COMPUTING MAXIMAL CONSTITUENTS OF PLETHYSMS

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Theorem 1.5 in [1] characterizes the maximal partitions λ such that s_λ appears with non-zero multiplicity in the plethysm $s_\nu \circ s_\mu$ of two Schur functions. The purpose of this note it to document some related Haskell [2] code and to give a small example showing how this theorem also gives information about certain non-maximal constituents.

1. Background

1.1. **Plethystic semistandard tableaux.** For a fixed partition μ, we define a total order $<$ on the set of semistandard μ-tableaux by $s < t$ if and only if, in the rightmost column where s and t differ, the largest integer entry not appearing in both tableaux lies in t. In [1, Definition 1.4] a *plethystic semistandard tableau* of *shape* μ^ν was defined to be a semistandard ν-tableau whose entries are semistandard μ-tableaux, where the semistandard μ-tableaux entries are ordered by $<$. The *weight* of a plethystic semistandard tableau is the sum of the weights (i.e. contents) of its μ-tableau entries. Let $\mathrm{PSSYT}(\mu^\nu, \lambda)$ denote the set of plethystic semistandard tableaux of shape μ^ν and weight λ. For example

$$\mathrm{PSSYT}\big((2)^{(3)}, (2,2,2)\big) = \left\{ \begin{array}{c} \boxed{\boxed{1}\boxed{1}}\boxed{\boxed{2}\boxed{2}}\boxed{\boxed{3}\boxed{3}} \\ \boxed{\boxed{1}\boxed{1}}\boxed{\boxed{2}\boxed{3}}\boxed{\boxed{2}\boxed{3}} \\ \boxed{\boxed{1}\boxed{2}}\boxed{\boxed{1}\boxed{2}}\boxed{\boxed{3}\boxed{3}} \\ \boxed{\boxed{1}\boxed{2}}\boxed{\boxed{1}\boxed{3}}\boxed{\boxed{2}\boxed{3}} \\ \boxed{\boxed{2}\boxed{2}}\boxed{\boxed{1}\boxed{3}}\boxed{\boxed{1}\boxed{3}} \end{array} \right\}.$$

Here the final tableau has weight $(0,2,0) + (1,0,1) + (1,0,1) = (2,2,2)$. These tableaux were enumerated using the Haskell [2] programs **PlethysticSemistandardSkewTableaux7.hs** and the command

```
display $ psssytsW ([3], []) ([2], []) [2,2,2].
```

The Appendix gives some further notes on this code, which may be obtained from the author. (Since it is a work in progress, I have not yet put it on the web.)

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2. Maximal plethystic semistandard tableaux

The output of

```
display $ maximalPSkewTableaux 6 ([1,1,1,1,1],[]) ([2,1],[])
```

shows that there are five plethystic skew-tableaux of shape $(2,1)^{(1^5)}$ and maximal weight, namely

$$\begin{array}{|c|}\hline \begin{array}{cc}1&1\\2&\end{array}\\\hline \begin{array}{cc}1&1\\3&\end{array}\\\hline \begin{array}{cc}1&1\\4&\end{array}\\\hline \begin{array}{cc}1&1\\5&\end{array}\\\hline \begin{array}{cc}1&1\\6&\end{array}\\\hline\end{array}\;,\quad
\begin{array}{|c|}\hline \begin{array}{cc}1&1\\2&\end{array}\\\hline \begin{array}{cc}1&1\\3&\end{array}\\\hline \begin{array}{cc}1&1\\4&\end{array}\\\hline \begin{array}{cc}1&1\\5&\end{array}\\\hline \begin{array}{cc}1&2\\2&\end{array}\\\hline\end{array}\;,\quad
\begin{array}{|c|}\hline \begin{array}{cc}1&1\\2&\end{array}\\\hline \begin{array}{cc}1&1\\3&\end{array}\\\hline \begin{array}{cc}1&1\\4&\end{array}\\\hline \begin{array}{cc}1&2\\2&\end{array}\\\hline \begin{array}{cc}1&3\\2&\end{array}\\\hline\end{array}\;,\quad
\begin{array}{|c|}\hline \begin{array}{cc}1&1\\2&\end{array}\\\hline \begin{array}{cc}1&1\\3&\end{array}\\\hline \begin{array}{cc}1&1\\4&\end{array}\\\hline \begin{array}{cc}1&2\\2&\end{array}\\\hline \begin{array}{cc}1&2\\3&\end{array}\\\hline\end{array}\;,\quad
\begin{array}{|c|}\hline \begin{array}{cc}1&1\\2&\end{array}\\\hline \begin{array}{cc}1&1\\3&\end{array}\\\hline \begin{array}{cc}1&2\\2&\end{array}\\\hline \begin{array}{cc}1&2\\3&\end{array}\\\hline \begin{array}{cc}1&3\\2&\end{array}\\\hline\end{array}\;.$$

