MT182 Matrix Algebra

Mark Wildon, mark.wildon@rhul.ac.uk

Administration:

- Workshops begin next week.
- Sign-in sheet. **Please return to the lecturer after each lecture.**
- Make sure you get the Part A Notes and Problem Sheet 1. **Please pass everything onwards, and eventually to the back, even if you the person you are passing to already has a copy.**
- Please take a clicker and use it!
- All handouts will be put on Moodle.
- **Lectures:** Monday noon (QBLT), Friday 10am (QBLT), Friday 3pm (BLT1).
- **Office hours in McCrea 240:** Monday 4pm, Wednesday 10am and Friday 4pm.

§1 Introduction to vectors

Definition 1.1

$\mathbb{R}^3$ is the set of all *ordered triples* (x, y, z) of real numbers.

The notation (x, y, z) means that we care about the order of the entries. For example $(1, 2, 3) \neq (2, 1, 3)$. Ordered triples are not sets.

Quiz: Exactly one of these statements is true. Which one?

(A) $\{1, 2, 1\} = (1, 2, 1)$

(B) $\{1, 2, 1\} \neq \{1, 2\}$

(C) $\{1, 2, 1\} = \{2, 1\}$

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Quiz: How many dimensions do we live in?

(A) 1 (B) 3 (C) 4 (D) 26 (E) None of these.

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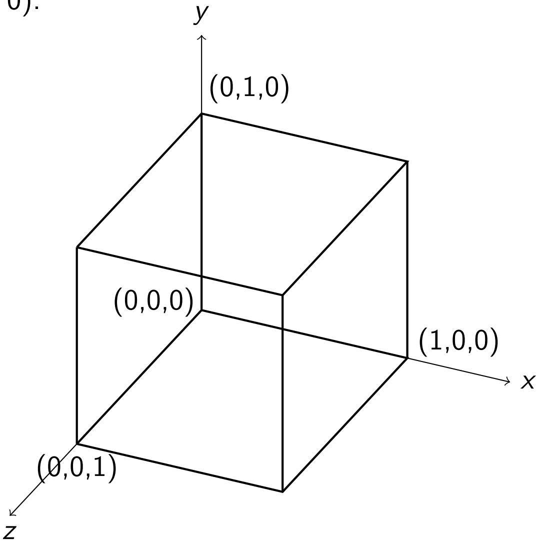
(With some room for debate.)

Three-dimensional Space

The diagram below shows a cube with one vertex at the *origin* $O = (0, 0, 0)$.



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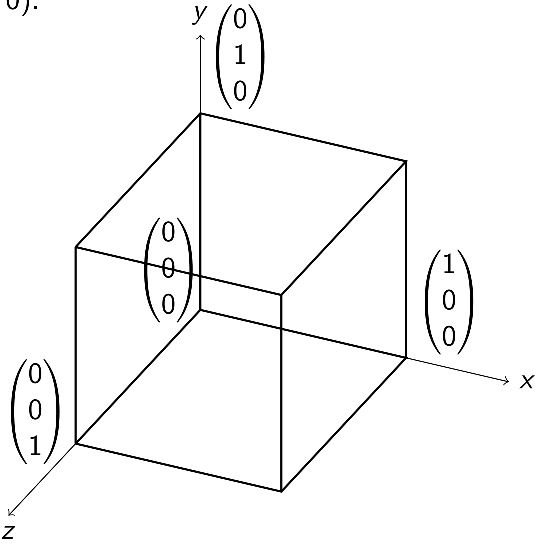

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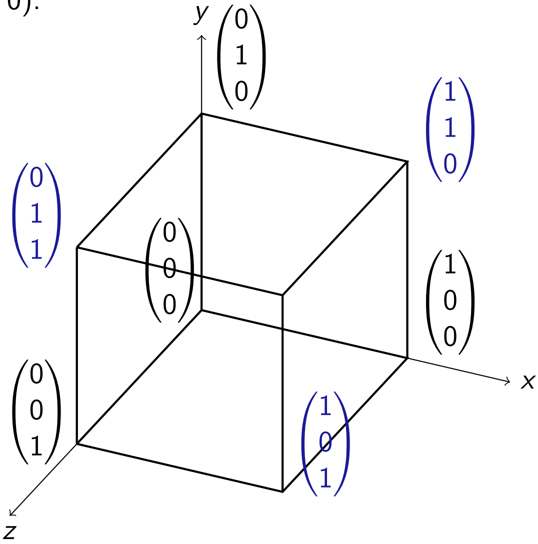

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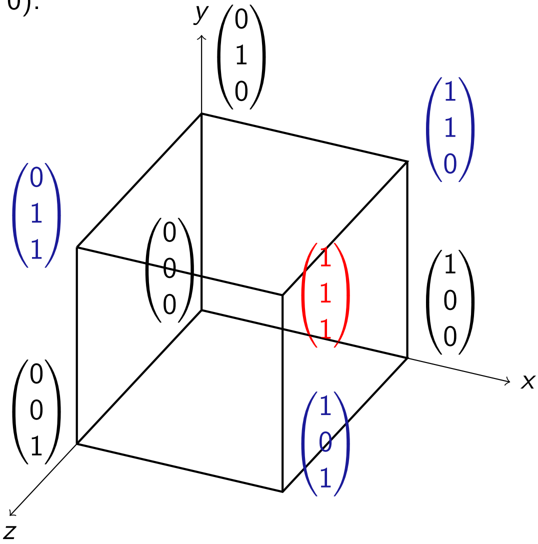


Displacement Vectors

Suppose that A and B are distinct points in $\mathbb{R}^3$. Starting from A we can walk to B. The *displacement vector* $\overrightarrow{AB}$, gives the distance we move in each coordinate direction.

Problem 1.3

(a) What is the displacement vector from $(1, 0, 0)$ to $(0, 0, 1)$?

(b) If we apply this displacement starting at $(1, 1, 0)$, where do we finish?

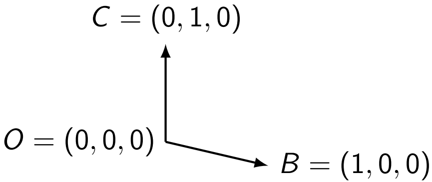
(c) If we finish at $(12, -3, 2)$ after applying this displacement, where must we have started?

(d) What is the displacement vector from the origin O to $(1, 1, 0)$?

Vector Sum

Problem 1.4

Let $B = (1, 0, 0)$ and let $C = (0, 1, 0)$. Start at the origin O and apply the displacement vector $\overrightarrow{OB}$. Where do we finish? Now apply $\overrightarrow{OC}$. Let D be the finishing point. Find D.



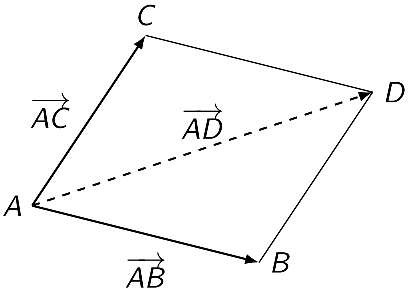
$$\overrightarrow{OB} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \overrightarrow{OC} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Definition 1.5

Let $\mathbf{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be vectors. We define the *sum* of $\mathbf{u}$ and $\mathbf{v}$ by $\mathbf{u} + \mathbf{v} = \begin{pmatrix} a + x \\ b + y \\ c + z \end{pmatrix}$.

Parallelogram Rule

Generalizing Problem 1.4, let A, B, C, D be points. If we start at A, and apply the displacement vectors $\overrightarrow{AB}$ then $\overrightarrow{AC}$, we end up at D where $\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{AC}$. This is called the *parallelogram rule*.



Exercise 1.6

Which displacement vectors label the sides BD and CD of the parallelogram? What is $\overrightarrow{AB} + \overrightarrow{BA}$? What is $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DA}$?

Parallelogram Rule: Quiz

Let A, B be distinct non-zero points in the plane. Suppose that $\overrightarrow{OA}$ is not parallel to $\overrightarrow{OB}$. How many points P are there such that $\{O, A, B, P\}$ is the set of vertices of a parallelogram?

(A) 1

(B) 2

(C) 3

(D) More information is required.

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Hint: sets are not ordered: $\{O, A, B, P\} = \{O, A, P, B\}$.

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Scalar Multiplication

Elements of $\mathbb{R}$ are often called *scalars*.

Definition 1.7

Let $\mathbf{v}$ be as in Definition 1.6, and let $\alpha \in \mathbb{R}$. We define the *scalar multiplication* of α and $\mathbf{v}$ by $\alpha\mathbf{v} = \begin{pmatrix} \alpha x \\ \alpha y \\ \alpha z \end{pmatrix}$.

Example 1.8

Let $A = (1, 0, 1)$ and $B = (1, 2, 3)$ and let $\mathbf{u}$ and $\mathbf{v}$ be the corresponding vectors. Then

$$2\mathbf{u} - 3\mathbf{v} = 2\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - 3\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ -6 \\ -7 \end{pmatrix} = -\begin{pmatrix} 1 \\ 6 \\ 7 \end{pmatrix}$$

Find $\mathbf{v} - \mathbf{u}$. Note that $\mathbf{v} - \mathbf{u} = \overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{AB}$. **This is often useful!** Find α such that the z-coordinate of $\alpha\mathbf{u} + \mathbf{v}$ is zero.

Linear Combinations

A sum, such as those in the example, of the form $\alpha\mathbf{u} + \beta\mathbf{v}$ is called a *linear combination* of the vectors $\mathbf{u}$ and $\mathbf{v}$. More generally, $\alpha_1\mathbf{u}_1 + \cdots + \alpha_r\mathbf{v}_r$ is a linear combination of the r vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$.

Problem 1.9

Let $\mathbf{u}$ and $\mathbf{v}$ be as in Example 1.8. Express $\mathbf{u}$ as a linear combination of the vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$. Express $\mathbf{v}$ as a linear combination of $\mathbf{i}, \mathbf{i} + \mathbf{j}, \mathbf{k}$.

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$$\mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \mathbf{i} + \mathbf{k}$$

and

$$\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = -\mathbf{i} + 2(\mathbf{i} + \mathbf{j}) + 3\mathbf{k}.$$

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Problem 1.10

There is a unique plane Π containing $(0, 0, 0)$, $(1, -1, 0)$, $(0, 1, -1)$. Is $(1, 0, -1)$ in Π? Is $(1, 1, -3)$ in Π?

Quiz: The correct answers are:

(A) TT (B) TF (C) FT (D) FF

Motivation for Dot Product

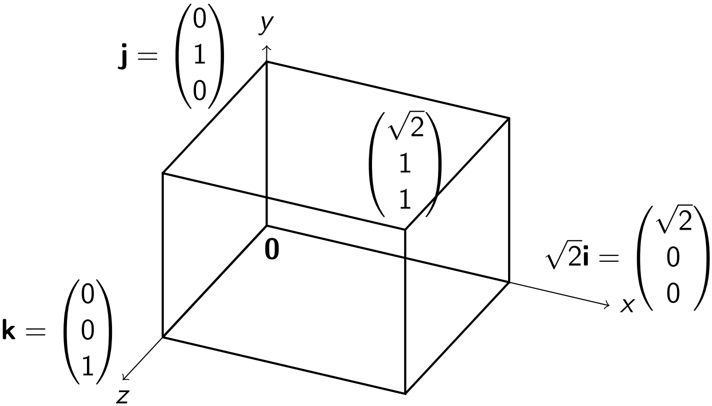
Definition 1.11

The *length* of a vector $\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is $||v|| = \sqrt{x^2 + y^2 + z^2}$.

The notation $|v|$ is also used. This can be confused with the absolute value of a real number, or the modulus of a complex number, so $||v||$ is probably better.

Example′ 1.12

Stretch the cube from Lecture 1 by $\sqrt{2}$ in the x direction.



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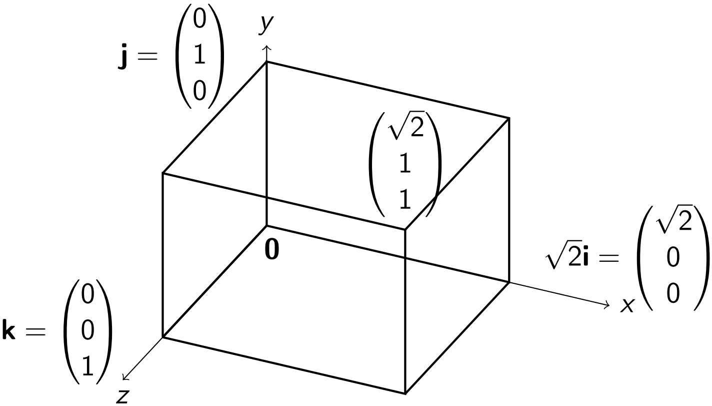

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The set of lengths is

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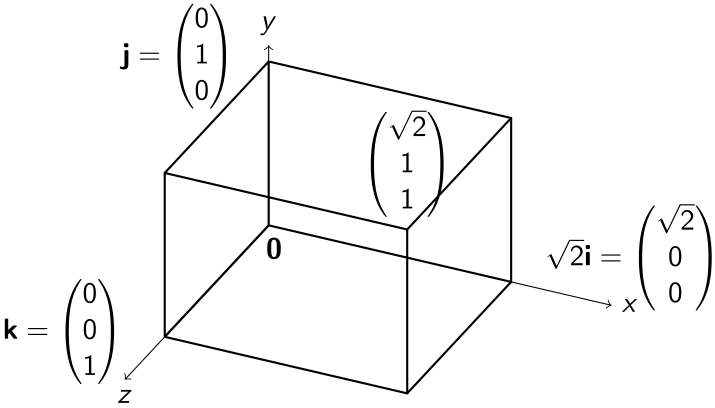

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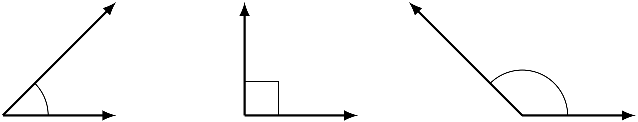
Angles Between Vectors

How can we compute the angle between two non-zero vectors? The angle is between 0 and π. It is $\pi/2$ if and only if the vectors are *orthogonal*.



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We will find the angle between $\sqrt{3}\mathbf{i} + \mathbf{j}$ and $\sqrt{3}\mathbf{i} - \mathbf{j}$ by considering the triangle with vertices at $(0, 0, 0)$, $(\sqrt{3}, 1, 0)$ and $(\sqrt{3}, -1, 0)$.

