IMPARTIAL GAMES WITH ENTAILMENT

This is a much expanded version of the basic theory of entailed games presented in [1] Chapter 12, pages 396 and 397, with some extra definitions to do with equality that were possibly not intended by the authors of [1].

1. Standard definitions

We define impartial games inductively as follows: $0 = \{\}$ is an impartial game. If $G_1, G_2, \ldots, G_k$ are impartial games then $\{G_1, G_2, \ldots, G_k\}$ is an impartial game, with *options* $G_1, G_2, \ldots, G_k$. (Note all the games defined in this way have finitely many options.) If G, G' are impartial games with options $G_1, G_2, \ldots, G_k$ and $G'_1, G'_2, \ldots, G'_{k'}$ respectively, then we define $G + G'$ to be the game

$$\{G_1 + G', \ldots, G_k + G', G + G'_1, \ldots, G + G'_k\}.$$

According to the usual definitions, an impartial game G is equal to 0 if it is a P-position (the **P**revious player wins) and equality is defined by

$$G = G' \iff G + G' = 0.$$

Note that games with different options may well be equal. For example, if we define (inductively) *nimbers* to be the games

$$g\star = \{0, \star, \ldots, (g-1)\star\}$$

for $g \in \mathbf{N}$ then

$$\{0, \star\} = \{0, \star, 3\star\}$$

as can easily be checked. (A move to $3\star$ is answered by a move to $2\star$, so the first player may as well restrict himself to the options 0, $\star$.)

The following lemma generalizes this example. In it, we define the *minimum excluded number* of a proper subset A of $\mathbf{N}_0$ by

$$\operatorname{mex} A = \min(\mathbf{N}_0 \backslash A).$$

Lemma 1. *Let G be a game with options equal to the nimbers $a_1\star, a_2\star, \ldots, a_k\star$. Then $G = a\star$ where $a = \operatorname{mex}\{a_1, \ldots, a_k\}$.*

Proof. Consider the game $G + a\star$. If $b < a$ then $b\star$ is an option of G. Hence a move in $a\star$ to $b\star$ can be answered by a move in G to $b\star$. A move in G to $b\star$ can be answered similarly. A move in G to $c\star$ where $c > a$ is answered by moving to $a\star$. So the new game after two moves is $a\star + a\star = 0$. Hence $G + a\star$ is a first player loss. □

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2. Entailed games

To allow for entailment we must reduce the options in a sum of games so that only moves in one particular component are allowed. (So our point of view is that the definition of an impartial game as a list of options implicitly includes games with entailment. However, the definitions of addition, and we will see, equality, must change.)

Definition 2. Suppose that G and G' are impartial games and that G has options $G_1, \ldots, G_k$. The *sum* $\underline{G} + G'$ *with a move in* G *entailed* has options $G_1 + G', \ldots, G_k + G'$.

For example, $2\star + 3\star = \star$ is an N-position (**N**ext person to play wins). But if a move in $2\star$ is entailed then we have

$$\underline{2\star} + 3\star = \{2\star, 3\star\}$$

so $\underline{2\star} + 3\star$ is a P-position.

We have to be very careful with equality. The definition given above of equality no longer makes sense for sums of games with entailed components. (The sum of two games each with entailed components would have to always be zero, since there are no legal moves, but we surely do not want to claim that any two games with an entailed component are equal.) There are also other problems.

For example, since $3\star + 2\star + 1\star$ is a P-position, the entailed game $(\underline{3\star} + 2\star) + \star$ is also a P-position. Is it correct to write $\underline{3\star} + 2\star = \underline{\star}$? If so, then the games $(\underline{3\star} + 2\star) + 2\star$ and $\underline{\star} + 2\star$ should behave similarly. (We can't say 'are equal' here, because of the remark in the previous paragraph.) But the first is an N-position (take the entire entailed heap) and the second is a P-position (the only legal move loses).

To get around this we will change the definition of equality.

3. Equality for entailed games

Definition 3 (*Full set of nim values*)**.** The full set of nim values of the zero game is $\mathbf{N}_0 = \{0, 1, 2, \ldots\}$. Suppose that G is an impartial game with options $G_1, \ldots, G_k, E_1, \ldots, E_\ell$, where the G_i are unentailed, but for each E_j, the next move must be made in E_j. Let g_i be the least value in the full set of nim values of G_i and let Γ_j be the full set of nim values of E_j. Then the full set of nim values of G is the complement in $\mathbf{N}_0$ of

$$\{g_1, \ldots, g_k\} \cup \Gamma_1 \cup \cdots \cup \Gamma_\ell.$$

Here g_i should be omitted if the full set of nim values of G_i is empty, and so has no least value.