Their weights are $(10,1^6)$, $(9,3,1,1,1)$, $(8,4,2,1)$, $(8,4,2,1)$ and $(7,5,3)$, respectively. (In particular we get an example where there is more than one tableaux of the same maximal weight.) By Theorem 1.5 in [1], these weights label the maximal constituents of the plethysm $s_{(1^5)} \circ s_{(2,1)}$ that have at most 6 rows. Thus

$$s_{(1^5)} \circ s_{(2,1)} = s_{(10,1,1,1,1,1,1)} + s_{(9,3,1,1,1)} + 2s_{(8,4,2,1)} + s_{(7,5,3)} + \cdots .$$

It is clear from inspection of the tableaux of weight $(10,1^6)$ that no plethystic semistandard tableau of shape $(2,1)^{(1^5)}$ and maximal weight can have an entry exceeding 6, therefore these are all the maximal constituents.

Example 2.1. Three of the plethystic semistandard skew-tableaux of shape $(2,1)^{(2^5)}$ and weight $(16,8,4,2)$ are obtained by concatenating two of the plethystic semistandard skew-tableaux of shape $(2,1)^{(1^5)}$ and weight $(8,4,2,1)$:

$$\begin{array}{|c|c|}\hline \begin{array}{cc}1&1\\2&\end{array}&\begin{array}{cc}1&1\\2&\end{array}\\\hline \begin{array}{cc}1&1\\3&\end{array}&\begin{array}{cc}1&1\\3&\end{array}\\\hline \begin{array}{cc}1&1\\4&\end{array}&\begin{array}{cc}1&1\\4&\end{array}\\\hline \begin{array}{cc}1&2\\2&\end{array}&\begin{array}{cc}1&2\\2&\end{array}\\\hline \begin{array}{cc}1&3\\2&\end{array}&\begin{array}{cc}1&3\\2&\end{array}\\\hline\end{array}\;,\quad
\begin{array}{|c|c|}\hline \begin{array}{cc}1&1\\2&\end{array}&\begin{array}{cc}1&1\\2&\end{array}\\\hline \begin{array}{cc}1&1\\3&\end{array}&\begin{array}{cc}1&1\\3&\end{array}\\\hline \begin{array}{cc}1&1\\4&\end{array}&\begin{array}{cc}1&1\\4&\end{array}\\\hline \begin{array}{cc}1&2\\2&\end{array}&\begin{array}{cc}1&2\\2&\end{array}\\\hline \begin{array}{cc}1&3\\2&\end{array}&\begin{array}{cc}1&2\\3&\end{array}\\\hline\end{array}\;,\quad
\begin{array}{|c|c|}\hline \begin{array}{cc}1&1\\2&\end{array}&\begin{array}{cc}1&1\\2&\end{array}\\\hline \begin{array}{cc}1&1\\3&\end{array}&\begin{array}{cc}1&1\\3&\end{array}\\\hline \begin{array}{cc}1&1\\4&\end{array}&\begin{array}{cc}1&1\\4&\end{array}\\\hline \begin{array}{cc}1&2\\2&\end{array}&\begin{array}{cc}1&2\\2&\end{array}\\\hline \begin{array}{cc}1&2\\3&\end{array}&\begin{array}{cc}1&2\\3&\end{array}\\\hline\end{array}\;.$$

