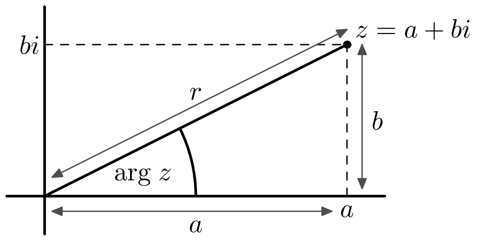
Complex Numbers: Extra examples and questions

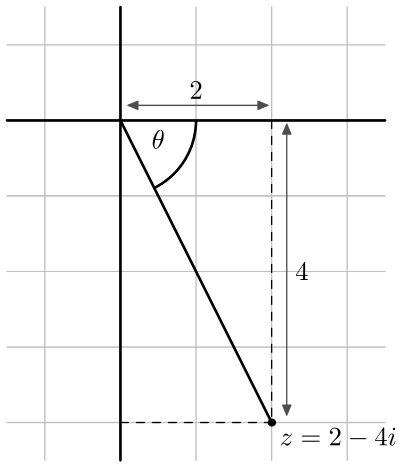
Here are some extra examples and questions on Part A, Complex Numbers. Answers will be put on Moodle on 1st November. Please see the lecturer if you would like to discuss anything on this sheet.

CARTESIAN, POLAR AND EXPONENTIAL FORMS. Let $z = a + bi \in \mathbb{C}$. The *Cartesian form* of z is $a+bi$. The polar form of z is $r(\cos\theta+i\sin\theta)$ where $r = |z|$ and $\theta = \arg z$. The *exponential form* is $re^{i\theta}$, with r and θ as in the polar form. So it is easy to convert between polar and exponential form. See pages 6 and 7 of the printed notes.



If $z = a + bi$ and $a > 0$ and $b > 0$ then $\arg z = \tan^{-1}(b/a)$. This should be clear from the diagram above. **Note that this formula only holds if $a > 0$ and $b > 0$.**

Example 1. Let $z = 2 - 4i \in \mathbb{C}$. We have $|z| = \sqrt{2^2 + 4^2} = \sqrt{20} = 2\sqrt{5}$. To find $\arg z$ it is best to start with a diagram.



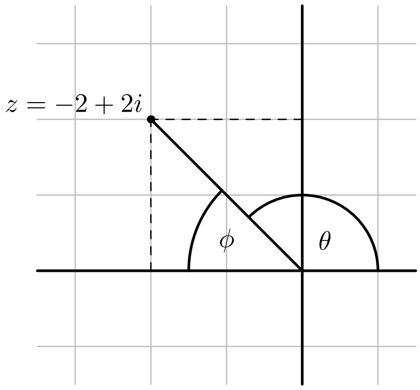
From the triangle shown, we see that $\tan\theta = 4/2 = 2$. So we have $\theta = \tan^{-1} 2$. But because the angle is clockwise (negative) from the real axis, we pick up a minus sign, so $\arg z = -\theta = -\tan^{-1} 2$.

In polar form

$$\begin{aligned} z &= 2\sqrt{5}(\cos(-\theta) + i\sin(-\theta)) \\ &= 2\sqrt{5}(\cos\theta - i\sin\theta) \end{aligned}$$

and in exponential form $z = 2\sqrt{5}e^{-i\theta}$.

Example 2. Let $w = -2 + 2i \in \mathbb{C}$. We have
$$|w| = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}.$$



The triangle shown with dashed lines has equal sides, so $\phi = \frac{\pi}{4}$ and $\theta = \frac{3\pi}{4}$. This time the angle is anticlockwise (positive) from the real axis so $\arg z = \theta$ and in polar form $z = 2\sqrt{2}(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})$. In exponential form $z = 2\sqrt{2}\mathrm{e}^{3\pi i/4}$.

Example 3. Let $z = 3\mathrm{e}^{5\pi i/4}$. In polar form $z = 3\cos\frac{5\pi}{4} + 3i\sin\frac{5\pi}{4}$. Since
$$\cos\frac{5\pi}{4} = \cos\left(\pi + \frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$$
and
$$\sin\frac{5\pi}{4} = \sin\left(\pi + \frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = -\frac{\sqrt{2}}{2},$$
the Cartesian form of z is
$$z = -\frac{3\sqrt{2}}{2} - i\frac{3\sqrt{2}}{2}.$$

Note that since $\frac{5\pi i}{4} - 2\pi = -\frac{3\pi i}{4}$, another value of the argument of z is $-\frac{3\pi i}{4}$, and so it is also correct to write $z = \mathrm{e}^{-3\pi i/4}$. The principal argument of a complex number, written $\mathrm{Arg}(z)$, is the unique argument such that $-\pi < \mathrm{Arg}\, z \leq \pi$. (See Definition 1.10.) So here we have $\mathrm{Arg}\, z = -\frac{3\pi i}{4}$.