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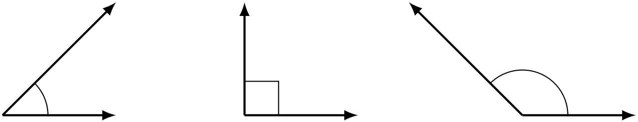

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We used that $||\sqrt{3}\mathbf{i}+\mathbf{j}|| = ||\sqrt{3}\mathbf{i}-\mathbf{j}|| = 2$, so the triangle is isosceles. We can reduce to this case by scaling each vector.

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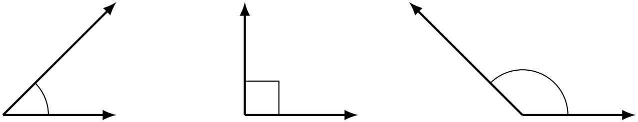

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Quiz: Conducting quizzes with clickers is a good use of lecturing time:

(A) Strongly disagree
(B) Somewhat disagree
(C) Slightly agree
(D) Strongly agree

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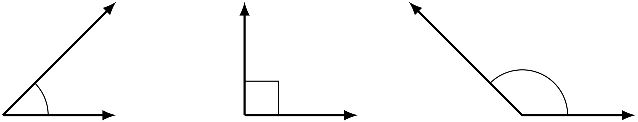

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Exercise 1.14

True or false? If $\mathbf{v} \in \mathbb{R}^3$ and $\alpha \in \mathbb{R}$ then $||\alpha v|| = \alpha||v||$.

(A) False (B) True

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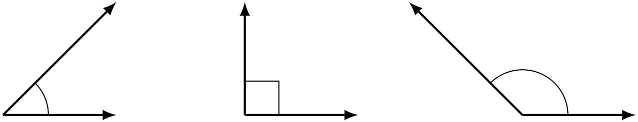

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General Angles

Theorem 1.15

Let $\mathbf{u} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ *and let* $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ *be non-zero vectors. Let* θ *be the angle between* $\mathbf{u}$ *and* $\mathbf{v}$*. Then*

$$\cos\theta = \frac{ax + by + cz}{||\mathbf{u}||||\mathbf{v}||}.$$

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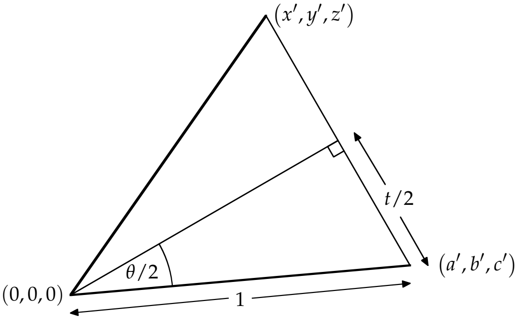
The diagram used in the proof is below:
$\widehat{\mathbf{u}} = \mathbf{u}/||\mathbf{u}|| = a'\mathbf{i} + b'\mathbf{j} + c'\mathbf{k}$ and $\widehat{\mathbf{v}} = \mathbf{v}/||\mathbf{v}|| = x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k}$.



Dot Product

Definition 1.16

Let $\mathbf{u} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ and $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \in \mathbb{R}^3$ be vectors. The *dot-product* of $\mathbf{u}$ and $\mathbf{v}$ is

$$\mathbf{u} \cdot \mathbf{v} = ax + by + cz.$$

Lemma 1.17

Let $\mathbf{n}, \mathbf{v} \in \mathbb{R}^3$ **be non-zero vectors [corrected in lecture]**. *Let* θ *be the angle between* $\mathbf{n}$ *and* $\mathbf{v}$.

(a) $\mathbf{v} \cdot \mathbf{v} = ||\mathbf{v}||^2$.

(b) $\mathbf{n} \cdot \mathbf{v} = ||\mathbf{n}|| \, ||\mathbf{v}|| \cos\theta$

(c) $\mathbf{n} \cdot \mathbf{v} = 0$ *if and only if* $\mathbf{n}$ *and* $\mathbf{v}$ *are orthogonal.*

(d) *Suppose that* $||\mathbf{n}|| = 1$. *Let* $\mathbf{w} = (\mathbf{n} \cdot \mathbf{v})\mathbf{n}$. *Then* $\mathbf{v} = \mathbf{w} + (\mathbf{v} - \mathbf{w})$ *where* $\mathbf{w}$ *is parallel to* $\mathbf{n}$ *and* $\mathbf{v} - \mathbf{w}$ *is orthogonal to* $\mathbf{n}$.

(e) *Let* $\mathbf{u} \in \mathbb{R}^3$ *and let* α, $\beta \in \mathbb{R}$. *Then*

$$\mathbf{n} \cdot (\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha\mathbf{n} \cdot \mathbf{u} + \beta\mathbf{n} \cdot \mathbf{v}.$$

Angle Between Lines

Two intersecting lines always make an angle between 0 and $\pi/2$.

Example 1.18

Let ℓ be the line through the origin and $(1, 0, -1)$. Let ℓ' be a line with direction $(-1, 1, 2)$. What is the angle between ℓ and ℓ'?

(A) $-\pi/6$ (B) $\pi/6$ (C) $\pi/4$ (D) $\pi/3$

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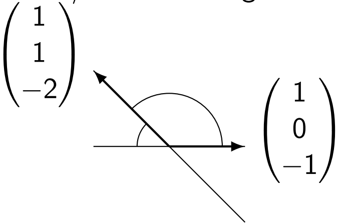
The angle between ℓ and ℓ' does not depend on where the intersection is. Since the direction of ℓ is $\mathbf{i} - \mathbf{k}$, θ satisfies

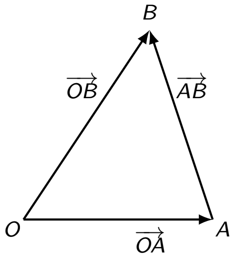
$$\cos\theta = \frac{(\mathbf{i} - \mathbf{k}) \cdot (\mathbf{i} + \mathbf{j} + 2\mathbf{k})}{||-\mathbf{i} + \mathbf{k}||\,||\mathbf{i} + \mathbf{j} + 2\mathbf{k}||} = \frac{-3}{\sqrt{2} \times \sqrt{6}} = \frac{-3}{2\sqrt{3}} = -\frac{\sqrt{3}}{2}$$

So the obtuse angle is $5\pi/6$ and the angle we want is $\pi/6$.



Cosine Rule

As an application of (a) and (b) we prove the cosine rule. We need part of Example 1.8: $\overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{AB}$.



Exercise 1.19

Let A, $B \in \mathbb{R}^3$ be points such that OAB is a triangle. Let $\mathbf{u} = \overrightarrow{OA}$ and $\mathbf{v} = \overrightarrow{OB}$ and let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$. Find the length of the side AB by using (a) to compute

$$||\overrightarrow{AB}||^2 = ||\overrightarrow{OB} - \overrightarrow{OA}||^2 = ||\mathbf{v} - \mathbf{u}||^2.$$

Angle between Planes

The plane through $\mathbf{a} \in \mathbb{R}^3$ with normal direction $\mathbf{n} \in \mathbb{R}^3$ is $\{\mathbf{v} \in \mathbb{R}^3 : \mathbf{n} \cdot \mathbf{v} = \mathbf{n} \cdot \mathbf{a}\}$.

Problem 1.20

What is the angle θ between the planes

$$\{\mathbf{v} \in \mathbb{R}^3 : (-\mathbf{i} + \mathbf{j} + 2\mathbf{k}) \cdot \mathbf{v} = 1\} \text{ and } \{\mathbf{v} \in \mathbb{R}^3 : (-\mathbf{i} + \mathbf{k}) \cdot \mathbf{v} = -1\}?$$

(A) $\pi/6$ (B) $\pi/4$ (C) $\pi/3$ (D) $5\pi/6$

A line always makes an angle between 0 and $\pi/2$ (a right angle) with a plane.

Lemma 1.21

Let Π be a plane with normal $\mathbf{n}$. Let ℓ be a line with direction $\mathbf{c}$ meeting Π at a unique point. The angle θ between Π and ℓ satisfies

$$\sin\theta = |\widehat{\mathbf{n}} \cdot \widehat{\mathbf{c}}|$$

where $\widehat{\mathbf{n}} = \mathbf{n}/||\mathbf{n}||$ and $\widehat{\mathbf{c}} = \mathbf{c}/||\mathbf{c}||$.

Angle between Planes

The plane through $\mathbf{a} \in \mathbb{R}^3$ with normal direction $\mathbf{n} \in \mathbb{R}^3$ is $\{\mathbf{v} \in \mathbb{R}^3 : \mathbf{n} \cdot \mathbf{v} = \mathbf{n} \cdot \mathbf{a}\}$.

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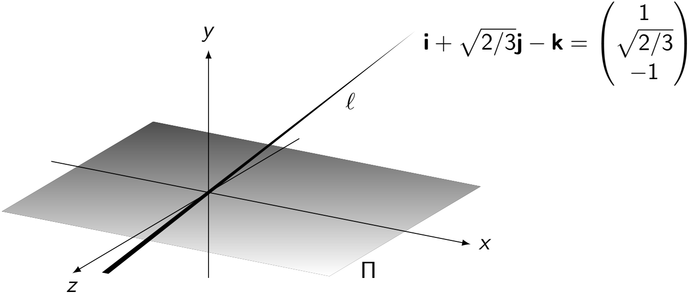
Lines and Planes

Problem 1.22

Let Π be the xz-plane, so $\Pi = \{x\mathbf{i} + z\mathbf{k} : x, z \in \mathbb{R}\}$. Let ℓ be the line passing through $\mathbf{0}$ and $\mathbf{i} + \sqrt{2/3}\mathbf{j} - \mathbf{k}$. What are the minimum and maximum angles between ℓ and a line in Π passing through $\mathbf{0}$?



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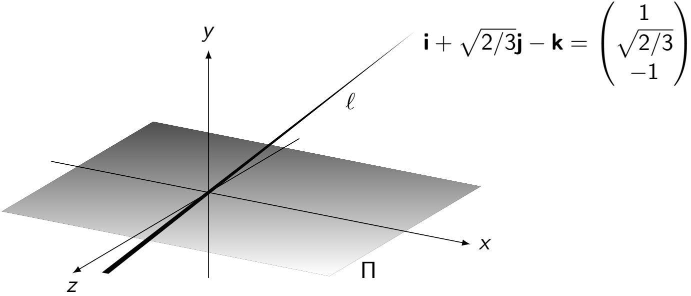

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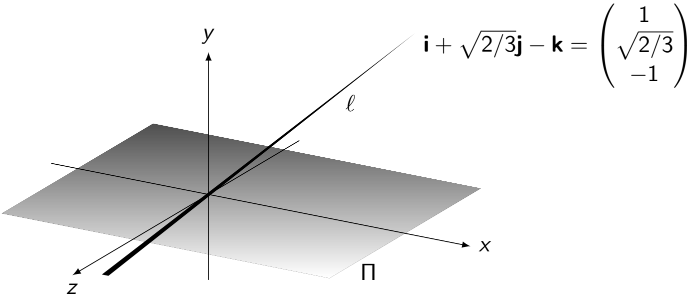

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Where does Π meet the line through $(-1, -1, 0)$ and $(0, 1, 1)$?

(A) $(1, 0, -1)$ (B) $(-\frac{1}{2}, 0, -\frac{1}{2})$ (C) $(\frac{1}{2}, 0, \frac{1}{2})$ (D) $(-\frac{1}{2}, 0, \frac{1}{2})$

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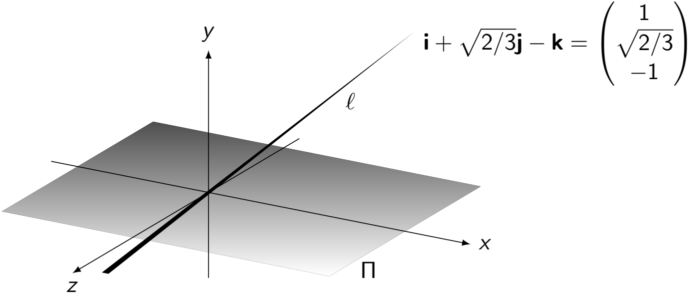

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Administration

- Please take the last installment of the Part A handout (pages 11–12)
- Please take Problem Sheet 2
- Please leave your answers to Sheet 1 at the end at the front
- **Please help to return clickers at end**

§2 The Vector Product

To solve Problem 1.22 we needed a vector orthogonal to the xz-plane. In this case the normal vector $\mathbf{j}$ was obvious.

Problem 2.1

Let Π be the plane containing $\mathbf{i}$, $\mathbf{2i} + \mathbf{j}$ and $4\mathbf{i} + 2\mathbf{j} + \mathbf{k}$. Find a normal vector to Π.

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Let Π be the plane containing $\mathbf{i}$, $\mathbf{2i} + \mathbf{j}$ and $4\mathbf{i} + 2\mathbf{j} + \mathbf{k}$. Find a normal vector to Π.

By Lemma 1.17(c), $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is orthogonal to both $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\begin{pmatrix} r \\ s \\ t \end{pmatrix}$ if and only if

$$ax + by + cz = 0$$
$$rx + sy + tz = 0.$$

Multiply the first equation by t, the second by c and subtract to get

$$(at - cr)x + (bt - cs)y = 0.$$

This suggests we might take $x = bt - cs$ and $y = cr - at$. Substituting in we find that both equations hold when $z = as - br$.

Definition of Vector Product

Definition 2.2

The *vector product* of vectors $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\begin{pmatrix} r \\ s \\ t \end{pmatrix}$ is defined by

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} r \\ s \\ t \end{pmatrix} = \begin{pmatrix} bt - cs \\ cr - at \\ as - br \end{pmatrix}.$$

If you remember the top entry is $bt - cs$ you can obtain the others using the cyclic permutations

$$a \mapsto b \mapsto c \mapsto a \quad \text{and} \quad r \mapsto s \mapsto t \mapsto r.$$

Properties of the Vector Product

Exercise 2.3

(a) Show that $\mathbf{v} \times \mathbf{v} = 0$ for all $\mathbf{v} \in \mathbb{R}^3$.

(b) Show that $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$.