If instead we concatenate the plethystic semistandard skew tableaux of shape $(2,1)^{(1^5)}$ and weights $(9,3,1,1,1)$ and $(7,5,3)$, we obtain

$$\begin{array}{|c|c|}\hline \begin{array}{|c|c|}\hline 1&1\\\hline 2&\\\cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 1&1\\\hline 2&\\\cline{1-1}\end{array} \\\hline \begin{array}{|c|c|}\hline 1&2\\\hline 2&\\\cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 1&2\\\hline 2&\\\cline{1-1}\end{array} \\\hline \begin{array}{|c|c|}\hline 1&1\\\hline 3&\\\cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 1&1\\\hline 3&\\\cline{1-1}\end{array} \\\hline \begin{array}{|c|c|}\hline 1&1\\\hline 4&\\\cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 1&2\\\hline 3&\\\cline{1-1}\end{array} \\\hline \begin{array}{|c|c|}\hline 1&1\\\hline 5&\\\cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 1&3\\\hline 2&\\\cline{1-1}\end{array} \\\hline \end{array}$$

which has weight $(16,8,4,1,1)$. Call this tableau T. Observe that decrementing any entry other than 1 in a $(2,1)$-tableau entry of T either creates a column repeat in this $(2,1)$-tableau or creates two equal $(2,1)$-tableaux in the same column of T. It follows that the GL-polytabloid $F(T)$, as defined in [1, Definition 2.3], is a highest-weight vector in $\nabla^{(2^5)}\nabla^{(2,1)}E$, where E is a 5-dimensional (or greater) complex vector space. (See Remark 3.2 for the general case, which is somewhat more technical.) Hence $s_{(2^5)} \circ s_{(2,1)}$ has $s_{(16,8,4,1,1)}$ as a non-maximal constituent. In fact

$$\begin{aligned} s_{(2^5)} \circ s_{(2,1)} = s_{(20,2,2,2,2,2)} &+ s_{(19,4,2,2,2,1)} + s_{(18,6,2,2,2)} \\ &+ 2s_{(17,7,3,2,1)} + 3s_{(16,8,4,2)} + 2s_{(15,9,5,1)} + s_{(14,10,6)} + \cdots \end{aligned}$$

where all other constituents are non-maximal.

This example shows that the methods in [1] also give information about non-maximal constituents of plethysms.

3. Appendix: computational notes

PlethysticSemistandardSkewTableaux7.hs. The algorithm employed by `psssytsW` to construct all plethystic semistandard tableaux of shape μ^ν and weight γ has some features of interest. As a first step, we make a list

$$\big(t_1, \mathrm{wt}(t_1)\big), \ldots, \big(t_k, \mathrm{wt}(t_k)\big)$$

where $t_1 < t_2 < \ldots < t_k$ are the semistandard μ-tableaux with entries from $\{1,\ldots,\ell(\gamma)\}$. We then enumerate all semistandard ν-tableaux with entries from $\{1,\ldots,k\}$; replacing i with t_i turns each such ν-tableau into a plethystic semistandard tableau. The ν-tableaux are constructed iteratively, starting at a removable box of $[\nu]$: if at any point the weight, considering the μ-tableau entries placed so far, exceeds γ in some position, then the partially constructed tableau is abandoned.

The special case when γ has one long part (corresponding to entries of 1) and a small number of short other parts arises when considering stability results. In this case the algorithm is efficient in practice. For example

```
[(p, monomialCoefficient [4,4] [3,2] p) | p <- stablePartitions 40 22 2]
```

finds $[x^\gamma](s_{(4,4)} \circ s_{(3,2)})$ for all partitions γ of 40 of the form $(18, \gamma_2, \gamma_3)$ in under a minute, by explicitly constructing the corresponding plethystic semistandard tableaux. The alternative using MAGMA would be to construct the entire plethysm $s_{(4,4)} \circ s_{(3,2)}$; while permitting further computations, this takes over an hour on my laptop.

