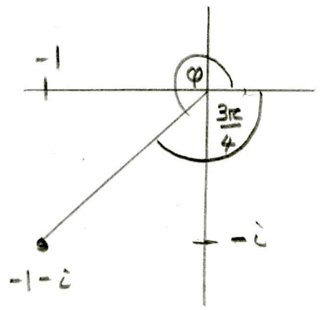
Complex Numbers: Answers to Revision Questions

Here are some answers to the questions on the sheet of revision examples and questions.

CARTESIAN, POLAR AND EXPONENTIAL FORMS.

1. *Write $-1-i$ in polar and exponential forms.*

First draw $-1-i$ on an Argand diagram to see roughly where it is.



The modulus of $-1-i$ is $\sqrt{(-1)^2+(-1)^2}=\sqrt{2}$. The angle θ on the diagram is $3\pi/4$ (one right-angle plus one-half of a right-angle). Since it is measured clockwise from the real axis, we get a minus sign. So $\arg(-1-i)=-3\pi/4$.

Hence

$$-1-i=\sqrt{2}\big(\cos(-3\pi/4)+i\sin(-3\pi/4)\big).$$

From the polar form it is trivial to convert to exponential form:

$$-1-i=\sqrt{2}\mathrm{e}^{-3i\pi/4}.$$

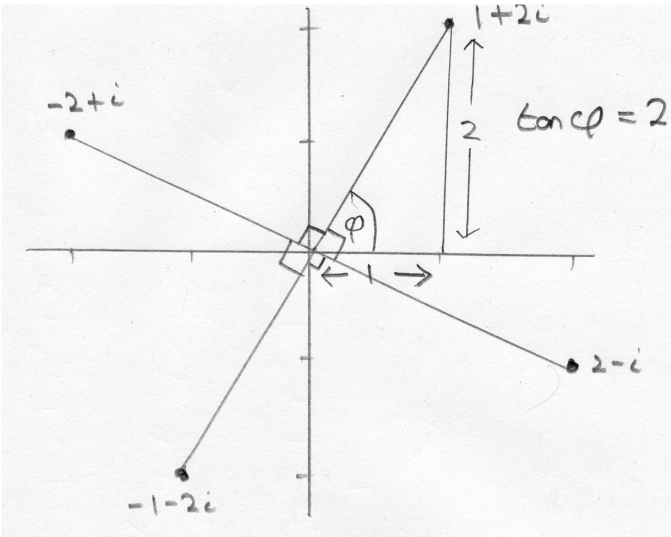
Note it would also be correct to write

$$-1-i=\sqrt{2}\big(\cos(5\pi/4)+i\sin(5\pi/4)\big)=\sqrt{2}\mathrm{e}^{5i\pi/4}.$$

You would get this expression if you instead used the angle ϕ, which is $5\pi/4$, measured anticlockwise from the real axis.

2. *Let $\phi=\tan^{-1}2$. Plot $1+2i$, $-2+i$, $-1-2i$ and $2-i$ on an Argand diagram, and convert these numbers to polar form, writing your answers in terms of ϕ.*

See the diagram below.



From this diagram we see that all the numbers have modulus $\sqrt{5}$ and that

$$\begin{aligned}\arg(1+2i) &= \phi\\ \arg(-2+i) &= \phi+\pi/2\\ \arg(-1-2i) &= \phi+\pi\\ \arg(2-i) &= \phi+3\pi/2.\end{aligned}$$

Hence

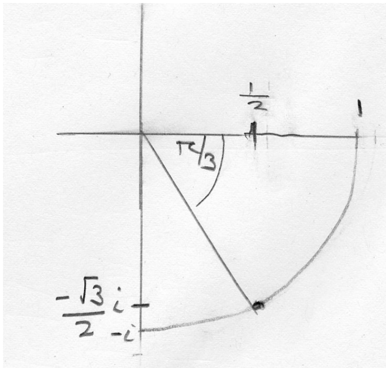
$$\begin{aligned}1+2i &= \sqrt{5}\big(\cos\phi + i\sin\phi\big)\\ -2+i &= \sqrt{5}\big(\cos(\phi+\pi/2) + i\sin(\phi+\pi/2)\big)\\ -1-2i &= \sqrt{5}\big(\cos(\phi+\pi) + i\sin(\phi+\pi)\big)\\ 2-i &= \sqrt{5}\big(\cos(\phi+3\pi/2) + i\sin(\phi+3\pi/2)\big).\end{aligned}$$

Again you could give different (but equivalent) answers by taking a different choice for the argument. For example, the principal argument of $2-i$ is $\phi-\pi/2$, so

$$2-i = \sqrt{5}\big(\cos(\phi-\pi/2) + i\sin(\phi-\pi/2)\big).$$

3. Let $z = \frac{1}{2} - i\frac{\sqrt{3}}{2}$. *Write z in polar and exponential forms.*

The modulus of z is $\sqrt{(\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} = \sqrt{\frac{1}{4}+\frac{3}{4}} = \sqrt{1} = 1$. From the diagram below we see that if $\arg z = \theta$ then $\cos\theta = 1/2$ (adjacent side is $1/2$, hypoteneuse is 1), so $\theta = \pi/3$.



Since θ is measured clockwise (negative) from the real axis, $\arg z = -\pi/3$. Hence

$$z = \cos(-\pi/3) + i\sin(-\pi/3) = \mathrm{e}^{-i\pi/3}.$$

It would be better style to tidy up the minus signs in the cosine and sine and write $z = \cos(\pi/3) - i\sin(\pi/3)$.