(c) Show that $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$ and $\mathbf{k} \times \mathbf{i} = \mathbf{j}$. Hence $(\mathbf{i} + \mathbf{j}) \times (\mathbf{i} - \mathbf{k})$ equals

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Note that in each case, if $\mathbf{u} \times \mathbf{v} = \mathbf{w}$ then the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ form a right-handed system with (as expected) $\mathbf{w}$ orthogonal to $\mathbf{u}$ and $\mathbf{v}$. Changing $\mathbf{u}$ and $\mathbf{v}$ by a small displacement does not change the orientation of the system, so the system is always right-handed.

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Quiz: True or false: $(\mathbf{i} \times \mathbf{j}) \times \mathbf{j} = \mathbf{i} \times (\mathbf{j} \times \mathbf{j})$?

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Science Festival Saturday 5th March

Volunteers needed! Help enthusiastic children and adults discover our puzzles, games, Enigma machines and other exhibits.

Why volunteer?

- £68 for the day
- 10 passport points
- Free lunch
- Free T-shirt
- It's fun and rewarding

Please contact Dr Mark Wildon: `mark.wildon@rhul.ac.uk`

Length of $\mathbf{u} \times \mathbf{v}$

Quiz: Suppose that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal non-zero vectors.

- What is $(\mathbf{u} \times \mathbf{v}) \times \mathbf{u}$?

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  (A) 4 (B) 6 (C) 7 (D) 8

Theorem 2.4

Let $\mathbf{u}$ and $\mathbf{v}$ be non-zero vectors. Let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$. Then

$$||\mathbf{u} \times \mathbf{v}|| = ||\mathbf{u}|| \, ||\mathbf{v}|| \sin \theta.$$

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Area of Triangles

The identity

$$||\mathbf{u} \times \mathbf{v}||^2 = ||\mathbf{u}||^2||\mathbf{v}||^2 - (\mathbf{u} \cdot \mathbf{v})^2$$

seen in the proof of Theorem 2.4 is useful and of independent interest.

Problem 2.5

Let $B = (1, 8, 2)$ and $C = (8, 2, 1)$. Let $\overrightarrow{OD} = \overrightarrow{OB} + \overrightarrow{OC}$.

(a) The area of the parallelogram $OBDC$ is

(A) 4085 (B) 69 (C) $\sqrt{4085}$ (D) something else

(b) What is the area of the triangle OBC?

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Suppose that ABC is a triangle with sides a, b, c, angles α, β, γ. Using Theorem 2.4 and the argument for (b), its area is

$$\tfrac{1}{2}||\overrightarrow{AB}||\,||\overrightarrow{AC}||\sin\alpha = \tfrac{1}{2}bc\sin\alpha.$$

Repeating this argument with the other sides gives $\frac{1}{2}bc\sin\alpha = \frac{1}{2}ca\sin\beta = \frac{1}{2}ab\sin\gamma$. Hence get sine rule.

Feedback on Sheet 1

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- Please take the feedback sheet. Model answers are on Moodle.
- Please see the lecturer if you want to discuss any of the questions.

Quiz:

- Let Π be the plane through $A = (1,0,0)$, $B = (0,0,1)$ and $C = (0,0,1)$. True or false: $\overrightarrow{AB} = -\mathbf{i} + \mathbf{j} \in \Pi$?

  (A) False (B) True

- True or false: if ℓ is a line with direction $\mathbf{i}$ then $\mathbf{i} \in \ell$?

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- Let ℓ be a line with direction $\mathbf{i}$. True or false: the angle θ between ℓ and Π satisfies $\cos\theta = \frac{1}{\sqrt{3}}$?

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  (A) False (B) True

Using the Vector Product to find a Normal

Example 2.6

Let Π be the plane through $A = (1, 2, 3)$, $B = (3, 1, 2)$ and $C = (2, 3, 1)$. Then

$$\Pi = \{\overrightarrow{OA} + \lambda\overrightarrow{AB} + \mu\overrightarrow{AC} : \lambda, \mu \in \mathbb{R}\}.$$

A normal vector to Pi is

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}.$$

We can scale the normal vector as we like, so take $\mathbf{n} = \mathbf{i} + \mathbf{j} + \mathbf{k}$. Since

$$\mathbf{n} \cdot \overrightarrow{OA} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 6$$

we have $\Pi = \{\mathbf{v} \in \mathbb{R}^3 : (\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot \mathbf{v} = 6\}$.

Intersection of Two Planes

Problem 2.7

Let Π be the plane through $\mathbf{0}$ with normal $\mathbf{i}+\mathbf{j}$. Let Π' be the plane through $\mathbf{j}$ with normal $\mathbf{i}+2\mathbf{j}+\mathbf{k}$. Find $\mathbf{a}$ and $\mathbf{c} \in \mathbb{R}^3$ such that

$$\ell = \{\mathbf{a} + \lambda\mathbf{c} : \lambda \in \mathbb{R}\}.$$

Shortest Distances and Projections

Problem 2.8

Let $A = (2, 3, 1)$. What is the shortest distance between A and a point P on the plane Π through $\mathbf{0}$ with normal direction $(1, 1, 1)$?

The solution using Lemma 1.17(d) needed that $\mathbf{0} \in \Pi$. One can always translate the plane and point to reduce to this case: see Question 4(b) on Sheet 2.

Quiz: True or false?

(a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$,

(A) False (B) True

(b) $\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{u}$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$,

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(c) There exist distinct $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ such that $\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{u}$.

(A) False (B) True

(d) $(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = \mathbf{0}$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$

(A) False (B) True

Shortest Distances and Projections

Problem 2.8

Let $A = (2,3,1)$. What is the shortest distance between A and a point P on the plane Π through $\mathbf{0}$ with normal direction $(1,1,1)$?

The solution using Lemma 1.17(d) needed that $\mathbf{0} \in \Pi$. One can always translate the plane and point to reduce to this case: see Question 4(b) on Sheet 2.

Quiz: True or false?

(a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ for all $\mathbf{u}$, $\mathbf{v} \in \mathbb{R}^3$,

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Shortest Distances between Two Lines

Problem 2.9

Let ℓ be the line through $-\mathbf{j}$ and $\mathbf{i} + \mathbf{k}$.

(a) True or false: ℓ meet the line through $\mathbf{i} + 2\mathbf{k}$ with direction $\mathbf{i} + \mathbf{j}$? [**Note this is a change from the printed notes.**]

(A) False (B) True

(b) True or false: ℓ meet the line ℓ' through $\mathbf{0}$ with direction $\mathbf{i} + \mathbf{j}$?

(A) False (B) True

(c) What is the shortest distance between ℓ and ℓ'?

Lemma 2.10

Let ℓ be the line through $\mathbf{a}$ with direction $\mathbf{c}$. Let ℓ' be the line through $\mathbf{a}'$ with direction $\mathbf{c}'$. The shortest distance between ℓ and ℓ' is

$$\left| \frac{\mathbf{c} \times \mathbf{c}'}{||\mathbf{c} \times \mathbf{c}'||} \cdot (\mathbf{a}' - \mathbf{a}) \right|.$$

Shortest Distances between Two Lines

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Shortest Distances between Two Lines

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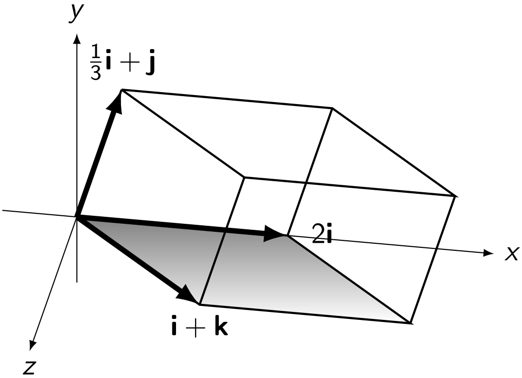
$$\left|\frac{\mathbf{c}\times\mathbf{c}'}{||\mathbf{c}\times\mathbf{c}'||}\cdot(\mathbf{a}'-\mathbf{a})\right|.$$

Volume of a Parallelepiped

Problem 2.11

What is the volume of the parallelepiped formed by the vectors $\mathbf{i}+\mathbf{k}, 2\mathbf{i}, \frac{1}{3}\mathbf{i}+\mathbf{j}$?



The Scalar Triple Product

Quiz: let $\overrightarrow{PC}$ be parallel to the vector $\widehat{\mathbf{n}}$ where $||\widehat{\mathbf{n}}|| = 1$. Let OPC be a right-angled triangle with hypotenuse OC. Write the length of the side PC as a dot product.

Theorem 2.12

Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c} \in \mathbb{R}^3$. *The volume of the parallelepiped formed by* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c} \in \mathbb{R}^3$ *is* $|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|$.

Exercise 2.13

Deduce from Theorem 2.12 that

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b}.$$

What is the geometric interpretation of the sign of $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$?

The scalar $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ is called the *scalar triple product* of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$.

Electoral Reform (Misuses of Mathematics)

The Government has proposed changing the rules for strike ballots so that 50% of the electorate must vote for a strike to be held. Of course a majority of those voting must favour a strike.

Suppose there are 1000 union members. What is the outcome in the following cases:

- 300 vote for strike, 10 vote against?
- 300 vote for strike, 290 vote against?

Any comments?

§3 Matrices and vectors

We now generalize the definition of 'vector' to mean an element of $\mathbb{R}^n$, for any $n \in \mathbb{N}$. If $\mathbf{u} \in \mathbb{R}^n$ we write u_i for the ith coordinate of $\mathbf{u}$. As usual vectors are written in column form.

Definition 3.1

The *length* of $\mathbf{u} \in \mathbb{R}^n$ is defined by

$$||\mathbf{u}|| = \sqrt{u_1^2 + \cdots + u_n^2}.$$

The *dot product* of vectors $\mathbf{u}$, $\mathbf{v} \in \mathbb{R}^n$ is defined by

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

The *sum* of vectors $\mathbf{u}$ and $\mathbf{v} \in \mathbb{R}^n$ and the *scalar multiplication* of $\mathbf{v} \in \mathbb{R}^n$ by $\alpha \in \mathbb{R}$ are defined by the obvious generalization of Definition 1.7.

Hypercubes

All the properties of the dot product proved in Lemma 1.17 hold in any dimension. In particular $||\mathbf{u}||^2 = \mathbf{u} \cdot \mathbf{u}$ for any vector $\mathbf{u}$, and if θ is the angle between $\mathbf{u}$ and $\mathbf{v}$ then $||\mathbf{u}||||\mathbf{v}|| \cos\theta = \mathbf{u} \cdot \mathbf{v}$.

Problem 3.2

Let $S = \{(x, y, z, w) : x, y, z, w \in \{0, 1\}\}$ be the set of vertices of the hypercube.

(a) What is the angle between $(1, 1, 0, 0)$ and $(0, 1, 1, 0)$?

(b) What is the angle between $(1, 1, 0, 0)$ and $(0, 0, 1, 1)$?

(c) What is the set of distances between distinct points in S?

(A) $\{1, \sqrt{2}, \sqrt{3}, 2\}$ (B) $\varnothing$ (C) $\{1, \sqrt{3}, 2\}$ (D) $\{0, 1, \sqrt{2}, \sqrt{3}, 2\}$

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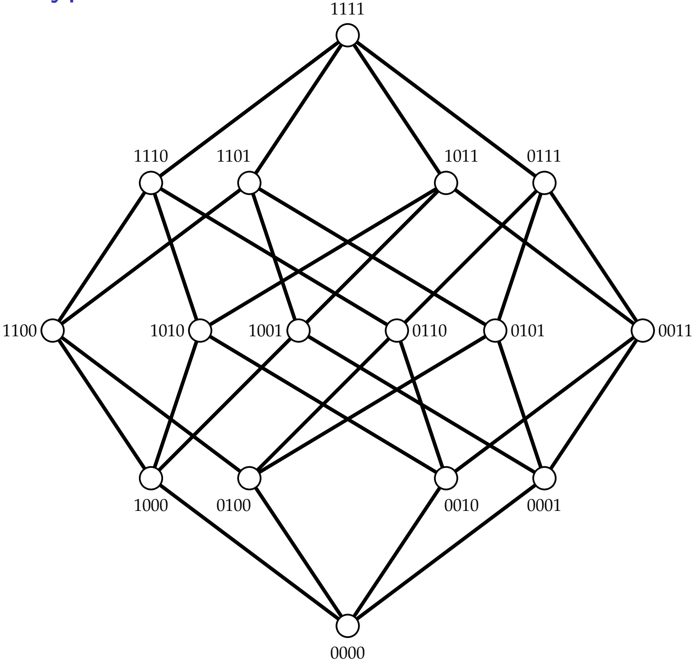
The Hypercube



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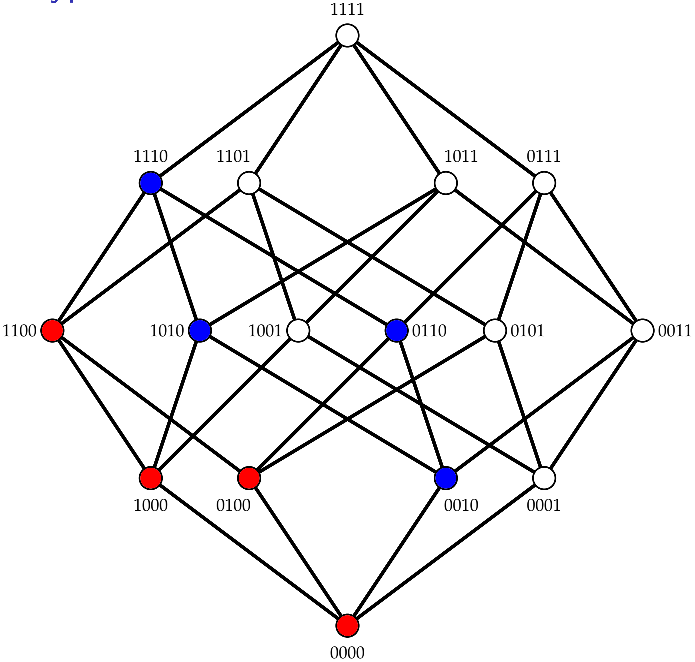

The Hypersphere

What proportion of the 10-dimensional hypercube of side length 1 is occupied by a hypersphere of diameter 1?

(A) 0.25%   (B) 5%   (C) 15.75%   (D) 25.8%

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Feedback on Sheet 2

- Please collect your work: A–G in yellow folder, H–R in green folder, S–Z in red folder.
- Please take the feedback sheet. Model answers are on Moodle.
- Please see the lecturer if you want to discuss any of the questions.