No very fast enumerative algorithm can be expected because the number of plethystic semistandard tableaux may be exponentially large. For example, $[x^{(1^{nm})}](s_{(1^n)} \circ s_{(1^m)})$ is the number of ways to partition $\{1, \ldots, mn\}$ into n disjoint subsets each of size m, namely $(mn)!/m!^n n!$. (This is also the degree of the Foulkes character $\phi^{(m^n)}$.) One might instead ask for an algorithm that gives the coefficient $[x^\gamma](s_\nu \circ s_\mu)$ without explicit enumeration. However, the problem of deciding, given the list of weights $\mathrm{wt}(t_1), \ldots, \mathrm{wt}(t_k)$, whether some subset has sum γ is NP-complete. (This is the multi-dimensional knapsack problem.) Therefore determining whether $[x^\gamma](s_{(1^n)} \circ s_\mu)$ is non-zero is an NP-complete problem. Determining the coefficient itself is, of course, at least as hard.

Another sign the enumeration problem is non-trivial is the incompatibility between the natural partial orders on semistandard μ-tableaux that respect weights (for example, the dominance order) and the total order $<$. For instance

$$\begin{array}{|c|c|c|c|}\hline 1&1&1&1\\\hline 2&2\\\cline{1-2}\end{array} < \begin{array}{|c|c|c|c|}\hline 1&1&1&2\\\hline a&b\\\cline{1-2}\end{array} < \begin{array}{|c|c|c|c|}\hline 1&1&2&2\\\hline 2&2\\\cline{1-2}\end{array}$$

for any $a, b \geq 2$, showing that tableaux of almost arbitrary weight can appear between tableaux of maximal, or near maximal, weight. Adapting this example, consider the plethystic semistandard tableau

$$\left[\;\begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2&2\\\cline{1-2}\end{array} \;\middle|\; \begin{array}{|c|c|c|}\hline 1&1&1\\\hline 3&3\\\cline{1-2}\end{array} \;\middle|\; \begin{array}{|c|c|c|}\hline 1&1&1\\\hline 3&3\\\cline{1-2}\end{array} \;\middle|\; \begin{array}{|c|c|c|}\hline 1&1&2\\\hline 2&2\\\cline{1-2}\end{array}\;\right]$$

of weight $(11, 5, 4)$, which has as a subtableau

$$\left[\;\begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2&2\\\cline{1-2}\end{array} \;\middle|\; \begin{array}{|c|c|c|}\hline 1&1&1\\\hline 3&3\\\cline{1-2}\end{array} \;\middle|\; \begin{array}{|c|c|c|}\hline 1&1&1\\\hline 3&3\\\cline{1-2}\end{array}\;\right]$$

of non-partition weight $(9, 2, 4)$. This rules out at least one 'greedy' algorithm for enumeration. It appears impossible to resolve this problem by another choice of $<$.

PlethysticSemistandardTableauxDecInc3. Let t be a semistandard μ-tableau t. We say that a semistandard μ-tableau t^- is a *decrement* of t if there exists (i,j) such that

$$t^-_{(i',j')} = \begin{cases} t_{(i',j')} & \text{if } (i',j') \neq (i,j) \\ t_{(i,j)} - 1 & \text{if } (i',j') = (i,j). \end{cases}$$

We define an increment t^+ of a semistandard μ-tableau t analogously.

Lemma 3.1. *Let T be a plethystic semistandard tableau of shape μ^ν and maximal weight. Let $(I,J) \in [\nu]$. Let $t = T_{(I,J)}$. If t has t^- as a decrement then t^- appears in column J of T.*

Proof. Let T^- be the plethystic tableau obtained from T by replacing the entry t in position (I,J) with t^-. If t^- is not already in this column then column standardising T^- gives a column standard plethystic tableau U, with semistandard μ-tableau entries, and of weight dominating $\mathrm{wt}(T)$. Let d be the maximum entry of a μ-tableau entry in T and let E be a d-dimensional complex vector space. By the model in §3 of [1], we may straighten the GL-polytabloid $F(U) \in \nabla^\nu \nabla^\mu E$ to write $F(U)$ as a non-zero integral linear combination of plethystic semistandard GL-polytabloids $F(S)$ each of weight dominating $\mathrm{wt}(T)$. Therefore T does not have maximal weight. □

Say that a plethystic semistandard tableau is *maximal* if it has maximal weight for its shape and *closed* if it has the property in Lemma 3.1. By this lemma, maximal plethystic semistandard tableaux are closed.