We shall write $N(G)$ for the full set of nim values of an impartial game G and $n(G)\star$ for the minimum of $N(G)$, taking care only to use $n(G)$ when $N(G) \neq \varnothing$. Extending the usual definition, we call $n(G)$ the *nimvalue* of the game G.

Definition 4. Let G and G' be impartial games. We say that G is *equal to* G', and write $G = G'$ if $N(G) = N(G')$.

Note that if $G = \{G_1, \dots, G_k\}$ is a game without any entailed options then $N(G)$ is exactly the complement of the set of minimum nim values $n(G_1), \dots, n(G_k)$ of options of G, and $n(G)$ is the minimum excluded number of the set $\{n(G_1), \dots, n(G_k)\}$. So for games without entailment, looking only at the least value of the full set of nim values recovers the usual definition of equality.

One more result is easy to prove.

Lemma 5. *Let G be an impartial game. Then G is a P-position (player to move loses) if and only if $0 \in N(G)$.*

Proof. The lemma is clearly true if $G = \{\}$ since then $N(G) = \mathbf{N}_0$.

Let G have options $G_1, \dots, G_k$ and assume inductively that the lemma holds for all options of G.

Suppose that $0 \in N(G)$. Then, by definition of $N(G)$, none of $G_1, \dots, G_k$ have 0 in their full set of nim values. So, by induction, any option of G is an N-position (player to move wins), and hence G is a P-position.

Now suppose that $0 \notin N(G)$. Again by definition of $N(G)$, there exists an option G_i of G such that $0 \in N(G_i)$. By induction, G_i is a P-position, and so G is an N-position. □

Equivalently, G is a first player win if and only if $n(G) \neq 0$. This agrees with the usual theory.

4. Examples

(1) We have $N(g\star) = \{g, g+1, \dots\}$.

(2) Let G be the game starting with a pile of two counters in which we can either take both counters or reduce to an entailed pile of one counter. The full set of nim values of the empty pile is $\mathbf{N}_0$ of which 0 is the least value, and the full set nim values of a pile of one counter is $\{1, 2, \dots\}$. So $N(G) = \varnothing$. Note that in any sum $G + G'$, the first player must win: if it good to move first in G' then play to the entailed pile in G, otherwise make the opponent move first in G' by taking both counters.

(3) More generally, let G be any game with all unentailed options. Let G^+ have options the same as G with one extra option of playing to G entailed. This extra option is the same as passing (with the opponent then forced to play). If $G_1, \dots, G_k$ are the options of G then $N(G^+)$ is the complement of

$$\{n(G_1) \dots, n(G_k)\} \cup N(G).$$

But by definition, $N(G)$ is the complement of $\{n(G_1), \dots, n(G_k)\}$ so $N(G^+) = \varnothing$. So as in (2), the first player will win.

(4) With only finitely many options, any full set of nim values either contains all but finitely many integers, or is finite. Either of these possibilities can be achieved. For example, to get $N(G) = \{0, 1, 4, 5\}$ make the options of G unentailed piles of sizes 2 and 3 and an entailed pile of size 6.

(5) Let G be the strings and coins game with two coins connected by a string, and one of these coins also connected to the ground. We show in Example 8 below that $N(G) = \varnothing$. Consider the (unentailed) game $\{G\}$ which has G as its unique option. Since G has no nim values, the full set of nim values of $\{G\}$ is $\mathbf{N}_0$.

5. Full set of nim values for a sum of games

We are now ready for the main result.

Theorem 6. *Let $A, B, \ldots, C$ be impartial games with non-empty full sets of nim values. Suppose that A has full set of nim values $a_0, a_1, a_2, \ldots$.*

(i) *The game $A + B + \cdots + C$ is a P-position if and only if*

$$n(A)\star + n(B)\star + \cdots + n(C)\star = 0.$$

(ii) *The game $\underline{A} + B + \cdots = C$ is a P-position if and only if there exists i such that*

$$a_i\star + n(B)\star + \cdots + n(C)\star = 0.$$

Note that the entailed game is a P-position more often than the unentailed game, as one would obviously expect. We will only prove (ii), but (i) follows along similar lines. In the proof, if $X \subseteq \mathbf{N}_0$ then $X\star$ denotes $\{m\star : m \in X\}$.