Quiz: Let $\mathbf{c}$ and $\mathbf{c}' \in \mathbb{R}^3$ be non-zero, and let $\mathbf{a}, \mathbf{a}' \in \mathbb{R}$. Let $\ell = \{\mathbf{a} + \lambda \mathbf{c} : \lambda \in \mathbb{R}\}$ and let $\ell' = \{\mathbf{a}' + \lambda \mathbf{c}' : \lambda \in \mathbb{R}\}$. Consider the propositions

P : the lines ℓ and ℓ' meet

Q : there exists $\lambda \in \mathbb{R}$ such that $\mathbf{a} + \lambda \mathbf{c} = \mathbf{a}' + \lambda \mathbf{c}'$.

True or false: $P \implies Q$?

(A) False (B) True

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(A) False (B) True

True or false: $Q \implies P$?

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Matrices as Containers

Definition 3.3

Let m, $n \in \mathbb{N}$. An $m \times n$ *matrix* is an array with m rows and n columns of real numbers.

Matrices are useful simply as containers.

Problem 3.4

The matrix below records the stock prices of British Land, Glencore and Whitbread in the first weeks of 2012 and 2016, all in pence.

$$\begin{array}{c} 2012 \\ 2016 \end{array} \begin{pmatrix} 360 & 395 & 1589 \\ 741 & 77 & 4130 \end{pmatrix}$$

Suppose an investor has a portfolio consisting of $(5, 2, 1)$ units of each stock. What is her portfolio worth in 2012? In 2016?

Linear Maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ and 2×2 Matrices

Definition 3.5

A *linear map* from $\mathbb{R}^2$ to $\mathbb{R}^2$ is a function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

where a, b, c, d are some fixed real numbers.

Problem 3.6

(a) Take $a = 0$, $b = 1$, $c = 1$, $d = 0$. We obtain the linear map

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ x \end{pmatrix}.$$

What, geometrically, does this linear map do to a vector $\mathbf{v} \in \mathbb{R}^2$?

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Problem 3.6

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined so that $f\begin{pmatrix} x \\ y \end{pmatrix}$ records the values, in 2012 and 2016, of a portfolio consisting of x Glencore shares and y Whitbread shares. Show that f is a linear map. What are the coefficients a, b, c, d?

Matrices Represent Linear Maps

The coefficients a, b, c, d in a linear map f can be recorded in a 2×2 matrix. We define the *product* of a 2×2 matrix and vector in $\mathbb{R}^2$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

By definition of the product, if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $\begin{pmatrix} x \\ y \end{pmatrix} \overset{f}{\mapsto} A \begin{pmatrix} x \\ y \end{pmatrix}$.

Example 3.7

The matrix $\begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}$ represents the linear map

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2x + y \\ 3x + y \end{pmatrix}$$

Notice that the entries a, b in the first row of the matrix appear in the first row of the result; c and d from the second row appear in the second row of the result.

Rotations

Problem 3.8

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined so that $f(\mathbf{v})$ is $\mathbf{v}$ rotated by $\pi/4$. For example,

$$f \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Find a formula for $f \begin{pmatrix} x \\ y \end{pmatrix}$ and show that f is a linear map.

The matrix representing f is $\frac{\sqrt{2}}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

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Quiz: What matrix represents the linear map 'double the x-coordinate'?

(A) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ (B) $\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$ (C) $\begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$ (D) $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

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Quiz: The composition of linear maps is linear. What matrix represents the linear map 'rotate by $\pi/4$, then double the x-coordinate'?

(A) $\frac{\sqrt{2}}{2}\begin{pmatrix}2 & 1\\2 & -1\end{pmatrix}$ (B) $\frac{\sqrt{2}}{2}\begin{pmatrix}1 & -2\\1 & 2\end{pmatrix}$

(C) $\frac{\sqrt{2}}{2}\begin{pmatrix}2 & -2\\1 & 1\end{pmatrix}$ (D) $\frac{\sqrt{2}}{2}\begin{pmatrix}2 & -1\\2 & 1\end{pmatrix}$

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Matrix Multiplication

Definition 3.9

We define the *product* of 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} r & s \\ t & u \end{pmatrix}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} r & s \\ t & u \end{pmatrix} = \begin{pmatrix} ar + bt & as + bu \\ cr + dt & cs + du \end{pmatrix}.$$

Example 3.10

Let $\theta, \phi \in \mathbb{R}$. The matrix representing rotation by $\theta + \phi$ is

$$M_{\theta+\phi} = \begin{pmatrix} \cos(\theta + \phi) & -\sin(\theta + \phi) \\ \sin(\theta + \phi) & \cos(\theta + \phi) \end{pmatrix}.$$

We can rotate by $\theta + \phi$ by first rotating by ϕ, then by θ. The corresponding matrices are

$$M_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{and} \quad M_\phi = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

Check that $M_{\theta+\phi} = M_\theta M_\phi$.

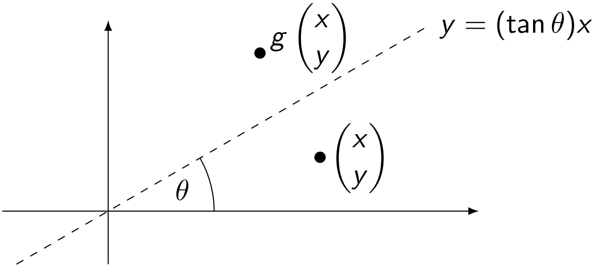
Products of Matrices and Composition of Linear Maps

Lemma 3.11

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}^2$ be linear maps represented by the matrices A and B, respectively. Then $gf : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map, represented by the matrix BA.

Problem 3.12

Let $\theta \in \mathbb{R}$. Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be defined so that $g(\mathbf{v})$ is $\mathbf{v}$ reflected in the line $y = (\tan\theta)x$. Show that g is a linear map and find the matrix representing g.



Inverses of 2×2 Matrices

Recall that if S and T are sets and $f : S \to T$ is a bijective function then the *inverse* of f is the function $f^{-1} : T \to S$ defined by $f^{-1}(t) = s \iff f(s) = t$.

True or false? The matrix $\begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix}$ represents an invertible linear map.

(A) False (B) True

Inverses of 2×2 Matrices

Recall that if S and T are sets and $f : S \to T$ is a bijective function then the *inverse* of f is the function $f^{-1} : T \to S$ defined by $f^{-1}(t) = s \iff f(s) = t$.

True or false? The matrix $\begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix}$ represents an invertible linear map.

(A) False (B) True

Let $M = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. The linear map represented by M is invertible.

(A) False (B) True

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Lemma 3.13

The linear map $f : \mathbb{R}^2 \to \mathbb{R}^2$ represented by the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible if and only if $ad - bc \neq 0$. In this case the inverse $f^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear map represented by $\frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Stretches, Rotations and Reflections

Problem 3.14

Let $\beta > 0$ be a real number. Let f, g, and $h : \mathbb{R}^2 \to \mathbb{R}^2$ be rotation by $\pi/4$, a stretch of β in the x-direction, and reflection in the x-axis.

(a) What is the matrix representing hgf?

(A) $\frac{\sqrt{2}}{2}\begin{pmatrix} \beta & \beta \\ -1 & -1 \end{pmatrix}$ (B) $\frac{\sqrt{2}}{2}\begin{pmatrix} \beta & -\beta \\ -1 & 1 \end{pmatrix}$

(C) $\frac{\sqrt{2}}{2}\begin{pmatrix} \beta & -\beta \\ -1 & -1 \end{pmatrix}$ (D) $\frac{\sqrt{2}}{2}\begin{pmatrix} \beta & -1 \\ -\beta & -1 \end{pmatrix}$

(b) What is the image of the square with vertices at $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$ under hgf?

(c) How does the area change?

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Determinants of 2×2 Matrices

Lemma 3.15

The image of the unit square under the linear transformation represented by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *has area* $|ad - bc|$*. Moreover,* $ad - bc > 0$ *if and only if the vectors*

$$\begin{pmatrix} a \\ b \\ 0 \end{pmatrix}, \begin{pmatrix} c \\ d \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

form a right-handed system.

We define the *determinant* of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ by $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$.

By Lemma 3.13, a 2×2 matrix M is invertible if and only if $\det M \neq 0$.

Feedback on Sheet 3

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- Please see the lecturer if you want to discuss any of the questions.

Quiz: Let $\mathbf{a}$, $\mathbf{c} \in \mathbb{R}^3$. Only one of the equations below makes correct use of mathematical notation. Which one?

(A) $\mathbf{a} \cdot \mathbf{c} = \mathbf{a}$ (B) $\mathbf{a} \times \mathbf{c} = \mathbf{ac}$ (C) $\mathbf{a} \times \mathbf{c} = \mathbf{c}$ (D) $\ell = \{\lambda \mathbf{c}\}$

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Quiz and Application of the Determinant

(a) True or false: if $M = 2\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $\det M = 2(ad - bc)$.

(A) False (B) True

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 \\ y^2 \end{pmatrix}$.

(b) True or false: f is linear.

(A) False (B) True

Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 1 \end{pmatrix}$.

(c) True or false: g is linear.

(A) False (B) True

The determinant gives a convenient criterion for a matrix to send a non-zero vector to $\mathbf{0}$.

Lemma 3.16

Let M be a 2×2-matrix. There is a non-zero vector $\mathbf{v}$ such that $M\mathbf{v} = \mathbf{0}$ if and only if $\det M = \mathbf{0}$.

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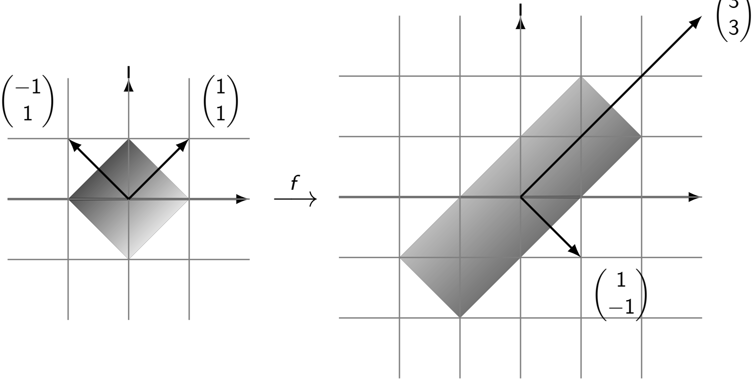
Eigenvectors of 2×2 Matrices: Motivation

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map defined by

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2y \\ 2x + y \end{pmatrix}$$

for $x, y \in \mathbb{R}$. Observe that

$$f\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad f\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = -\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$



Eigenvectors and Eigenvalues of 2×2 matrices

Definition 3.17

Let A be a 2×2-matrix. We say that a non-zero vector $\mathbf{v} \in \mathbb{R}^2$ is an *eigenvector* of A with *eigenvalue* λ if $A\mathbf{v} = \lambda\mathbf{v}$.

Exercise: why do we require $\mathbf{v}$ to be non-zero in Definition 3.17?

Let I denote the 2×2 *identity matrix* $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. We have

$$\begin{aligned} \mathbf{v} \text{ is an eigenvector of } A \text{ with eigenvalue } \lambda &\iff A\mathbf{v} = \lambda\mathbf{v} \\ &\iff (\lambda I - A)\mathbf{v} = 0 \end{aligned}$$

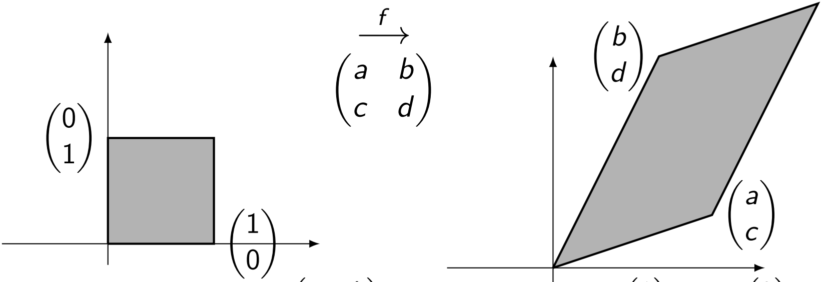
Hence, by Lemma 3.16,

$$A \text{ has an eigenvector with eigenvalue } \lambda \iff \det(\lambda I - A) = 0.$$

Correction

- Please take Problem Sheet 5 and pages 19–22 of the printed notes.
- Please hand in answers to Sheet 4.
- **No 182 lecture next Monday 15th.** No office hour on Monday or Wednesday. Workshops as usual.

In the diagram I drew for the first quiz question in the previous lecture, some vectors were mislabelled. The correct labels are below. To correct your notes, swap b and c.



Columns of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ are the images of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Finding Eigenvectors and Eigenvalues

Example′ 3.18

(Another very similar example is given in full in the printed notes.)

Let $A = \begin{pmatrix} -3 & 6 \\ -4 & 7 \end{pmatrix}$. Then

$$\begin{aligned} \det(\lambda I - A) &= \det \begin{pmatrix} \lambda + 3 & 6 \\ -4 & \lambda - 7 \end{pmatrix} \\ &= (\lambda + 3)(\lambda - 7) + 24 \\ &= \lambda^2 - 4\lambda + 3 \\ &= (\lambda - 1)(\lambda - 3). \end{aligned}$$

Hence A has eigenvalues 1 and 3.

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Exercise 3.19. True or false: $\mathbf{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ is the unique eigenvector of A with eigenvalue 1?

(A) False (B) True

Finding Eigenvectors and Eigenvalues

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Exercise 3.19. True or false: $\mathbf{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ is the unique eigenvector of A with eigenvalue 1?

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Diagonalization

Let $A = \begin{pmatrix} -3 & 6 \\ -4 & 7 \end{pmatrix}$ be as in Example' 3.18 and let

$$P = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$$

be the matrix whose columns are the eigenvectors $\mathbf{u}$ and $\mathbf{v}$ of A.

Problem 3.20

Compute A^{2016}.

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Problem 3.20
Compute A^{2016}.

Quiz: A 2×2 matrix A has eigenvalues 3 and -2. Which of the following is a possible formula for the entry in the top left of A?

(A) $(-2)^n + 3^n$ (B) $2^n + 3^n$ (C) 1 (D) $\sqrt{2}^n$

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Proportional Vectors

Not all matrices can be diagonalized. A necessary and sufficient condition is that the matrix has two non-proportional eigenvectors, as defined below. (If time permits, this will be proved in Part E.)