Remark 3.2. The closure property just defined is not sufficient to guarantee that $F(T)$ is a highest weight vector, because one must also consider decrements of T that do not preserve the semistandardness of its μ-tableau entries. This technical point is illustrated in Example 7.4 of [1], where the 11 semistandard $(2,2)$-tableaux shown form a plethystic semistandard tableau T of shape $(2,2)^{(1^{11})}$ which is closed in the sense just defined, but $F(T)$ is not a highest-weight vector, and correspondingly, $\mathrm{wt}(T)$ is not maximal.

Lemma 3.1 leads to the following algorithm for constructing closed semistandard plethystic tableaux of shape μ^ν and entries from $\{1,\ldots,d\}$. Say that a box of a skew diagram $[\nu]/[\nu^\star]$ is *inner* if it is an addable box of $[\nu^\star]$.

(1) Suppose, inductively, that the μ-tableau entries in a candidate closed semistandard plethystic tableau has been chosen in boxes in $\nu^\star$ where $[\nu^\star] \subseteq [\nu]$. If $\nu^\star = \nu$ then stop. Otherwise choose an inner box (I,J) of $[\nu]\backslash[\nu^\star]$. By Lemma 3.1, if there is a closed plethystic semistandard tableau having the semistandard μ-tableau t in position (I,J) then
   - *either* $t = t_\mu$, the unique μ-tableau with no decrement;

- *or* $t = s^+$ for some s already in column J, and moreover any decrement t^- of t is already in column J.

(2) For each t passing the test in $(1a)$, check whether putting t in position (I, J) gives a plethystic semistandard tableau T^+. (The shape of T^+ is ${\nu^\star_+}^\mu$ where $[\nu^\star_+] = [\nu^\star] \cup \{(I, J)\}$.) Continue (1) with each such T^+.

To obtain maximal tableaux one then filters the list of closed tableaux, taking those that really do have maximal weight. For example,

```
display $ closedPSkewTableaux 6 ([1,1,1,1,1],[]) ([2,1],[])
```

gives the five plethystic tableaux of shape $(2, 1)^{(1^5)}$ shown in §2 and the three further non-maximal but closed tableaux

$$\begin{array}{|c|}\hline \begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}\\\hline \begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array}\\\hline \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\\\hline \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\\\hline \begin{array}{|c|c|}\hline 1&4\\\hline 2\\\cline{1-1}\end{array}\\\hline \end{array}\,, \quad \begin{array}{|c|}\hline \begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}\\\hline \begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array}\\\hline \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\\\hline \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\\\hline \begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1}\end{array}\\\hline \end{array}\,, \quad \begin{array}{|c|}\hline \begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}\\\hline \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\\\hline \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\\\hline \begin{array}{|c|c|}\hline 1&4\\\hline 2\\\cline{1-1}\end{array}\\\hline \begin{array}{|c|c|}\hline 1&5\\\hline 2\\\cline{1-1}\end{array}\\\hline \end{array}\,.$$

of weights $(7, 5, 2, 1)$, $(6, 6, 3)$ and $(6, 6, 1, 1, 1)$, respectively. This shows that the final filtering step is necessary. For a smaller example with the same behaviour, replace $(2, 1)^{(1^5)}$ with $(2, 1)^{(1^3)}$; the closed plethystic tableaux are then

$$\begin{array}{|c|}\hline \begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}\\\hline \begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array}\\\hline \begin{array}{|c|c|}\hline 1&1\\\hline 4\\\cline{1-1}\end{array}\\\hline \end{array}\,, \quad \begin{array}{|c|}\hline \begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}\\\hline \begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array}\\\hline \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\\\hline \end{array}\,, \quad \begin{array}{|c|}\hline \begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}\\\hline \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}\\\hline \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\\\hline \end{array}$$

of weights $(6, 1, 1, 1)$, $(5, 3, 1)$ and $(4, 4, 1)$, and the final one is not maximal.