Questions. See also Question 3 on Sheet 2.

1. Write $-1 - i$ in polar and exponential forms.

2. Let $\phi = \tan^{-1} 2$. Plot $1 + 2i$, $-2 + i$, $-1 - 2i$ and $2 - i$ on an Argand diagram, and convert these numbers to polar form, writing your answers in terms of ϕ.

3. Let $z = \frac{1}{2} - i\frac{\sqrt{3}}{2}$. Write z in polar and exponential forms.

4. What are $\mathrm{Arg}\, i$ and $\mathrm{Arg}(-i)$? Put i and $-i$ in exponential form.

5. Convert $\mathrm{e}^{-\pi i/6}$ to Cartesian form.

Modulus, argument and circles.

Example 4. Let X be the set of $z \in \mathbb{C}$ such that $|z-(2+i)| = 1$. Geometrically, this condition means that z is distance 1 from $2+i$. So X is the circle with centre at $2+i$ and radius 1.

This can also be seen algebraically. Let $z = a+bi$. Then

$$|z-(2+i)| = |a+bi-2-i| = |a-2+(b-1)i| = \sqrt{(a-2)^2+(b-1)^2}.$$

So an equivalent condition for $a+bi$ to be in X is

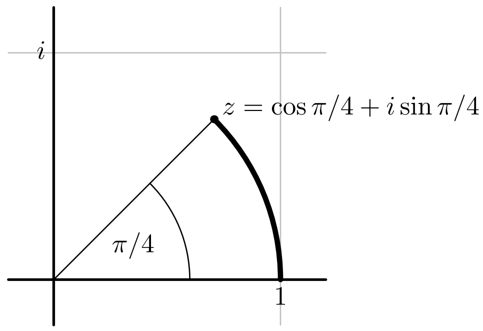
$$(a-2)^2+(b-1)^2 = 1.$$

This is the equation of a circle with centre at $2+i$ and radius 1.

Example 5. Let Y be the set of $z \in \mathbb{C}$ of the form $\mathrm{e}^{i\theta}$ where $0 \le \theta \le \pi/4$. Converting to polar form, we see that Y consists of all complex numbers of the form

$$\cos\theta + i\sin\theta$$

where $0 \le \theta \le \pi/4$. So Y is the arc of the circle with centre 0 and radius 1 shown below.



Questions. See also Question 6 on Sheet 2.

1. Draw on the same Argand diagram the set of all $z \in \mathbb{C}$ such that $|z| = 2$, and the set of all $w \in \mathbb{C}$ such that $|w-2| = 2$.

2. Let T be the set of $z \in \mathbb{C}$ such that $|z| = 1$ and $0 \le \operatorname{Arg} z \le \pi/2$. Draw T on an Argand diagram.

3. Draw the set of complex numbers of the form $1+\mathrm{e}^{-i\theta}$ where $0 \le \theta \le \pi/6$ on an Argand diagram.

Logic and sets: Extra examples and questions

NEGATION. Negation of statements involving $\forall$ and $\exists$ was discussed on page 26 of the printed notes. To show that a proposition is false, it can sometimes be easier to prove that its negation is true. For example, if R is the proposition in Question 5(c) of Sheet 6:

$$(\exists n \in \mathbb{N})(\forall m \in \mathbb{N})(m \text{ divides } n)$$

then $\neg R$ is logically equivalent to

$$(\forall n \in \mathbb{N})\neg(\forall m \in \mathbb{N})(m \text{ divides } n)$$

which in turn is logically equivalent to

$$(\forall n \in \mathbb{N})(\exists m \in \mathbb{N})\neg(m \text{ divides } n)$$

and so to

$$(\forall n \in \mathbb{N})(\exists m \in \mathbb{N})(m \text{ does not divide } n).$$

Now $\neg R$ is true, because given $n \in \mathbb{N}$, if we take $m = n + 1$ (or any other number greater than m), then m does not divide n.

1. Negate the following propositions

(a) $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(y^2 = x)$
(b) $(\forall x \geq 0)(\exists y \in \mathbb{R})(y^2 = x)$
(c) $(\forall x \in \mathbb{R})(\exists n \in \mathbb{N})(n \geq x)$
(d) $(\exists n \in \mathbb{N})(\forall x \in \mathbb{R})(n \geq x)$

Which are true are which are false? Justify your answers.