Definition 3.21

Let $n \in \mathbb{N}$. Vectors $\mathbf{u}$ and $\mathbf{v} \in \mathbb{R}^n$ are *proportional* if there exist α and $\beta \in \mathbb{R}$, not both zero, such that $\alpha\mathbf{u} = \beta\mathbf{v}$.

Quiz: True or false:

(a) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ are proportional.

(A) False (B) True

(b) $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ are proportional.

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Two Distinct Eigenvalues Implies Diagonalizable: Preliminary Lemma

Lemma 3.22

Let A be a 2×2 matrix. Suppose that A has distinct real eigenvalues λ and μ with eigenvectors $\mathbf{u}$ and $\mathbf{v}$, respectively. Then $\mathbf{u}$ and $\mathbf{v}$ are not proportional.

Quiz:

(a) Rotation by $\pi/4$ has an eigenvector.

(A) False (B) True

(b) Rotation by π has an eigenvector.

(A) False (B) True

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Two Distinct Eigenvalues Implies Diagonalizable

By definition, $\begin{pmatrix} a \\ c \end{pmatrix}$, $\begin{pmatrix} b \\ d \end{pmatrix} \in \mathbb{R}^2$ are proportional if and only if there exist α, $\beta \in \mathbb{R}$, not both zero, such that

$$\alpha \begin{pmatrix} a \\ c \end{pmatrix} = \beta \begin{pmatrix} b \\ d \end{pmatrix}.$$

Quiz: Let $\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}$ be non-zero. Which logical connective can correctly replace $\star$ below?

$$\begin{pmatrix} a \\ c \end{pmatrix} \text{ and } \begin{pmatrix} b \\ d \end{pmatrix} \text{ are proportional} \quad \star \quad \frac{c}{a} = \frac{d}{b}.$$

(A) $\Longrightarrow$ (B) $\Longleftarrow$ (C) $\Longleftrightarrow$ (D) $\vee$

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$$\alpha \begin{pmatrix} a \\ c \end{pmatrix} = \beta \begin{pmatrix} b \\ d \end{pmatrix}.$$

Lemma 3.23

If $\mathbf{u}$ *and* $\mathbf{v} \in \mathbb{R}^2$ *are not proportional then the matrix* $(\mathbf{u} \ \ \mathbf{v})$ *with columns* $\mathbf{u}$ *and* $\mathbf{v}$ *is invertible.*

Proposition 3.24

Let A *be a* 2×2 *matrix. Suppose that* A *has distinct eigenvalues* λ *and* μ *with eigenvectors* $\mathbf{u}$ *and* $\mathbf{v}$*, respectively. Let* $P = (\mathbf{u} \ \ \mathbf{v})$ *be the matrix formed by these eigenvectors. Then* P *is invertible and*

$$A = P \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} P^{-1}.$$

A Very Simple Weather Model

If today is sunny then tomorrow is sunny with probability 3/4. If today is rainy then tomorrow is equally likely to be sunny and rainy.

Example 3.25

Suppose Monday is sunny. Then Tuesday is sunny with probability 3/4. The probability that Wednesday is sunny is

(A) 5/8 (B) 11/16 (C) 3/4 (D) 13/16

A Very Simple Weather Model

If today is sunny then tomorrow is sunny with probability $3/4$. If today is rainy then tomorrow is equally likely to be sunny and rainy.

Example 3.25

Suppose Monday is sunny. Then Tuesday is sunny with probability $3/4$. The probability that Wednesday is sunny is

(A) $5/8$ (B) $11/16$ (C) $3/4$ (D) $13/16$

More generally,

$$\mathbf{P}[\text{day } n+1 \text{ is sunny}] = \mathbf{P}[\text{day } n \text{ is sunny}]\tfrac{3}{4} + \mathbf{P}[\text{day } n \text{ is rainy}]\tfrac{1}{2}.$$

So, setting $p_n = \mathbf{P}[\text{day } n \text{ is sunny}]$, we get $p_{n+1} = \frac{3}{4}p_n + \frac{1}{2}(1 - p_n)$.

Exercise 3.26

Show that $1 - p_{n+1} = \frac{1}{4}p_n + \frac{1}{2}(1 - p_n)$.

Feedback on Sheet 4

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Quiz: Let $f : X \to Y$, $g : Y \to Z$ be functions.

- True or false? If f and g are invertible then $(gf)^{-1} = g^{-1}f^{-1}$.
  (A) False (B) True
- True or false? If gf is invertible then f is invertible.
  (A) False (B) True
- True or false? If gf is invertible then f is injective.
  (A) False (B) True

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§4 Matrices and linear maps in higher dimensions

Matrices were defined in Definition 3.3. If A is an $m \times n$ matrix we write A_{ij} for the entry of A in row i and column j. For example if A is the 2×3 matrix

$$\begin{pmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$$

then $A_{23} = 5$ and $A_{ij} = i + j$ for all $i \in \{1, 2\}$ and $j \in \{1, 2, 3\}$.

Definition 4.1

Let m, $n \in \mathbb{N}$ and let A be an $m \times n$ matrix. Let $\mathbf{v} \in \mathbb{R}^n$. We define the *product* of A and $\mathbf{v}$ by

$$A \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} A_{11}v_1 + A_{12}v_2 + \cdots + A_{1n}v_n \\ A_{21}v_1 + A_{22}v_2 + \cdots + A_{2n}v_n \\ \vdots \\ A_{m1}v_1 + A_{m2}v_2 + \cdots + A_{mn}v_n \end{pmatrix}$$

Equivalently, $(A\mathbf{v})_i = \sum_{j=1}^{n} A_{ij} v_j$ for each $i \in \{1, 2, \ldots, m\}$.

Definition of Linear Maps

Exercise 4.2

Check that Definition 4.1 generalizes the definition on page 14 of the product of a 2×2 matrix and a vector in $\mathbb{R}^2$.

We also generalize Definition 3.5.

Definition 4.3

Let $m, n \in \mathbb{N}$. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is a *linear map* if there is an $m \times n$ matrix A such that $f(\mathbf{v}) = A\mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^n$.

Note that $\mathbb{R}^n$ is the domain of f and $\mathbb{R}^m$ is the codomain of f.

Examples of Linear Maps

If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map represented by the matrix A then column j of A is the image of the vector in $\mathbb{R}^n$ with 1 in position j and zero in all other positions.

Example 4.4

Let M be the stock price matrix in Problem 3.4. Suppose an investor has x, y and z shares in each company. The function $f : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$f \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 360 & 395 & 1589 \\ 741 & 77 & 4130 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 360x + 395y + 1589z \\ 741x + 77y + 4130z \end{pmatrix}$$

is linear; the top component is the value of the portfolio in January 2012, and the bottom component is the value of the portfolio in January 2016.

Note that the columns of the matrix are $f(\mathbf{i})$, $f(\mathbf{j})$ and $f(\mathbf{k})$.

Quiz on Matrix times Vector

After the great stock market crash of March 2016, the new prices of the three stocks BLND, GLEN, WHIT are in the third row of the matrix below.

$$\begin{pmatrix} 360 & 395 & 1589 \\ 741 & 77 & 4130 \\ 500 & 10 & 3000 \end{pmatrix}.$$

- What is a portfolio with 5 BLND, 100 GLEN and 1 WHIT stock worth?
  (A) 6000 (B) 6500 (C) 7000 (D) 7500
- A portfolio with 10 BLND and 1 WHIT is now worth £100. How many Glencore stocks does it have?
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Vector Product as a Linear Map

Example 4.5

Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $f(\mathbf{v}) = (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \times \mathbf{v}$. We will find the matrix A representing f.

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$$A = \begin{pmatrix} 0 & -3 & 2 \\ 3 & 0 & -1 \\ -2 & 1 & 0 \end{pmatrix}.$$

Note that the columns of A are $f(\mathbf{i})$, $f(\mathbf{j})$ and $f(\mathbf{k})$.

Vector Product as a Linear Map

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Now let $g : \mathbb{R}^3 \to \mathbb{R}$ be defined by $g(\mathbf{v}) = (\mathbf{i} - \mathbf{j}) \cdot \mathbf{v}$. Which of these is the matrix B representing g?

(A) $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ (B) $\begin{pmatrix} 1 & 1 & 0 \end{pmatrix}$ (C) $\begin{pmatrix} 0 & 1 & -1 \end{pmatrix}$ (D) $\begin{pmatrix} 1 & -1 & 0 \end{pmatrix}$

(a) What is the matrix representing the composition gf?

(b) What is the matrix representing the composition ff?

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Product of Matrices

Definition 4.6

If B is an $m \times n$ matrix and A is an $n \times p$ matrix then the *product* BA is the $m \times p$ matrix defined by

$$(BA)_{ij} = \sum_{k=1}^{n} B_{ik} A_{kj} \quad \text{for } 1 \leq i \leq m \text{ and } 1 \leq j \leq p.$$

Written out without the Sigma notation,

$$(BA)_{ij} = B_{i1}A_{1j} + B_{i2}A_{2j} + \cdots + B_{in}A_{nj}.$$

So to calculate $(BA)_{ij}$ go left-to-right along row i of B and top-to-bottom down column j of A, multiplying each pair of entries. Then take the sum.

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Alternatively, if column j of A is $\mathbf{v}_j$, so $A = (\mathbf{v}_1\ \mathbf{v}_2 \ldots \mathbf{v}_n)$, then $BA = (B\mathbf{v}_1\ B\mathbf{v}_2 \ldots B\mathbf{v}_n)$.

Composition of Linear Maps and Matrix Multiplication

Exercise 4.7

Continuing Example 4.5, check that the matrix found in (a) representing gf is BA and the matrix found in (b) representing ff is A^2.

Composition of Linear Maps and Matrix Multiplication

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Continuing Example 4.5, check that the matrix found in (a) representing gf is BA and the matrix found in (b) representing ff is A^2.

In each case matrix multiplication corresponds to composition of linear maps.

The general result is as follows. The proof is non-examinable and will probably be skipped; please see the printed notes (on Moodle this evening). The 2×2 case is Lemma 3.11.

Lemma 4.8

Let $f : \mathbb{R}^p \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ be linear maps, represented by the $n \times p$ matrix A and the $m \times n$ matrix B, respectively. Then $gf : \mathbb{R}^p \to \mathbb{R}^m$ is a linear map, represented by the matrix BA.

Properties of Matrix Multiplication

Proposition 4.9

(i) *If A, B, C are matrices then $C(BA) = (CB)A$, whenever either side is defined. (Associativity.)*

(ii) *If A is an $n \times p$ matrix and B and C are $m \times n$ matrices then $(B + C)A = BA + CA$. (Distributivity.)*

The set of all $n \times n$ matrices forms a ring, in the sense defined in 181 Number Systems. The zero element is the all-zeros matrix, and the one element is the *identity* matrix

$$I = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

Quiz: The equation $A^2 = I$ has

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Quiz: The ring of 2×2 matrices is commutative, i.e. $AB = BA$ for all 2×2 matrices A and B.

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Quiz: If A and B are square matrices and $AB = 0$ (meaning the all-zeros matrix) then $BA = 0$.

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Feedback on Sheet 5

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Quiz: The matrix A has $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ as eigenvectors with eigenvalues 2 and 1.

(a) The eigenvalues of $A - I$ are

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(b) The eigenvalues of A^{-1} are

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Revision of Matrix Multiplication

Quiz: Let $f : \mathbb{R} \to \mathbb{R}^3$ be the linear map $f(x) = \begin{pmatrix} x \\ 2x \\ 3x \end{pmatrix}$. Let $g : \mathbb{R}^3 \to \mathbb{R}$ be the linear map $g(\mathbf{v}) = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cdot \mathbf{v}$.

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Revision of Matrix Multiplication

Quiz: Let $f : \mathbb{R} \to \mathbb{R}^3$ be the linear map $f(x) = \begin{pmatrix} x \\ 2x \\ 3x \end{pmatrix}$. Let $g : \mathbb{R}^3 \to \mathbb{R}$ be the linear map $g(\mathbf{v}) = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cdot \mathbf{v}$.

(a) The matrix representing f is:

(A) $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ (B) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ (C) $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ (D) something else

(b) True or false? The matrix representing g is $\begin{pmatrix} -1 & 0 & 1 \end{pmatrix}$.

(A) False (B) True

(c) The matrix representing gf is

(A) (2) (B) (6) (C) $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ (D) $\begin{pmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -3 & 0 & 3 \end{pmatrix}$

(d) The matrix representing fg is

(A) (2) (B) (6) (C) $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ (D) $\begin{pmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -3 & 0 & 3 \end{pmatrix}$

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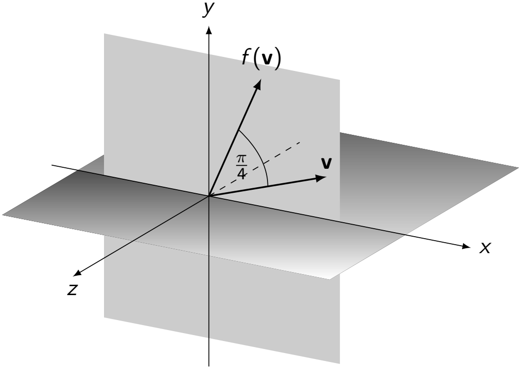
Eigenvectors and Eigenvalues

Definition 4.10

Let A be a $n \times n$ matrix. A non-zero vector $\mathbf{v} \in \mathbb{R}^n$ is an *eigenvector* of A with *eigenvalue* λ if $A\mathbf{v} = \lambda\mathbf{v}$.

Example 4.11

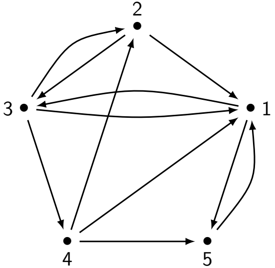
Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be rotation by $\pi/4$ with axis $\mathbf{k}$ and $g : \mathbb{R}^3 \to \mathbb{R}^3$ be rotation by $\pi/2$ with axis $\mathbf{j}$.



Google's Pagerank Algorithm

Example 4.12

The diagram below shows a tiny part of the World Wide Web. Each dot represents a website; an arrow from site A to site B means that site A links to site B.



Exercise 4.13

Simulate a surfer who clicks 10 links. Where do you finish?