The function `closedPlethysticSkewTableauxFrom d bs gs ts` implements steps (1) and (2) using breadth first searching to avoid repeatedly generating the same plethystic semistandard tableaux. Here `bs` is the list for $[\nu]\backslash[\nu^\star]$, `gs` is the singleton list representing $\{t_\mu\}$, and `ts` is the list of partially constructed candidate closed plethystic semistandard tableaux. (The reason for the generality in `gs` will be seen shortly.)

Plethystic semistandard tableaux of arbitrary depth. In fact **Plethystic-SemistandardTableauxDecInc3** solves a more general problem. Define a *plethystic semistandard tableau* t of shape (μ) and *depth* 1 to be a semistandard μ-tableau with integer entries, and a plethystic semistandard tableau of *depth* D and *shape* $(\mu^{(1)}, \ldots, \mu^{(D-1)}, \nu)$ to be a semistandard ν-tableau whose entries are inductively defined plethystic semistandard tableaux of depth $D-1$ and shape $(\mu^{(1)}, \ldots, \mu^{(D-1)})$. The entries, of depth $D-1$, are ordered by the total order, defined inductively. For example,

$$\boxed{\boxed{\boxed{1 \mid 2}\;\boxed{2 \mid 3}}\;\boxed{\boxed{2 \mid 2}\;\boxed{2 \mid 3}}} < \boxed{\boxed{\boxed{2 \mid 2}\;\boxed{1 \mid 3}}\;\boxed{\boxed{1 \mid 3}\;\boxed{2 \mid 3}}}$$

since, comparing on the rightmost entries, each plethystic semistandard tableaux of shape $\big((2),(2)\big)$, we have

$$\boxed{\boxed{2 \mid 2}\;\boxed{2 \mid 3}} < \boxed{\boxed{1 \mid 3}\;\boxed{2 \mid 3}}.$$

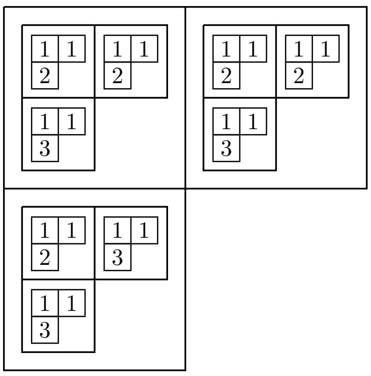
The definition of weight extends in the obvious way. Note that if $D = 2$ then plethystic semistandard tableaux of shape (μ, ν) are the same as plethystic semistandard tableaux of shape μ^ν, as already defined.

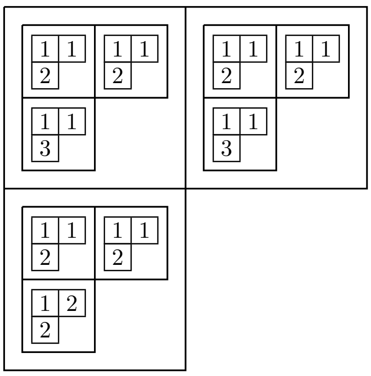
The definitions of increment and decrement extend to these more general objects: note however that the unique undecrementable tableau of shape μ is t_μ, but when $D > 2$ there are usually many undecrementable tableaux of shape $(\mu^{(1)}, \ldots, \mu^{(D-1)})$. The algorithm as presented above must therefore be modified by replacing the condition $t \neq t_\mu$ in (1) with $t \notin \mathcal{T}$, where $\mathcal{T}$ is the set of all undecrementable plethystic semistandard tableaux of shape $(\mu^{(1)}, \ldots, \mu^{(D-1)})$. This set is generated as a one-off preliminary computation in a new step (0).

