm{e}^{-x}$.]

APPLICATION TO DIAGONAL RAMSEY NUMBERS.

Theorem 16.6. *Let $n, s \in \mathbf{N}$. If $s \geq 3$ and*

$$\mathrm{e}\binom{s}{2}\binom{n-2}{s-2}2^{1-\binom{s}{2}} < 1$$

then there is a red-blue colouring of the complete graph K_n with no red K_s or blue K_s and so $R(s,s) > n$.

Proof. Define the events A_S as at the start of this section. We remarked that if S is an s-subset of $\{1, 2, \ldots, n\}$ then the event E_S is independent of the events A_T for those s-subsets T such that $S \cap T \leq 1$. We now estimate the number of s-subsets T such that $|S \cap T| \geq 2$. We can choose two elements to be in $S \cap T$ in $\binom{s}{2}$ ways, and then choose any $s-2$ of the remaining $n-2$ elements of $\{1, 2, \ldots, n\}$ to complete T. This gives $\binom{s}{2}\binom{n-2}{s-2}$. The set S is counted exactly $\binom{s}{2}$ times; avoiding this overcounting (while ignoring all the other overcounting) gives

$$\begin{aligned}\big|\{T : T \subseteq \{1, 2, \ldots, n\}, |S \cap T| \geq 2\}\big| &\leq \binom{s}{2}\binom{n-2}{s-2} - \binom{s}{2} + 1 \\ &\leq \binom{s}{2}\binom{n-2}{s-2} - 1.\end{aligned}$$

Therefore we let $d = \binom{s}{2}\binom{n-2}{s-2} - 1$. Since

$$\mathbf{P}[E_S] = 2^{1-\binom{n}{s}}.$$

for all S, we take $p = 2^{1-\binom{n}{s}}$. Then we can apply the Symmetric Lovász Local Lemma, provided that $\mathrm{e}p(d+1) \leq 1$, which is one of the hypotheses of the theorem we are proving. Hence

$$\mathbf{P}\Big[\bigcap_S \overline{A_S}\Big] > 0$$

and so there is a red-blue colouring with no monochromatic K_s, as required. □

Theorem 16.6 is stronger than Theorem 15.5, roughly by a factor of s.

Proposition 16.7. *If $s \geq 2$ then*

$$R(s,s) \geq 2^{(s-1)/2}s/\mathrm{e}.$$

Proof. To apply Theorem 16.6 we must choose n such that

$$\mathrm{e}\binom{s}{2}\binom{n-2}{s-2}2^{1-\binom{s}{2}} \leq 1.$$

We estimate the left-hand side as follows

$$\begin{aligned}
\mathrm{e}\binom{s}{2}\binom{n-2}{s-2}2^{1-\binom{s}{2}} &= \mathrm{e}\frac{s(s-1)}{2}\frac{(n-2)^{s-2}}{(s-2)!}2.2^{-s(s-1)/2} \\
&\leq \mathrm{e}\frac{s(s-1)}{(s-2)!}\frac{1}{(n-2)^2}\Big(\frac{(n-2)}{2^{(s-1)/2}}\Big)^s \\
&\leq \mathrm{e}\frac{s^4}{s!(n-2)^2}\Big(\frac{n}{2^{(s-1)/2}}\Big)^s \\
&\leq \mathrm{e}\frac{s^4}{(n-2)^2}\Big(\frac{n\mathrm{e}}{2^{(s-1)/2}s}\Big)^s
\end{aligned}$$

using that $s! \geq (s/\mathrm{e})^s$. (See Question 9 on Sheet 7.) If we take

$$n = \lfloor 2^{(s-1)/2}s/\mathrm{e} \rfloor.$$

then we need $\mathrm{e}s^4/(n-2)^2 \leq 1$, which holds provided $s \geq 13$. It can be checked directly that with this definition of n the first inequality holds for $2 \leq s \leq 12$. □

Proposition 5.3.1 in Alon and Spencer gives the slightly stronger asymptotic result that $R(s,s) \geq (\sqrt{2}/\mathrm{e})(1+\epsilon_s)2^{s/2}s$ where $\epsilon_s \to 0$ as $s \to \infty$. The following numerical example shows the improvement of Proposition 16.7 over Theorem 15.5.

Example 16.8. When $s = 15$, the largest n such that

$$\binom{n}{15}2^{1-\binom{15}{2}} < 1$$

is $n = 792$. So Theorem 15.5 tells us that $R(15,15) > 792$. But

$$\mathrm{e}\binom{15}{2}\binom{n-2}{15-2}2^{1-\binom{15}{2}} < 1$$

provided $n \leq 947$. Theorem 16.2 therefore gives the stronger result that $R(15,15) > 947$.

EDGE DISJOINT PATHS IN A GRAPH. Suppose that G is a graph and that $\mathcal{F}_1, \ldots, \mathcal{F}_n$ are sets of paths in G of length at least m such that if $i \neq j$ and $P \in \mathcal{F}_i$ then P shares an edge with at most k paths from $\mathcal{F}_j$. **[Hypothesis corrected after lecture.]**

Proposition 16.9. *If*

$$n \leq \frac{m}{2k\mathrm{e}}$$

then there are paths $P_1 \in \mathcal{F}_1, \ldots, P_n \in \mathcal{F}_n$ *such that* $P_1, \ldots, P_n$ *are edge disjoint.*

Outline proof. Choose the paths at random: given $P_i \in \mathcal{F}_i$, there are at most k paths $P_j \in \mathcal{F}_j$ meeting P_i, so if A_{ij} is the event that the paths P_i and P_j have an edge in common then $\mathbf{P}[A_{ij}] \leq k/m$. Moreover the events A_{ij} and $A_{i'j'}$ are independent unless $\{i,j\} \cap \{i',j'\} \neq \varnothing$. There are $n-2$ subsets $\{i,k\}$ with $k \neq j$ and $n-2$ subsets $\{j,k\}$ with $k \neq i$, so the degree of the dependency graph is $2n-3$. Now apply the symmetric Lovász Local Lemma. □

APPLICATION TO $R(3,t)$. The Asymmetric Lovász Local Lemma can be used to get the bound

$$R(3,t) \geq \frac{Ct^2}{(\log t)^2}$$

for some constant C. For an outline of the proof and references to further results, see Alon and Spencer, Chapter 5, Section 3. By Question 4 on Sheet 7, we have $R(3,t) \leq t(t+1)/2$, hence

$$\log R(3,t) \sim 2\log t \quad \text{as } n \to \infty.$$

More precise asymptotic results are known: see J. H. Kim, *The Ramsey number* $R(3,t)$ *has order of magnitude* $t^2/\log t$, Random Structures Algorithms **7** (1995), 173–207, but are well beyond the scope of this course.