(A) Site 1 (B) Site 2 (C) Site 3 (D) Site 4

Google's Pagerank Algorithm

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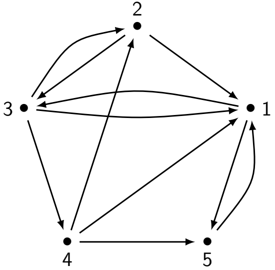

If $p(n)_i$ is the probability that we are at site i after n steps, then $p(2)_4 =$ and $p(3)_3 =$

(A) 1/3 (B) 1/6 (C) 1/12 (D) 0

(A) 1/2 (B) 5/12 (C) 1/3 (D) 1/4

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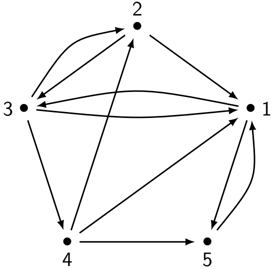

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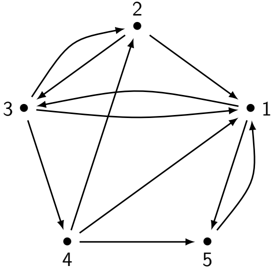

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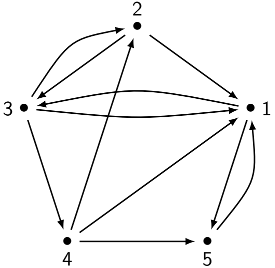

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Part C: Solving equations

§5 Row operations

Example 5.1

Suppose that $2x + y + 3z = 11$ and $2x + 4y + 6z = 20$. Then $x - y = ?$

(A) -1 (B) 0 (C) 1 (D) can't say

Example′ 5.2

$$\begin{pmatrix} 2 & 1 & 3 \\ 2 & 4 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{0} \tag{1}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{0} \tag{2}$$

Solve the system of your choice. (But please, some people, do (1).)

Part C: Solving equations

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Solve the system of your choice. (But please, some people, do (1).)

Elementary Row Operations

Fix m, $n \in \mathbb{N}$ and let A be an $m \times n$ matrix.

Definition 5.3

An elementary row operation (ERO) on an $m \times n$ matrix A is one of the following:

(a) Multiply a row of A by a non-zero scalar.

(b) Swap two rows of A.

(c) Add a multiple of one row of A to another row of A.

Quiz: True or false? A' can be obtained from A by a sequence of EROs where

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 2 & 4 & 6 \end{pmatrix}, \quad A' = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

(A) False (B) True

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Quiz: True or false? A' can be obtained from A by a sequence of EROs where

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad A' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

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(A) False (B) True

Matrices for EROs

Elementary row operations on an $m \times n$ matrix are performed by *left* multiplication by suitable $m \times m$ matrices. Importantly, these matrices are all invertible.

Lemma 5.4

Let A be an $m \times n$ matrix. Let $1 \leq k, \ell \leq m$ with $k \neq \ell$. Let I be the identity $m \times m$ matrix. Let $\alpha \in \mathbb{R}$, with $\alpha \neq 0$.

(a) *Let S be I with the 1 in row k replaced with α. Then SA is A with row k scaled by α.*

(b) *Let P be I with rows k and ℓ swapped. Then PA is A with rows k and ℓ swapped.*

(c) *Let Q be I with the entry in row k and column ℓ changed from 0 to α. Then QA is A with row k replaced with the sum of row k and α times row ℓ.*

Moreover, S, P and Q are invertible.

EROs Preserve Solution Sets

Corollary 5.5

Let A be an $m \times n$ matrix and let A' be obtained from A by a sequence of elementary row operations. Then

$$\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = 0\} = \{\mathbf{x} \in \mathbb{R}^n : A'\mathbf{x} = 0\}.$$

The more general result for $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} \in \mathbb{R}^n$ is on Problem Sheet 7. Hint for proof: adapt the proof of Corollary 5.5.

Echelon Form and Row-Reduced Echelon Form

Example 5.2 suggests what to aim for when doing row operations.

Definition 5.6

Let A be an $m \times n$ matrix. We say that A is in *echelon form* if both of the following conditions hold.

(i) All zero rows are at the bottom.

(ii) Suppose $i < m$. If row i of A is non-zero, and the leftmost non-zero entry of row i is in column j then all the non-zero entries of row $i+1$ of A are in columns $j+1, \ldots, n$.

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We say that A is in *row-reduced echelon form* if A is in echelon form and

(iii) If row i is non-zero and the lefmost non-zero entry is in column j, then $A_{ij} = 1$, and this 1 is the unique non-zero entry in column j.

Thus the matrix in Example 5.2(2) is in row-reduced echelon form.

Quiz and Example of Row Reduction

Let

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 1 & 1 & 3 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

True or false? A is in echelon form:

(A) False (B) True

B is in row-reduced echelon form:

(A) False (B) True

C is in row-reduced echelon form:

(A) False (B) True

Example 5.2 (continued)

We will put

$$A = \begin{pmatrix} 4 & 8 & 0 & -4 \\ 2 & 4 & -1 & -1 \\ -1 & -2 & 3 & -2 \end{pmatrix}$$

into row-reduced echelon form.

Quiz and Example of Row Reduction

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$$A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 1 & 1 & 3 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

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Feedback on Sheet 6

- Please collect your work: A–G in green folder, H–R in clear folder, S–Z in red folder.
- Please take feedback sheet. Model answers are on Moodle.
- Please see the lecturer to discuss any of the questions.
- If you want to check your continuous assessment 0/1 marks, please email lecturer, `mark.wildon@rhul.ac.uk`.

Question. Show that if $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ then $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$.

Answer. Take $\mathbf{u} = \mathbf{v} = \mathbf{0}$. Then $\mathbf{u} \times \mathbf{v} = \mathbf{0} = -\mathbf{v} \times \mathbf{u}$.

Quiz: Is this an acceptable answer?

(A) False (B) True

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Looking at examples and special cases is a very good idea. **But make it clear in your answers that you know you are not working in general.** Try to give the general proof as well.

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2. Let $A = \frac{1}{3}\begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix}$. Let $\mathbf{n} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and let Π be the plane through the origin with normal $\mathbf{n}$.

(a) Show that $A\mathbf{n} = -\mathbf{n}$.

(b) Let $\mathbf{u} = \mathbf{i} - \mathbf{j}$. Find $A\mathbf{u}$.

(c) Find a vector $\mathbf{v}$ such that $\Pi = \{\alpha\mathbf{u} + \beta\mathbf{v} : \alpha, \beta \in \mathbb{R}\}$.

(d) Show that if $\mathbf{w} \in \Pi$ then $A\mathbf{w} = \mathbf{w}$.

(e) What, geometrically, is the linear map $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $f(\mathbf{w}) = A\mathbf{w}$?

Example 5.2 (continued)

Row operations putting A into row-reduced echelon form begin as follows:

$$
\begin{aligned}
A = \begin{pmatrix} 4 & 8 & 0 & -4 \\ 2 & 4 & -1 & -1 \\ -1 & -2 & 3 & -2 \end{pmatrix} &\xmapsto{①\mapsto ①/4} \begin{pmatrix} 1 & 2 & 0 & -1 \\ 2 & 4 & -1 & -1 \\ -1 & -2 & 3 & -2 \end{pmatrix} \\
&\xmapsto{②\mapsto ② - 2①} \begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ -1 & -2 & 3 & -2 \end{pmatrix} \\
&\xmapsto{③\mapsto ③ + ①} \begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 3 & -3 \end{pmatrix}.
\end{aligned}
$$

General result on $A\mathbf{x} = \mathbf{0}$

Proposition 5.7

Let A be an $m \times n$ matrix in row-reduced echelon form. Suppose that the first r rows of A are non-zero and the rest are zero. Thus A has r pivot columns and $n - r$ non-pivot columns, say columns $\ell_1, \ldots, \ell_{n-r}$. Given any $\alpha_1, \ldots, \alpha_{n-r} \in \mathbb{R}$ there is a unique solution $\mathbf{x} \in \mathbb{R}^n$ to the equation

$$A\mathbf{x} = \mathbf{0}$$

such that $x_{\ell_s} = \alpha_s$ for each s with $1 \leq s \leq n - r$.

Stated informally, the solution space of a system of n equations in m variables is $n - r$ dimensional, for some $r \leq m, n$.

Example 5.8

See printed notes. (This is the continuation of Example 5.2, done earlier in lecture.)

Row Reduction for $A\mathbf{x} = \mathbf{b}$

Problem 5.9

Find all solutions to the system

$$\begin{aligned}
x_1 \qquad\;\; - \;\; x_3 - x_4 &= -4 \\
x_1 + x_2 \qquad\quad\;\; - x_4 &= -1 \\
x_1 - x_2 - 2x_3 \qquad &= -1 \\
2x_1 - 3x_2 - 5x_3 \qquad &= \beta
\end{aligned}$$

where $\beta \in \mathbb{R}$.

Problem 5.9 (continued)

Row reducing the augmented matrix

$$\begin{pmatrix} 1 & 0 & -1 & -1 & -4 \\ 1 & 1 & 0 & -1 & -1 \\ 1 & -1 & -2 & 0 & -1 \\ 2 & -3 & -5 & 0 & \beta \end{pmatrix}$$

we get . . .

Problem 5.9 (continued)

Row reducing the augmented matrix

$$\begin{pmatrix} 1 & 0 & -1 & -1 & -4 \\ 1 & 1 & 0 & -1 & -1 \\ 1 & -1 & -2 & 0 & -1 \\ 2 & -3 & -5 & 0 & \beta \end{pmatrix}$$

we get ...

$$\begin{pmatrix} \mathbf{1} & 0 & -1 & 0 & 2 \\ 0 & \mathbf{1} & 1 & 0 & 3 \\ 0 & 0 & 0 & \mathbf{1} & 6 \\ 0 & 0 & 0 & 0 & \beta + 5 \end{pmatrix}$$

Problem 5.9 (continued)

Row reducing the augmented matrix

$$\begin{pmatrix} 1 & 0 & -1 & -1 & -4 \\ 1 & 1 & 0 & -1 & -1 \\ 1 & -1 & -2 & 0 & -1 \\ 2 & -3 & -5 & 0 & \beta \end{pmatrix}$$

we get ...

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The equation for the final row is $0x_1 + 0x_2 + 0x_3 + 0x_4 = \beta + 5$. So if $\beta \neq -5$ then the equations are inconsistent and there is no solution.

Problem 5.9 (continued)

Row reducing the augmented matrix

$$\begin{pmatrix} 1 & 0 & -1 & -1 & -4 \\ 1 & 1 & 0 & -1 & -1 \\ 1 & -1 & -2 & 0 & -1 \\ 2 & -3 & -5 & 0 & \beta \end{pmatrix}$$

we get ...

$$\begin{pmatrix} \mathbf{1} & 0 & -1 & 0 & 2 \\ 0 & \mathbf{1} & 1 & 0 & 3 \\ 0 & 0 & 0 & \mathbf{1} & 6 \\ 0 & 0 & 0 & 0 & \beta + 5 \end{pmatrix}$$

The equation for the final row is $0x_1 + 0x_2 + 0x_3 + 0x_4 = \beta + 5$. So if $\beta \neq -5$ then the equations are inconsistent and there is no solution.

Suppose $\beta = 5$. The pivot columns are 1, 2 and 4, so x_1, x_2 and x_4 are determined by x_3. Intuitively, the family of solutions is 1 dimensional.

Quiz: Intuition for Dimension

(a) Let A be a 1×3 matrix. What are the possible dimensions of

$$\{\mathbf{x} \in \mathbb{R}^3 : A\mathbf{x} = \mathbf{0}\}?$$

(A) 1 (B) 2 (C) 3 (D) 2, 3

(b) Let A be a 2×3 matrix. What are the possible dimensions of

$$\{\mathbf{x} \in \mathbb{R}^3 : A\mathbf{x} = \mathbf{0}\}?$$

(A) 1 (B) 1, 2 (C) 1, 2, 3 (D) 2, 3

Dimension will be defined formally in Part E of the course.

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Dimension will be defined formally in Part E of the course.

Application: Inverting a Matrix

Problem 5.10

Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 1 & 2 & 4 \end{pmatrix}$. What is A^{-1}? We form the augmented matrix $(A \,|\, I)$ and perform row operations to put it in row-reduced echelon form, getting $(I \,|\, B)$.

Application: Inverting a Matrix

Problem 5.10

Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 1 & 2 & 4 \end{pmatrix}$. What is A^{-1}? We form the augmented matrix $(A \mid I)$ and perform row operations to put it in row-reduced echelon form, getting $(I \mid B)$.

Example 5.11

By row operations we can transform

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix} \quad \text{to} \quad A' = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

in row-reduced echelon form. Quiz: True or false: A' is surjective?

(A) False (B) True

A' is invertible?

(A) False (B) True

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A' is invertible?

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§6 Permutations and determinants

We saw in Lemma 3.15 that if A is a 2×2 matrix then the image of the unit square with vertices at

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

under A has area $|\det A|$.

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under A has area $|\det A|$.

Now suppose that A is a 3×3 matrix, with columns $\mathbf{a}, \mathbf{b}, \mathbf{c}$. As seen in Example 4.5, the columns of A are the images of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ under A. By Theorem 2.12 and Exercise 2.13 the volume of the parallelepiped formed by $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$.

Definition 6.1

Let A be a 3×3 matrix with columns $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. We define

$$\det A = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}).$$

Other Expressions for the 3×3 Determinant

Lemma 6.2

Let A be a 3×3 matrix. Then

$$
\begin{aligned}
&\det A \\
&= A_{11} \det \begin{pmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{pmatrix} - A_{21} \det \begin{pmatrix} A_{12} & A_{13} \\ A_{32} & A_{33} \end{pmatrix} + A_{31} \det \begin{pmatrix} A_{12} & A_{13} \\ A_{22} & A_{23} \end{pmatrix} \\
&= A_{11}A_{22}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32} \\
&\qquad\qquad - A_{12}A_{21}A_{33} - A_{13}A_{22}A_{31} - A_{11}A_{23}A_{32}.
\end{aligned}
$$

$$
\begin{array}{ccc}
A_{11} & A_{12} & A_{13} \\
A_{21} & A_{22} & A_{23} \\
A_{31} & A_{32} & A_{33}
\end{array}
\qquad\qquad
\begin{array}{ccc}
A_{11} & A_{12} & A_{13} \\
A_{21} & A_{22} & A_{23} \\
A_{31} & A_{32} & A_{33}
\end{array}
$$

Permutations

To define the determinant for $n \times n$ matrices we need to know the correct sign to attach to each bijection

$$\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}.$$

We call such bijections *permutations*.