Example 3.3. The function `maximalQSkewTableau d zs ys xs` constructs all plethystic semistandard tableaux of depth 3 of shape (μ, ν, ξ), corresponding to $[\texttt{zs}, \texttt{ys}, \texttt{xs}]$. For example the output of

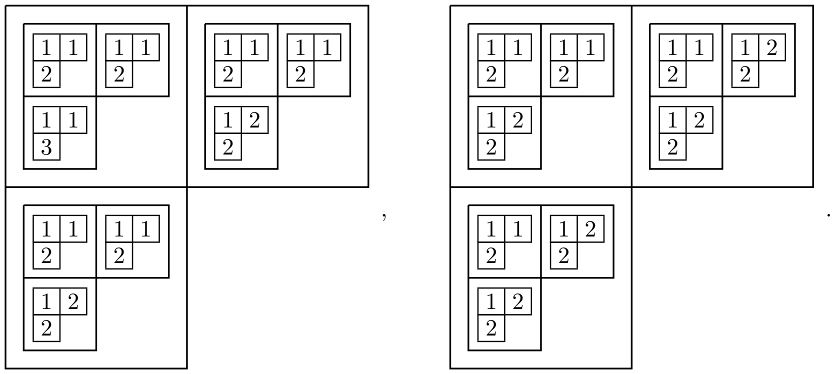
```
display $ maximalQSkewTableaux 3 ([2,1],[]) ([2,1],[]) ([2,1],[])
```

shows the four maximals below:



,


,



of weights $(18, 5, 4)$, $(17, 8, 2)$, $(16, 10, 1)$ and $(14, 13)$, respectively. Computation in MAGMA shows that

$$s_{(2,1)} \circ s_{(2,1)} \circ s_{(2,1)} = s_{(18,5,4)} + s_{(17,8,2)} + s_{(16,10,1} + s_{(14,13)} + s_{(18,6,2,1)} + \cdots$$

where the final maximal constituent may be found by replacing 3 with 4 (to allow entries from $\{1, 2, 3, 4\}$) above.

Further implementation notes. The function

```
sssyts :: (Ord a) => [a] -> SkewPartition -> [SSYT a]
```

enumerates all semistandard tableaux with entries from a given totally ordered set, of the given shape. The algorithm used builds from the rightmost removable box of the partition, working leftwards: conveniently this gives semistandard tableaux in the total order $<$. For example, the plethystic semistandard tableaux of shape $\big((2), (2), (2)\big)$ above appear in the output of `sssyts (sssyts (sssyts [1,2,3::Int] ([2],[])) ([2],[])) ([2],[])`. There are in total 231 such tableaux.

The function

```
display :: ShowT a => a -> IO ()
```

is defined in `ShowTableaux2.hs` using instances for the type class `ShowT`, whose unique member is `showT :: a -> String`. These instances have to be defined individually for each possible depth because of the different types involved. For example, plethystic semistandard tableaux of shape $\big((2), (2), (2)\big)$ have Haskell type `SSYT (SSYT (SSYT Int))`. Once the `ShowT` instances are defined, there is a unified way to write the `display` function:

```
class Display a where
  displayC :: a -> IO ()

data DType a = D a
instance (ShowT a) => Display (DType a) where
  displayC (D x) = putStrLn $ showT x

display :: (ShowT a) => a -> IO ()
display x = displayC (D x)
```

It is necessary to introduce the auxiliary type `DType` because `instance (ShowT a) => Display a)` is not decidable by the type-checker. (Consider a later declaration: `instance (Display a) => Show T a`; clearly this creates a potential loop.)

References

[1] M. de Boeck, R. Paget and M. Wildon, *Plethysms of symmetric functions and highest weight representations*, Submitted. ArXiv:1810.03448 (September 2018), 35 pages.

[2] Simon Peyton Jones et al., *The Haskell 98 language and libraries: The revised report*, Journal of Functional Programming **13** (2003), no. 1, 0–255, `http://www.haskell.org/definition/`.