PROPOSITIONS AND TRUTH TABLES. Recall from Exercise 5.8 that $A \implies B$ is logically equivalent to $(\neg A) \vee B$; that is $A \implies B$ is true if and only if either A is false or B is true. The truth table is shown below.

A	B	$A \implies B$
T	T	T
T	F	F
F	T	T
F	F	T

For more examples of truth tables see pages 27 and 28 of the printed notes or Chapter 6 of *How to Think Like a Mathematician.*

2. Recall that a proposition is a tautology if it is always true. Using truth tables, or by arguing directly (see the answer to Question 4 on Sheet 6 for an example of this approach) decide which of the following propositions are tautologies:

(i) $P \implies (Q \implies P)$,
(ii) $(P \implies Q) \implies P$,
(iii) $(P \implies Q) \implies \big((Q \implies R) \implies (P \implies R)\big)$.

3. Let P, Q, R be propositions. Let M be the proposition below

$$(P \wedge Q) \vee (Q \wedge R) \vee (R \wedge P).$$

(a) Show that M is true if and only if at least two of P, Q and R are true.

(b) Show that M is logically equivalent to

$$(P \vee Q) \wedge (Q \vee R) \wedge (R \vee P).$$

4. Show that the following propositions formed from propositions P, Q and R are logically equivalent:

(a) $(P \implies Q)$ and $(\neg Q \implies \neg P)$ [**corrected Q to $\neg Q$ on 29th November**]

(b) $\neg(P \vee Q \vee R)$ and $\neg P \wedge \neg Q \wedge \neg R$

(c) $\neg(P \vee Q) \vee R$ and $\neg\big((P \wedge Q) \wedge \neg R\big)$

The first is used in proof by the contrapositive: to show $P \implies Q$ you can instead show that $\neg Q \implies \neg P$ (see the bottom of page 25 of the printed notes).

Sets. Sets were introduced on page 4 and were the subject of §6.

5. Let X be the set $\big\{1, \pi, \{42, \sqrt{2}\}, \{\{1, 3\}\}\big\}$. Decide which of the following statements are true and which are false.

(i) $\pi \in X$;

(ii) $\{\pi\} \notin X$;

(iii) $\{42, \sqrt{2}\} \in X$;

(iv) $\{1\} \subseteq X$;

(v) $\{1, \sqrt{2}\} \subseteq X$;

(vi) $\{1, \pi\} \subseteq X$;

(vii) $(\exists A \in X)(1 \in A)$;

(viii) $\{1, 3\} \subseteq X$;

(ix) $\{1, 3\} \in X$

(x) $(\exists A \in X)(\{1, 3\} \in A)$;

6. Define subsets X, Y and Z of the natural numbers as follows:

$$X = \{n \in \mathbb{N} : 6 \mid (n-1)\}$$
$$Y = \{n \in \mathbb{N} : 3 \mid (n-1)\}$$
$$Z = \{n \in \mathbb{N} : 3 \mid (n^2-1)\}$$

Show that $X \subseteq Y$ and $Y \subseteq Z$. Deduce that $X \subseteq Z$.

Functions. Recall from Definition 7.2 that a function $f : X \to Y$ is

(i) *injective* if for each $y \in Y$ there exists *at most one* $x \in X$ such that $f(x) = y$,

(ii) *surjective* if for each $y \in Y$ there exists some $x \in X$ such that $f(x) = y$,

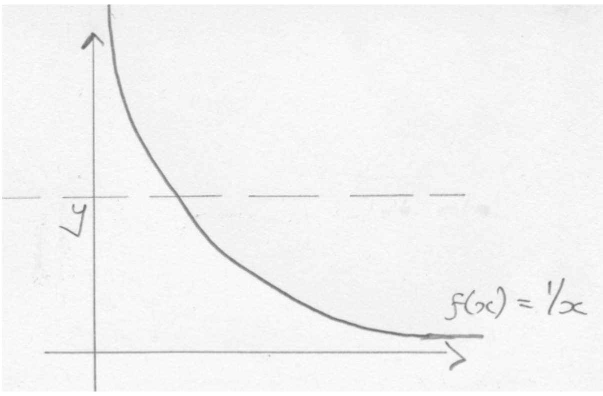
(iii) *bijective* if f is injective and surjective.

Equivalently, f is injective if

$$f(x) = f(x') \Longrightarrow x = x'$$

and bijective if for each $y \in Y$ there exists a unique $x \in X$ such that $f(x) = y$.