Proof. Let A have unentailed options $A_1, \ldots, A_k$ and entailed options $E_1, \ldots, E_\ell$. Consider the following chain of implications:

$\underline{A} + B + \cdots + C$ is an N-position

$\iff$ Either there exists an unentailed option A_i of A such that $A_i + B + \cdots + C$ is a P-position, or there exists an entailed option E_i of A such that $\underline{E_i} + B + \cdots + C$ is a P-position

$\iff$ Either there exists an unentailed option A_i of A such that $n(B)\star + \cdots + n(C)\star = n(A_i)\star$ or there exists an entailed option E_i of A such that $n(B)\star + \cdots + n(C)\star \in N(E_i)\star$

$\iff$ $n(B)\star + \cdots + n(C)\star \notin N(A)\star$

$\iff$ For all j we have $a_j\star + n(B)\star + \cdots + n(C)\star \neq 0$

Here the second implication uses induction on the sums $A_i + B + \cdots + C$, and $\underline{E_i} + B + \cdots = C$, and the third uses the definition of $N(A)$. □

Notice that the condition in the theorem depends only on the full set of nim values of the games, and not on the games themselves. This justifies the definition of equality for impartial games with entailment, and shows that equal games can be freely substituted for one another in sums.

We end with a lemma that deals with the case excluded from Theorem 6.

Lemma 7. *Let G be a game with empty full set of nim values. Then $G + G'$ is an N-position for any impartial game G'.*

Proof. By hypothesis, G has options having every nim value $0\star, \star, \ldots$. Play in G to a game with value $n(G')$. □

6. Nimstring example

We shall give the Nimstring rules in a non-standard way so that the game is obviously an impartial game in the sense defined above.

A Nimstring game consists of a number of coins connected by strings. Coins can also be connected by strings to the ground. In each move a string is cut. If cutting a string frees one or more coins then the new game has a single option: pass and let the opponent play on (if they can).

Example 8.

(i) The empty Nimstring game is 0, with full set of nim values $\mathbf{N}_0$.

(ii) The game G with a single coin and one string to the ground has as its unique option the entailed game $\underline{\{0\}}$. The full set of nim values of $\underline{\{0\}}$ is $\{1, 2, \ldots\}$ so we have $N(G) = \{0\}$.

(iii) The game G' with a single coin and two strings to the ground has G (unentailed) as its unique option. So $N(G') = \{1, 2, \ldots\}$.

(iv) Let T be the game with two coins connected by a string. This game has the entailed game $\underline{\{0\}}$ as its single option. So $N(T)$ is the complement of $N(\underline{\{0\}})$. Since $N(\underline{\{0\}}) = \{1, 2, \ldots\}$ this shows that $N(T) = \{0\}$.

(iv) Let C be the game with two coins connected by a string, with one coin also connected to the ground. The options of C are the entailed game $\underline{\{G\}}$ and the game T just seen. Now $N(\underline{\{G\}}) = \{1, 2, \ldots\}$. So $N(C)$ is the complement of $\{0\} \cup \{1, 2, \ldots\}$, that is $N(C) = \varnothing$.

Observe any Nimstring game having C as a component is, by Lemma 7, a P-position. In the terminology of [1], C is a loony position, and any move that creates a new component of C is loony (losing under any circumstance).

Example (ii) shows that in any Nimstring game containing a single coin connected to the ground, together with some other components, the single coin is irrelevant (full set of nim values is $\{0\}$, so in any winning move calculation as in Theorem 6, it contributes 0). The same result for two coins connected by a string follows from Example (iv), and it also true for three coins in a line connected by two strings.

7. Complimenting moves

Generalizing the examples seen in Nimstring, suppose that G is a game with unentailed options $g_1\star, \ldots, g_k\star$ and entailed options $\underline{\{E_1\}}, \ldots \underline{\{E_\ell\}}$, where $\ell \geq 1$. So if the entailed option $\underline{\{E_j\}}$ is chosen then the opponent has no choice (even if there are other components) but to move to E_j.