Example 6.3 (Two-line notation)

See board.

Definition 6.4

A permutation σ of $\{1, 2, \ldots, n\}$ is an *r-cycle* if there exist distinct $x_1, x_2, \ldots, x_r \in \{1, 2, \ldots, n\}$ such that

$$\sigma(x_1) = x_2,\ \sigma(x_2) = x_3,\ \ldots\ \sigma(x_r) = x_1$$

and $\sigma(y) = y$ for all $y \notin \{x_1, x_2, \ldots, x_r\}$. We write

$$\sigma = (x_1, x_2, \ldots, x_r).$$

Feedback on Sheet 7

- Please collect your work: A–G in blue folder, H–R in green folder, S–Z in red folder.
- Please take feedback sheet. Model answers are on Moodle.
- Please see the lecturer to discuss any of the questions.
- If you want to check your continuous assessment 0/1 marks, please email lecturer, `mark.wildon@rhul.ac.uk`.

Quiz: True or False?

- $x = 2 \implies x^2 = 4$

  (A) False (B) True

- $\{2\} = \{x \in \mathbb{R} : x^2 = 4\}$

  (A) False (B) True

The forwards implication '$\Longrightarrow$' is equivalent to

$$\{2\} \subseteq \{x \in \mathbb{R} : x^2 = 4\}.$$

Note '$\Longleftarrow$' does not hold, and the sets are not equal.

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Quiz on Composing Permutations

A 2-cycle is called a *transposition*. Quiz: True or false?

(a) Let $\sigma = (1, 3, 2)$. Then $\sigma = (2, 1, 3)$.
(A) False (B) True

(b) Let $\sigma = (1, 3, 2)$. Then $\sigma^3 = (1, 2, 3)$.
(A) False (B) True

(c) If $\sigma = (1, 2)$ and $\tau = (2, 3)$ then $\sigma\tau = (1, 2, 3)$.
(A) False (B) True

(d) Every permutation of $\{1, 2, \ldots, n\}$ is a composition of transpositions.
(A) False (B) True

(e) Let $\sigma = (1, 2)(1, 3)(1, 4)(1, 5)$, defined by composing the four transpositions $(1, 2)$, $(1, 3)$, $(1, 4)$, $(1, 5)$. Which of the following cycles is equal to σ?
(A) (1,2,3,4,5) (B) (1,5,4,3,2) (C) (5,4,3,2,1) (D) (5,1,2,3,4)

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Sign of a Permutation

Quiz: True or False? $(1, 2, 3, 4, 5)$ is equal to a composition of

(a) 4 transpositions;

(A) False (B) True

(b) 6 transpositions;

(A) False (B) True

(c) 5 transpositions;

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Definition 6.5

Let σ be a permutation. We define the *sign* of σ by

- $\operatorname{sgn}(\sigma) = 1$ if σ is equal to a composition of an even number of transpositions;
- $\operatorname{sgn}(\sigma) = -1$ if σ is equal to a composition of an odd number of transpositions.

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Sign is well defined: no permutation can be written as a composition of both evenly many and oddly many transpositions.

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Quiz: True or False? $(1, 2, 3, 4, 5)$ is equal to a composition of

(a) 4 transpositions;

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(b) 6 transpositions;

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Products of Signs

The sliding block puzzle played in last Friday's lecture had these initial and target positions:

1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

Products of Signs

The sliding block puzzle played in last Friday's lecture had these initial and target positions:

1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	$\square$

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	$\square$

Here $\square$ shows the empty space.

Claim 6.6

There is no sequence of slides going from the initial position to the target position.

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1	2	3	4
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13	14	15	$\square$

Here $\square$ shows the empty space.

Claim 6.6

There is no sequence of slides going from the initial position to the target position.

Lemma 6.7

Let σ, τ be permutations of $\{1, 2, \ldots, n\}$. Then

$$\operatorname{sgn}(\sigma\tau) = \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau).$$

Disjoint Cycle Decomposition of a Permutation

We say that two cycles $(x_1, \ldots, x_k)$ and $(y_1, \ldots, y_\ell)$ are *disjoint* if $\{x_1, \ldots, x_k\} \cap \{y_1, \ldots, y_\ell\} = \varnothing$.

Quiz: True or false? The cycles $(1, 6, 4)$ and $(3, 5)$ are disjoint.

(A) False (B) True

Let $\tau = (4, 5)$, $\sigma = (1, 4, 7)(2, 5)(6, 8)$. Which is $\tau\sigma$?

(A) $(1, 5, 2, 4, 7, 6, 8)$ (B) $(1, 4, 7)(2, 5)(6, 8)$

(C) $(4, 7, 1, 5, 2)(6, 8)$ (D) $(1, 5, 2, 4, 7)$.

Exercise 6.8

Write the permutation of $\{1, 2, 3, 4, 5, 6\}$ defined by $\sigma(1) = 6$, $\sigma(2) = 1$, $\sigma(3) = 5$, $\sigma(4) = 4$, $\sigma(5) = 3$, $\sigma(6) = 2$ as a composition of disjoint cycles.

Lemma 6.9

A permutation σ of $\{1, 2, \ldots, n\}$ can be written as a composition of disjoint cycles. The cycles in this composition are uniquely determined by σ.

Disjoint Cycle Decomposition of a Permutation

We say that two cycles $(x_1, \ldots, x_k)$ and $(y_1, \ldots, y_\ell)$ are *disjoint* if $\{x_1, \ldots, x_k\} \cap \{y_1, \ldots, y_\ell\} = \varnothing$.

Quiz: True or false? The cycles $(1, 6, 4)$ and $(3, 5)$ are disjoint.

(A) False (B) True

Let $\tau = (4, 5)$, $\sigma = (1, 4, 7)(2, 5)(6, 8)$. Which is $\tau\sigma$?

(A) $(1, 5, 2, 4, 7, 6, 8)$ (B) $(1, 4, 7)(2, 5)(6, 8)$

(C) $(4, 7, 1, 5, 2)(6, 8)$ (D) $(1, 5, 2, 4, 7)$.

Exercise 6.8

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Lemma 6.9

A permutation σ of $\{1, 2, \ldots, n\}$ can be written as a composition of disjoint cycles. The cycles in this composition are uniquely determined by σ.

Computing the Sign of a Permutation

Lemma 6.10

Let $\tau = (x_1, x_2, \ldots, x_\ell)$ *be an* ℓ*-cycle. Then* $\mathrm{sgn}(\tau) = (-1)^{\ell-1}$.

Thus cycles of odd length have sign 1 and cycles of even length have sign -1.

Corollary 6.11

If σ *is a permutation of* $\{1, 2, \ldots, n\}$ *then*

$$\mathrm{sgn}(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ has an even number of cycles of even length} \\ -1 & \text{if } \sigma \text{ has an odd number of cycles of even length} \end{cases}$$

in its disjoint cycle decomposition.

Exercise 6.12

We saw earlier that if $\tau = (4, 5), \sigma = (1, 4, 7)(2, 5)(6, 8)$ then

$$\tau\sigma = (4, 7, 1, 5, 2)(6, 8) = (1, 5, 2, 4, 7)(6, 8).$$

Quiz: True or False?

(a) $\mathrm{sgn}(\tau) = 1$?

(A) False (B) True

(b) $\mathrm{sgn}(\sigma) = 1$?

(A) False (B) True

(c) $\mathrm{sgn}(\sigma^{-1}) = 1$?

(A) False (B) True

(d) $\mathrm{sgn}(\tau\sigma) = 1$?

(A) False (B) True

Exercise 6.12

We saw earlier that if $\tau = (4,5), \sigma = (1,4,7)(2,5)(6,8)$ then

$$\tau\sigma = (4,7,1,5,2)(6,8) = (1,5,2,4,7)(6,8).$$

Quiz: True or False?

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(A) False (B) True

(b) $\mathrm{sgn}(\sigma) = 1$?

(A) False (B) True

(c) $\mathrm{sgn}(\sigma^{-1}) = 1$?

(A) False (B) True

(d) $\mathrm{sgn}(\tau\sigma) = 1$?

(A) False (B) True

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(A) False (B) True

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(A) False (B) True

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(A) False (B) True

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(A) False (B) True

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We saw earlier that if $\tau = (4,5), \sigma = (1,4,7)(2,5)(6,8)$ then

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(A) False (B) True

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(A) False (B) True

(d) $\operatorname{sgn}(\tau\sigma) = 1$?

(A) False (B) True

Definition of the determinant

Definition 6.13

Let A be an $n \times n$ matrix. We define the *determinant* of A by

$$\det A = \sum_{\sigma} \operatorname{sgn}(\sigma) A_{1\sigma(1)} A_{2\sigma(2)} \dots A_{n\sigma(n)}$$

where the sum is over all permutations σ of $\{1, 2, \dots, n\}$.

Exercise 6.14

(a) Check that if A is a 2×2 matrix then Definition 6.13 agrees with the definition given immediately after Lemma 3.15.

(b) List all permutations of $\{1, 2, 3\}$ and state their signs. Hence check that Definition 6.13 agrees with Lemma 6.2.

(c) Compute $\det \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ \beta & 0 & 0 & 1 \end{pmatrix}$ for each $\beta \in \mathbb{R}$.

[*Hint:* only a few of the $4! = 24$ permutations of $\{1, 2, 3, 4\}$ give a non-zero summand.]

On Moodle you can find:

- Answers to Sheet 8
- A mock MT182 exam, with the same style and format as the real thing.
- Examination guidance.

§7 Properties of the determinant

Recall that A^T denotes the transpose of A, defined on Question 3 of Sheet 6 by $(A^T)_{ij} = A_{ji}$.

Proposition 7.1

Let A be an $n \times n$ matrix.

(i) *If A' is obtained from A by swapping two rows then* $\det A' = -\det A$.

(ii) *If A' is obtained from A by scaling a row by $\alpha \in \mathbb{R}$ then* $\det A' = \alpha \det A$.

(iii) $\det A^T = \det A$.

Exercise 7.2

Let A be a 3×3 matrix with columns $\mathbf{a}, \mathbf{b}, \mathbf{c}$. Let $A' = (\mathbf{a} \;\; \mathbf{c} \;\; \mathbf{b})$ and let $A'' = (\mathbf{b} \;\; 3\mathbf{c} \;\; 2\mathbf{a})$ Express $\det A'$ and $\det A''$ in terms of $\det A$.

Feedback on Sheet 8

- Please collect your work: A–G in blue folder, H–R in green folder, S–Z in red folder.
- Please take feedback sheet. Model answers are on Moodle.
- Please see the lecturer to discuss any of the questions.
- If you want to check your continuous assessment 0/1 marks, please email lecturer, `mark.wildon@rhul.ac.uk`.

Quiz: Let $\sigma = (1,6,2)(3,5)$, as in Question 4. What is σ^3?

(A) id (B) $(3,5)$ (C) $(1,6,2)$ (D) $(1,2,6)$

Feedback on Sheet 8

- Please collect your work: A–G in blue folder, H–R in green folder, S–Z in red folder.
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Quiz: Let $\sigma = (1,6,2)(3,5)$, as in Question 4. What is σ^3?

(A) id (B) $(3,5)$ (C) $(1,6,2)$ (D) $(1,2,6)$

Adding One Row to Another: Lemmas

Lemma 7.3

If A is an $n \times n$ matrix with two equal rows then $\det A = 0$.

Quiz: Which permutation of $\{1, 2, 3, 4\}$ corresponds to taking the red entries below?

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix}$$

(A) $(1, 2, 3, 4)$ (B) $(1, 2, 3)$ (C) $(1, 3, 2)$ (D) $(1, 3)$

Which permutation cancels its contribution to the determinant?

(A) $(1, 2, 3, 4)$ (B) $(1, 2, 3)$ (C) $(1, 3, 2)$ (D) $(1, 3)$

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(A) $(1, 2, 3, 4)$ (B) $(1, 2, 3)$ (C) $(1, 3, 2)$ (D) $(1, 3)$

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(A) $(1, 2, 3, 4)$ (B) $(1, 2, 3)$ (C) $(1, 3, 2)$ (D) $(1, 3)$

Which permutation cancels its contribution to the determinant?

(A) $(1, 2, 3, 4)$ (B) $(1, 2, 3)$ (C) $(1, 3, 2)$ (D) $(1, 3)$

Adding One Row to Another: Lemmas

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If A is an $n \times n$ matrix with two equal rows then $\det A = 0$*.*

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(A) $(1, 2, 3, 4)$ (B) $(1, 2, 3)$ (C) $(1, 3, 2)$ (D) $(1, 3)$

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(A) $(1, 2, 3, 4)$ (B) $(1, 2, 3)$ (C) $(1, 3, 2)$ (D) $(1, 3)$

Which permutation cancels its contribution to the determinant?

(A) $(1, 2, 3, 4)$ (B) $(1, 2, 3)$ (C) $(1, 3, 2)$ (D) $(1, 3)$

The cancelling permutation is shown in purple.

Adding One Row to Another: Lemmas

Lemma 7.3

If A is an $n \times n$ matrix with two equal rows then $\det A = 0$.

Lemma 7.4

Let $A_{ij} \in \mathbb{R}$ for $2 \leq i \leq n$ and $1 \leq j \leq n$. If $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $\beta_1, \beta_2, \ldots, \beta_n \in \mathbb{R}$ then

$$\det \begin{pmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_n \\ A_{21} & A_{22} & \ldots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \ldots & A_{nn} \end{pmatrix} + \det \begin{pmatrix} \beta_1 & \beta_2 & \ldots & \beta_n \\ A_{21} & A_{22} & \ldots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \ldots & A_{nn} \end{pmatrix} =$$

$$\det \begin{pmatrix} \alpha_1 + \beta_1 & \alpha_2 + \beta_2 & \ldots & \alpha_n + \beta_n \\ A_{21} & A_{22} & \ldots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \ldots & A_{nn} \end{pmatrix}.$$

Adding One Row to Another

Quiz: what row operation does $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$ perform on 2×2 matrices?

(A) add 3 times row 1 to row 2 (B) add 3 times row 2 to row 1

What is the inverse of this matrix?

