MT354/454/5454 Combinatorics: Preliminary Problem Sheet

The purpose of this sheet is to remind you of the notation for sets, tuples and functions, and to give you some short and straightforward questions on which to practice the Basic Counting Principles.

Answers will be posted to Moodle on 10th October. Please do Question 2 before the lecture on 6th October.

You are very welcome to ask the lecturer about the problem sheets. You can do this either after lectures, or in office hours. You do not have to wait until the answers are posted to Moodle.

1. A menu has 3 starters, 4 main courses and 6 desserts.

(a) How many ways are there to order a starter, main course and dessert? [*Hint:* multiply choices.]

(b) How many ways are there to order a two course meal, including exactly one main course?

2. Let

$$\begin{aligned} X &= \{(a_1, a_2, a_3) : 1 \leq a_1, a_2, a_3 \leq 10,\ a_1, a_2, a_3 \text{ distinct}\}, \\ Y &= \{A \subseteq \{1, \ldots, 10\} : |A| = 3\}. \end{aligned}$$

Define a function $f : X \to Y$ by $f\big((a_1, a_2, a_3)\big) = \{a_1, a_2, a_3\}$.
For example, $(3, 2, 5) \in X$ and $f\big((3, 2, 5)\big) = \{2, 3, 5\} \in Y$.

(a) Find $|X|$.

(b) Find the number of tuples $(a_1, a_2, a_3) \in X$ such that $f\big((a_1, a_2, a_3)\big) = \{2, 3, 5\}$.

(c) By generalizing the idea in (b), find $|Y|$.

[*Hint:* the point of this question is to show the ideas in one proof of the formula for binomial coefficients. So please do not assume this formula is true in (c).]

3. A deck consists of 52 cards. There are four Aces, four Kings, four Queens and four Jacks. How many hands of five cards are there that

(a) have at least one Ace, King, Queen and Jack? [*Hint:* first count hands of the form AKQJx, then hands of the form AAKQJ, and so on.]

(b) have at least one Ace, King and Queen?

Remark: In Section 5 of the course we will see the Principle of Inclusion and Exclusion: it gives a quick, unified, way to solve both problems.

4. Fix $n \in \mathbf{N}$. Let $X = \{(a, b) : 1 \leq a \leq b \leq n\}$. Find a simple formula for $|X|$ in terms of n.

5. Let

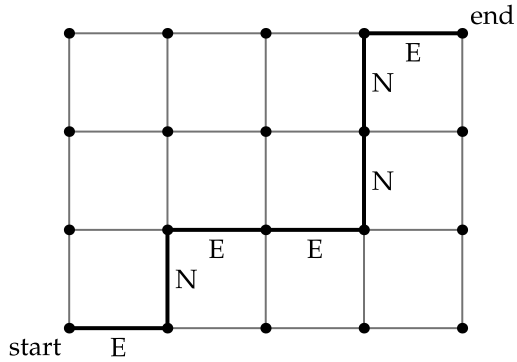
$$X = \left\{ \begin{array}{c} \text{placements of 4 indistinguishable balls into 7} \\ \text{numbered urns so that each ball is in a different urn} \end{array} \right\}$$

$$Y = \left\{ \begin{array}{c} \text{ways to walk 4 blocks East and 3 blocks North} \\ \text{on a New York grid, moving only East and North} \end{array} \right\}$$

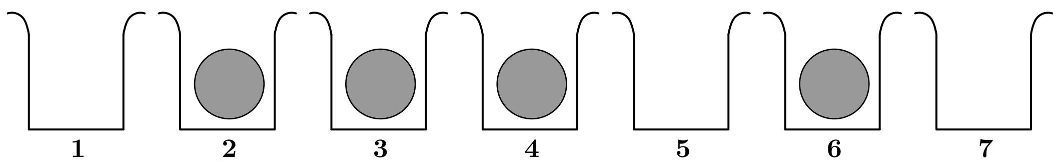
$$Z = \big\{ A \subseteq \{1, 2, \ldots, 7\} : |A| = 4 \big\}$$

(a) Define explicit bijective maps $f : X \to Z$ and $g : Y \to Z$. [*Hint:* it might help to first work out what you want the answers to (b) and (c) to be.]

(b) Which element of Z corresponds to the walking route ENEENNE $\in Y$?



(c) Which walking route corresponds to the ball-and-urn placement shown below?



(d) Find a binomial coefficient equal to $|X|$ and $|Y|$.