A SHORT PROOF OF THE EXISTENCE OF JORDAN NORMAL FORM

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Let V be a finite-dimensional complex vector space and let $T : V \to V$ be a linear map. A fundamental theorem in linear algebra asserts that there is a basis of V in which T is represented by a matrix in *Jordan normal form*

$$\begin{pmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_k \end{pmatrix}$$

where each J_i is a matrix of the form

$$\begin{pmatrix} \lambda & 1 & \dots & 0 & 0 \\ 0 & \lambda & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda & 1 \\ 0 & 0 & \dots & 0 & \lambda \end{pmatrix}$$

for some $\lambda \in \mathbf{C}$.

We shall assume that the usual reduction to the case where some power of T is the zero map has been made. (See [1, §58] for a characteristically clear account of this step.) After this reduction, it is sufficient to prove the following theorem.

Theorem 1. *If $T : V \to V$ is a linear transformation of a finite-dimensional vector space such that $T^m = 0$ for some $m \geq 1$, then there is a basis of V of the form*

$$u_1, Tu_1, \dots, T^{a_1-1}u_1, \dots, u_k, Tu_k, \dots, T^{a_k-1}u_k$$

where $T^{a_i}u_i = 0$ for $1 \leq i \leq k$.

At this point all the proofs the author has seen (even Halmos' in [1, §57]) become unnecessarily long-winded. In this note we present a simple proof which leads to a straightforward algorithm for finding the required basis.

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Proof. We work by induction on $\dim V$. For the inductive step we may assume that $\dim V \geq 1$. Clearly $T(V)$ is properly contained in V, since otherwise $T^m(V) = \cdots = T(V) = V$, a contradiction. Moreover, if T is the zero map then the result is trivial. We may therefore assume that $0 \subset T(V) \subset V$. By applying the inductive hypothesis to the map induced by T on $T(V)$ we may find $v_1, \ldots, v_l \in T(V)$ so that

$$v_1, Tv_1, \ldots, T^{b_1-1}v_1, \ldots, v_l, Tv_l, \ldots, T^{b_l-1}v_l$$

is a basis for $T(V)$ and $T^{b_i}v_i = 0$ for $1 \leq i \leq l$.

For $1 \leq i \leq l$ choose $u_i \in V$ such that $Tu_i = v_i$. Clearly $\ker T$ contains the linearly independent vectors $T^{b_1-1}v_1, \ldots, T^{b_l-1}v_l$; extend these to a basis of $\ker T$, by adjoining the vectors $w_1, \ldots, w_m$ say. We claim that

$$u_1, Tu_1, \ldots, T^{b_1}u_1, \ldots, u_l, Tu_l, \ldots, T^{b_l}u_l, w_1, \ldots, w_m$$

is a basis for V. Linear independence may easily be checked by applying T to a given linear relation between the vectors. To show that they span V, we use dimension counting. We know that $\dim \ker T = l + m$ and that $\dim T(V) = b_1 + \ldots + b_l$. Hence, by the rank-nullity theorem,

$$\dim V = (b_1 + 1) + \ldots + (b_l + 1) + m,$$

which is the number of vectors in our claimed basis. We have therefore constructed a basis for V in which T is in Jordan normal form. □

We end by remarking that this proof can be modified to avoid the preliminary reduction. Let λ be an eigenvalue of T. By induction we may find a basis of $(T - \lambda I)V$ in which the map induced by T on $(T - \lambda I)V$ is in Jordan normal form. This basis can then be extended in a similar way to before to obtain a basis for V in which T is in Jordan normal form.

References

[1] P. R. Halmos, *Finite-dimensional vector spaces*, 2nd ed. Undergraduate Texts in Mathematics. Springer, 1987.

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