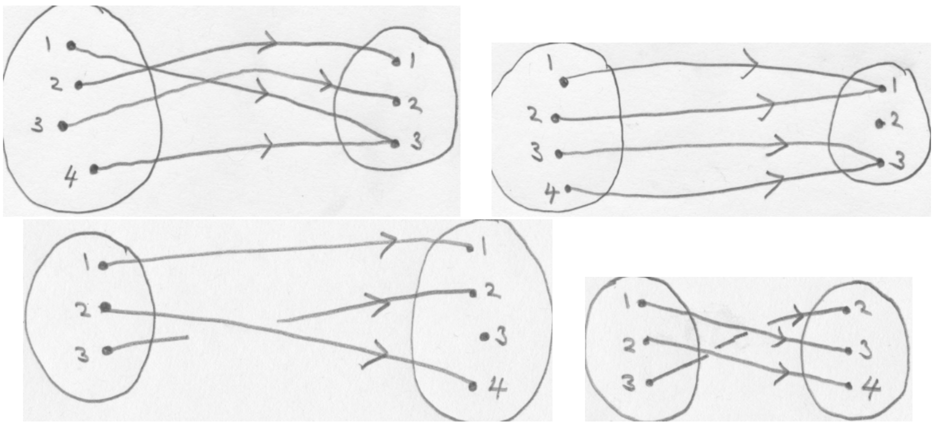
These properties can be recognized from the graph of a function. For example, let $\mathbb{R}^{>0} = \{x \in \mathbb{R} : x > 0\}$ and consider $f : \mathbb{R}^{>0} \to \mathbb{R}^{>0}$ defined by $f(x) = 1/x$. The graph is shown below.



Since for each $y \in \mathbb{R}^{>0}$ the horizontal line of height y meets the graph at a unique point, the function is bijective.

If instead we define $g : \mathbb{R}^{>0} \to \mathbb{R}$ by $g(x) = 1/x$ then g has the same graph as f, but g is no longer surjective. For instance, -1 is in the codomain of g and $g(x) \neq -1$ for any $x \in \mathbb{R}^{>0}$. Correspondingly, the horizontal line of height -1 does not meet the graph above.

7. For each of the diagrams below decide whether the function it represents is (1) injective, (2) surjective, (3) bijective.

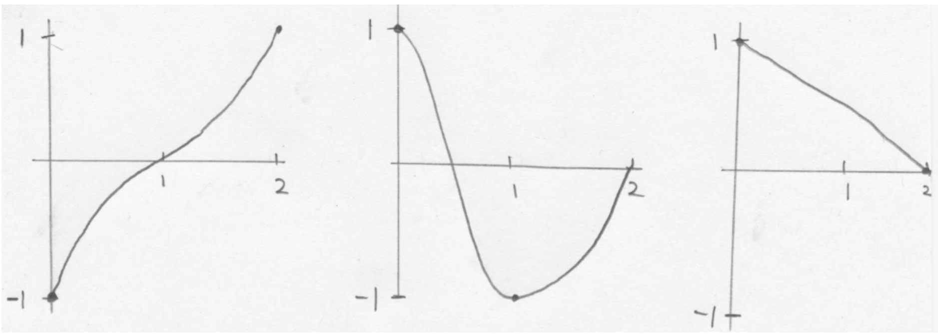


For example, the top left diagram shows the function

$$f : \{1, 2, 3, 4\} \to \{1, 2, 3\}$$

defined by $f(1) = 3$, $f(2) = 1$, $f(3) = 2$, $f(4) = 3$.

8. The graphs below show functions $f : [0, 2] \to [-1, 1]$. Decide for each each graph whether the function it shows is (1) injective, (2) surjective, (3) bijective.



9. Let $f : X \to Y$ be a function. In symbols, the condition that f is surjective is

$$(\forall y \in Y)(\exists x \in X)(f(x) = y).$$

Write down the negation of this proposition.

10. Let $f : [0, \infty) \to [1, \infty)$ be the function defined by $f(x) = (x + 1)^3$.

(a) What is the domain of f? What is the codomain of f?

(b) Show that f is injective. Start your answer:

'suppose that $x, x' \in [0, \infty)$ and $f(x) = f(x')$. Then ...'

(c) Show that f is surjective.

(d) What are the domain and codomain of the inverse function f^{-1}?

(e) Find a formula for $f^{-1}(y)$ where y is in the domain of f^{-1}.

11. Let $X = \{x \in \mathbb{R} : x \neq -1\}$ and let $Y = \{y \in \mathbb{R} : y \neq 0\}$. Let $g : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$g(x) = \frac{1}{x + 1}$$

Show that $g : X \to Y$ is bijective and find a formula for $g^{-1} : Y \to X$.