Suppose that some E_j has empty full set of nim-values. Then the full set of nim values of $\underline{\{E_j\}}$ is $\mathbf{N}_0$ and so the full set of nim values of G is empty.

Now suppose that every E_j has non-empty full set of nim-values. The full set of nim values of $\underline{\{E_j\}}$ is then $\mathbf{N}_0 \backslash \{n(E_j)\}$. Hence the full set of nim values of G is the complement of

$$\bigcup_j (\mathbf{N}_0 \backslash \{n(E_j)\}) \ \cup \ \{g_1, \ldots, g_k\}.$$

Thus

$$N(G) = \bigcap_j \{n(E_j)\} \ \cap \ (\mathbf{N}\backslash\{g_1, \dots, g_k\}).$$

If there are two entailed options E_j and $E_{j'}$ with $n(E_j) \neq n(E_{j'})$ then we see that $N(G) = \varnothing$. If all entailed options have the same least nim value g then we still have $N(G) = \varnothing$, except when $g \neq g_i$ for all i. In this final case, $N(G) = \{g\}$. Thus G is loony (empty set of nim values) or $N(G) = \{g\}$.

In the case where $n(E_1) = \ldots = n(E_\ell) = g$ and $g \in \{g_1, \ldots, g_k\}$ then $N(G) = \varnothing$, and the unique winning move in the game $G + g\star$ is the unentailed move in G to $g\star$. (By assumption this is an unentailed option.) So even when G is loony, it may not be correct to play an entailed move.

Example 9. An important special case occurs when the options of G are the entailed game $\{g\star\}$ and $g\star$. Then $N(G)$ is the complement of

$$N(\{g\}) \cup \{g\}$$

and since $N(\{g\} = \mathbf{N}_0\backslash\{g\}$, we see that $N(G) = \varnothing$. Intuitively: if, as part of a larger game, it's good to play to $g\star$ in G, then the first player can do so. Otherwise replacing G with $g\star$ must give a first player win, and the first player can play to the entailed game $\{g\}$ to force the second player to put him into the same game with G replaced with $g\star$.

Note that this argument does not show that an entailed move is always winning.

Example 10. For instance, if G is the strings and coins game with two coins connected by a string and one of these coins connected to the ground, then G is a loony position (empty full set of nim values) and the unique entailed move in G loses.

Example 11. Let G be the game with an entailed option $\{\star\}$ and an unentailed option $2\star$. Then the full set of nim values is the complement of

$$N(\{\star\}) \cup \{2\} = \{0, 2, 3, \ldots\}$$

namely $\{1\}$. This agrees with the full set of nim values of $\{\star\}$. So the first player cannot do worse by taking the entailed option. For instance, if we play $G + 2\star$, then it might seem natural to play in G to $2\star$, leaving the opponent with the zero game. But if instead the first player takes the entailed option, on his next turn he plays in $\star + 2\star$, and he can win in this game instead.

The discussion at the start of this section shows that any Nimstring game can be analysed just using normal nimvalues, together with a special symbol, conventionally ☾, to denote a game with empty full set of nim values.

Example 12. Let G be a Nimstring game with capturable coins $c_1, \ldots, c_m$ and let H_j be the game with coin c_j removed. We may suppose that $N(H_j)$ is non-empty if and only if $j \in \{1, \ldots, \ell\}$, where $\ell < m$.

We have seen that $N(G)$ is empty if any entailed option has empty full set of nim values. Suppose then that $\ell = m$. We have seen that if $n(H_j) \neq n(H_{j'})$ for any distinct j and j' then the full set of nim values of G is empty. In the remaining case we have $n(H_1) = \ldots = n(H_\ell)$. Moreover, we know that $g_i \neq g$ for any unentailed nimber $g_i\star$ that is an option of G, and $n(G) = n(H_1) = \ldots = n(H_\ell)$.

Note that, as promised, this analysis did not require the full set of nim values of the options, only their minimum value, or the fact that they were empty.

References

1. Elwyn R. Berlekamp, John H. Conway, and Richard K. Guy, *Winning ways for your mathematical plays*, 2nd ed., vol. 2, A K Peters, 2003.
2. ______, *Winning ways for your mathematical plays*, 2nd ed., vol. 3, A K Peters, 2003.