(A) $\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$ (B) $\begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}$ (C) $\frac{1}{3}\begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}$ (D) $\begin{pmatrix} 1 & \frac{1}{3} \\ 0 & 1 \end{pmatrix}$

Proposition 7.5

Let A be an $n \times n$ matrix. Let $1 \leq k, \ell \leq n$ with $k \neq \ell$. Let A' be obtained from A by replacing row k with the sum of row k and α times row ℓ. Then $\det A' = \det A$.

In particular, the elementary matrix corresponding to the row operation in Proposition 7.5 has determinant 1.

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In particular, the elementary matrix corresponding to the row operation in Proposition 7.5 has determinant 1.

Determinants and Matrix Products

Example′ 7.6

We will use column operations to find

$$\det\begin{pmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \end{pmatrix}$$

Quiz: What is $\det\begin{pmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \\ 7 & 7 & 7 \end{pmatrix}$?

(A) 0 (B) -7 (C) 7 (D) 2

Determinants and Matrix Products

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$$\det \begin{pmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \end{pmatrix}$$

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(A) 0 (B) -7 (C) 7 (D) 2

Administration

- Sign in for Monday's lecture (if and only if you were there).
- Please hand in answers to Sheet 9 at the end.
- Please take the final installment of the Part D handout.
- Spare copies of Problem Sheet 9 and feedback on Sheet 8 at front.

$\det AB = \det A \det B$

Corollary 7.7

Let A and B be $n \times n$ matrices.

(i) *If E is one of the elementary matrices in Lemma 5.4 then* $\det EB = \det E \det B$.

(ii) *If Q is a product of elementary $n \times n$ matrices then* $\det QB = \det Q \det B$.

(iii) *If A is an $n \times n$ matrix then* $\det AB = \det A \det B$.

(iv) $\det B \neq 0$ *if and only if B is invertible.*

Laplace Expansion

Definition 7.8

Let A be an $n \times n$ matrix with $n \geq 2$ and let $1 \leq k, \ell \leq n$. The *minor of A in row k, column ℓ*, denoted $M(k, \ell)$, is the $(n-1) \times (n-1)$ matrix obtained from A by deleting row k and column ℓ.

For example, a general 3×3 matrix A has minors

$$M(1,1) = \begin{pmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{pmatrix}, M(1,2) = \begin{pmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{pmatrix}, \ldots.$$

Proposition 7.9

Let A be an $n \times n$ matrix. Then

$$\det A = A_{11} \det M(k,1) - A_{12} \det M(1,2) + \cdots + (-1)^{n+1} A_{1n} \det M(1,n).$$

Adjugate Matrix

Definition 7.10

Let A be an $n \times n$ matrix. Let A be an $n \times n$ matrix. The *adjugate of A* is the $n \times n$ matrix defined by

$$(\operatorname{adj} A)_{k\ell} = (-1)^{\ell+k} M(\ell, k).$$

Note that the k and ℓ are swapped on the right-hand side.

Quiz: Let $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. The diagonal entries of $\operatorname{adj} A$ are all equal to:

(A) -1 (B) 0 (C) 1 (D) 2

The off-diagonal entries of $\operatorname{adj} A$ are all equal to:

(A) -1 (B) 0 (C) 1 (D) 2

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The off-diagonal entries of $\operatorname{adj} A$ are all equal to:

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Note that the k and ℓ are swapped on the right-hand side.

Example 7.11

Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 4 \\ 3 & 5 & 6 \end{pmatrix}$ be the matrix in Sheet 8, Question 1(a). We have

$$\operatorname{adj} A = \begin{pmatrix} \det\begin{pmatrix} 4 & 4 \\ 5 & 6 \end{pmatrix} & -\det\begin{pmatrix} 1 & 1 \\ 5 & 6 \end{pmatrix} & \det\begin{pmatrix} 1 & 1 \\ 4 & 4 \end{pmatrix} \\ -\det\begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix} & \det\begin{pmatrix} 1 & 1 \\ 3 & 6 \end{pmatrix} & -\det\begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix} \\ \det\begin{pmatrix} 2 & 4 \\ 3 & 5 \end{pmatrix} & -\det\begin{pmatrix} 1 & 1 \\ 3 & 5 \end{pmatrix} & \det\begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 4 & -1 & 0 \\ 0 & 3 & -2 \\ -2 & -2 & 2 \end{pmatrix}.$$

Hence $\operatorname{adj} A = 2A^{-1} = (\det A)A^{-1}$.

The Adjugate and the Inverse

Proposition 7.12

If A is an $n \times n$ matrix then

$$A(\operatorname{adj} A) = (\det A)I.$$

Hence if $\det A \neq 0$ *then* $A^{-1} = \frac{1}{\det A} \operatorname{adj} A$*.*

In printed notes: $(\operatorname{adj} A)A = (\det A)I$. Also true, but form above fits better with Proposition 7.9 (row expansion).

Exercise 7.13

Find $\operatorname{adj} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and hence check that Proposition 7.12 holds in the 2×2 case.

Eigenvalues and Eigenvectors

Proposition 7.14

Let A be a square matrix. Then λ is an eigenvalue of A if and only if $\det(\lambda I - A) = 0$.

Quiz:

(1) What are the eigenvalues of $\begin{pmatrix} 1 & -1 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$?

(A) -1,2,3 (B) 1,-1,3 (C) 0,1,2 (D) 1,2,3

(2) Let A be the 3×3 matrix representing rotation by $\pi/4$ with axis $\mathbf{i}$. True or false? 1 is the only eigenvalue of A.

(A) False (B) True

Eigenvalues and Eigenvectors

Proposition 7.14

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Quiz:

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(A) $-1,2,3$ (B) $1,-1,3$ (C) $0,1,2$ (D) $1,2,3$

(2) Let A be the 3×3 matrix representing rotation by $\pi/4$ with axis $\mathbf{i}$. True or false? 1 is the only eigenvalue of A.

(A) False (B) True

Eigenvalues and Eigenvectors

Proposition 7.14

Let A be a square matrix. Then λ is an eigenvalue of A if and only if $\det(\lambda I - A) = 0$.

Example 7.15

Let $\alpha \in \mathbb{R}$ and let $A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & \alpha \\ 0 & 0 & 1 \end{pmatrix}$. So $\lambda I - A = \begin{pmatrix} \lambda - 2 & 0 & 0 \\ -1 & \lambda - 1 & -\alpha \\ 0 & 0 & \lambda - 1 \end{pmatrix}$ and

$$\det(\lambda I - A) = (\lambda - 2) \det \begin{pmatrix} \lambda - 1 & -\alpha \\ 0 & \lambda - 1 \end{pmatrix} = (\lambda - 2)(\lambda - 1)^2$$

Hence the eigenvalues of A are 1 and 2. When $\lambda = 2$ we have

$$(2I - A) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & -\alpha \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -x + y - \alpha z \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

if and only if $z = 0$ and $x = y$. So $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector with eigenvalue 2.

Feedback on Sheet 9

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- Please take feedback sheet and Part E printed notes. Model answers are on Moodle.
- **Please eat all the chocolates.**
- Please see the lecturer to discuss any of the questions: office hours this week are Monday 4pm, Wednesday 10am, or by appointment.
- If you want to check your continuous assessment 0/1 marks, please email lecturer, `mark.wildon@rhul.ac.uk`.

§8 Vector spaces and subspaces

Definition 8.1

A vector space is a set V of *vectors* with an addition rule and zero element $\mathbf{0}$ such that

(A1) $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$

(A2) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(A3) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$. Moreover there is a rule for multiplying a vector by a scalar such that

(M1) $\alpha(\beta\mathbf{v}) = (\alpha\beta)\mathbf{v}$

(M2) $1\mathbf{v} = \mathbf{v}$

(M3) $0\mathbf{v} = \mathbf{0}$

(D) $(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$

for all $\mathbf{v} \in V$ and $\alpha, \beta \in \mathbb{R}$.

Examples

Example 8.2

(1) Let $n \in \mathbb{N}$. Then $\mathbb{R}^n$ is a vector space.

(2) Let $n \in \mathbb{N}$. The set of all $n \times n$ matrices is a vector space.

(3) The set of all real polynomials, i.e.

$$\mathbb{R}[x] = \{a_0 + a_1 x + \cdots + a_d x^d : a_0, a_1, \ldots, a_d \in \mathbb{R}, d \in \mathbb{N}_0\}$$

is a vector space.

(4) The line $\ell = \{\lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix} : \lambda \in \mathbb{R}\}$ is a vector space.

(5) The translated line $\ell' = \{\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix} : \lambda \in \mathbb{R}\}$ is not a vector space.

More Examples of Vector Spaces

Quiz: True or false?

- $\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : x + z = 0 \right\}$ is a vector space.

  (A) False (B) True

- $\mathbb{Q}$ is a vector space

  (A) False (B) True

- The set of continuous functions from $\mathbb{R}$ to $\mathbb{R}$ is a vector space.

  (A) False (B) True

- The set of continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) \geq 0$ for all $x \in \mathbb{R}$ is a vector space.

  (A) False (B) True

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  (A) False (B) True

Feedback

Please vote on the following aspects of MT182 using:

(A) Definitely agree

(B) Mostly agree

(C) Neutral

(D) Mostly disagree

(E) Definitely disagree

- The clicker quizzes were useful for me.

Feedback

Please vote on the following aspects of MT182 using:

(A) Definitely agree

(B) Mostly agree

(C) Neutral

(D) Mostly disagree

(E) Definitely disagree

- The clicker quizzes were useful for me.
- The feedback sheets were useful for me.

Feedback

Please vote on the following aspects of MT182 using:

(A) Definitely agree

(B) Mostly agree

(C) Neutral

(D) Mostly disagree

(E) Definitely disagree

- The clicker quizzes were useful for me.
- The feedback sheets were useful for me.
- The comments on my work from the markers or lecturer were useful for me.

Feedback

Please vote on the following aspects of MT182 using:

(A) Definitely agree

(B) Mostly agree

(C) Neutral

(D) Mostly disagree

(E) Definitely disagree

- The clicker quizzes were useful for me.
- The feedback sheets were useful for me.
- The comments on my work from the markers or lecturer were useful for me.

The pace of lectures (maybe not including this one) was

(A) Much too fast

(B) A bit too fast

(C) About right

(D) A bit too slow

(E) Much too slow

Linear Independence, Spanning and Basis

Definition 8.3

Let V be a vector space. Vectors $\mathbf{u}_1, \dots, \mathbf{u}_r \in V$

(i) are *linearly dependent* if there exist $\alpha_1, \alpha_2, \dots, \alpha_r \in \mathbb{R}$, not all equal to zero, such that

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots + \alpha_r \mathbf{u}_r = \mathbf{0}.$$

(ii) are *linearly independent* if they are not linearly dependent.

(iii) *span* V if for all $\mathbf{v} \in V$ there exist $\beta_1, \dots, \beta_r \in \mathbb{R}$ such that

$$\mathbf{v} = \beta_1 \mathbf{u}_1 + \beta_2 \mathbf{u}_2 + \cdots + \beta_r \mathbf{u}_r.$$

(iv) are a *basis* for V if they are linearly independent and span V.

Finally if V has a basis of size d we say that V has *dimension* d.

It will be proved in the second year linear algebra course that any two bases of a vector space have the same size. So dimension is well-defined.

Key Example

Example 8.4

Let $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : x + z = 0 \right\}$ be the plane in $\mathbb{R}^3$ through the origin with normal $\mathbf{i} + \mathbf{k}$. Let

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \ \mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ \mathbf{u}_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

Then

(i) $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly dependent;

(ii) $\mathbf{u}_1, \mathbf{u}_2$ are linearly independent;

(iii) $\mathbf{u}_1, \mathbf{u}_2$ span V;

(iv) $\mathbf{u}_1, \mathbf{u}_2$ are a basis of W, so V has dimension 2.

Quiz on Example 8.4

Quiz: True or false:

(a) $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ span V.

(A) False (B) True

(b) $\mathbf{u}_1, \mathbf{u}_3$ are linearly dependent.

(A) False (B) True

(c) $\mathbf{0}, \mathbf{u}_1, \mathbf{u}_3$ span V.

(A) False (B) True

(d) $\mathbf{0}, \mathbf{u}_1, \mathbf{u}_3$ are linearly independent.

(A) False (B) True

(e) V has a unique basis.

(A) False (B) True

Quiz on Example 8.4

Quiz: True or false:

(a) $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ span V.

(A) False (B) True

(b) $\mathbf{u}_1, \mathbf{u}_3$ are linearly dependent.

(A) False (B) True

(c) $\mathbf{0}, \mathbf{u}_1, \mathbf{u}_3$ span V.

(A) False (B) True

(d) $\mathbf{0}, \mathbf{u}_1, \mathbf{u}_3$ are linearly independent.

(A) False (B) True

(e) V has a unique basis.

(A) False (B) True

Quiz on Example 8.4

Quiz: True or false:

(a) $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ span V.

(A) False (B) True

(b) $\mathbf{u}_1, \mathbf{u}_3$ are linearly dependent.

(A) False (B) True

(c) $\mathbf{0}, \mathbf{u}_1, \mathbf{u}_3$ span V.

(A) False (B) True

(d) $\mathbf{0}, \mathbf{u}_1, \mathbf{u}_3$ are linearly independent.

(A) False (B) True

(e) V has a unique basis.

(A) False (B) True

Quiz on Example 8.4

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(a) $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ span V.

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(A) False (B) True

(e) V has a unique basis.

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(d) $\mathbf{0}, \mathbf{u}_1, \mathbf{u}_3$ are linearly independent.

(A) False (B) True

(e) V has a unique basis.

(A) False (B) True

Subspaces

Definition 8.5

Let W be a vector space. A subset V of W is a *subspace of W* if

(i) $\mathbf{0} \in V$ *and*

(ii) if $\mathbf{u}, \mathbf{v} \in V$ and $\alpha, \beta \in \mathbb{R}$ then $\alpha\mathbf{u} + \beta\mathbf{v} \in V$.

Example 8.6

Let $\mathbf{n} \in \mathbb{R}^3$ be non-zero and let

$$V = \left\{ \mathbf{v} \in \mathbb{R}^3 : \mathbf{n} \cdot \mathbf{v} = 0 \right\}$$

be the plane in $\mathbb{R}^3$ through the origin with normal $\mathbf{n}$. Then V is a subspace of $\mathbb{R}^3$ of dimension 2.

Proposition 8.7

If V is a subspace of a vector space W then V is a vector space.

For example, the plane in Example 8.4 is now proved to be a vector space.