4. *What are* Arg i *and* Arg$(-i)$*? Put* i *and* $-i$ *in exponential form.*

Note that when written with a capital 'A', Arg refers to the principal argument, chosen so that $-\pi < \operatorname{Arg} z \leq \pi$.

Here $\operatorname{Arg} i = \pi/2$ and $\operatorname{Arg} -i = -\pi/2$. Hence $i = \mathrm{e}^{i\pi/2}$ and $-i = \mathrm{e}^{-i\pi/2}$.

5. *Convert* $\mathrm{e}^{-\pi i/6}$ *to Cartesian form.*

By definition of the exponential function (see Definition 2.1),

$$\mathrm{e}^{-\pi/6} = \cos\big(-\frac{\pi}{6}\big) + i\sin\big(-\frac{\pi}{6}\big).$$

Now using that $\cos(-\theta) = -\cos\theta$ and $\sin(-\theta) = -\sin\theta$ we get

$$\mathrm{e}^{-\pi/6} = \cos\frac{\pi}{6} - i\sin\frac{\pi}{6}.$$

You should know the values of cos and sin for the angles $0, \pi/6, \pi/4, \pi/3, \pi/2$. Using the values for $\pi/6$ gives the simpler form

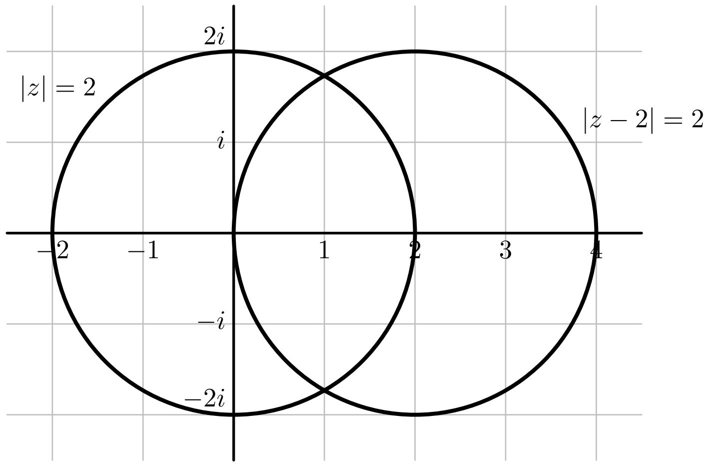
$$\mathrm{e}^{-\pi/6} = \frac{\sqrt{3}}{2} - \frac{i}{2}.$$

Modulus, argument and circles.

1. *Draw on the same Argand diagram the set of all $z \in \mathbb{C}$ such that $|z| = 2$, and the set of all $w \in \mathbb{C}$ such that $|w - 2| = 2$.*

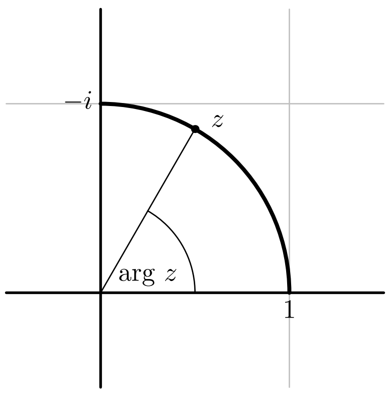
The complex numbers of modulus 2 are exactly the complex numbers on the circle of radius 2 about 0.

The condition $|w - 2| = 2$ means that w is distance 2 from the point $2 + 0i$ on the Argand diagram. So we get a circle of radius 2 about 2.



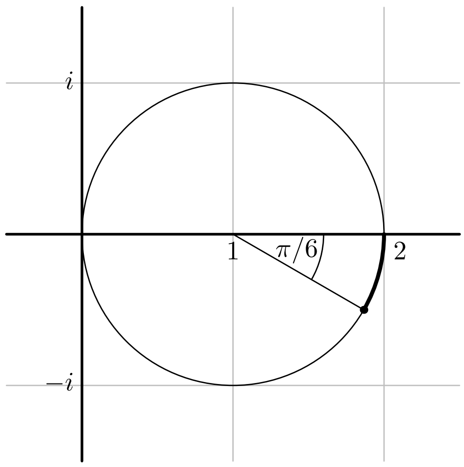
2. *Let T be the set of $z \in \mathbb{C}$ such that $|z| = 1$ and $0 \leq \operatorname{Arg} z \leq \pi/2$. Draw T on an Argand diagram.*

The condition $|z| = 1$ means that z is on the unit circle; the condition $0 \leq \operatorname{Arg} z \leq \pi/2$ means that the marked angle θ is between 0 and $\pi/2$. Therefore we get the arc shown below.



3. *Draw the set of complex numbers of the form* $1 + e^{-i\theta}$ *where* $0 \leq \theta \leq \pi/6$ *on an Argand diagram.*

The points of the form $1 + e^{-i\theta}$ are distance 1 from 1, so lie on the circle of radius 1 with centre 1, shown in the diagram below. As θ varies from 0 to $\pi/6$ we get the thick arc.



Logic and sets: Answers to revision questions

Here are some answers to the questions on the sheet of revision examples and questions.

1. Negate the following propositions

(a) $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(y^2 = x)$
(b) $(\forall x \geq 0)(\exists y \in \mathbb{R})(y^2 = x)$
(c) $(\forall x \in \mathbb{R})(\exists n \in \mathbb{N})(n \geq x)$
(d) $(\exists n \in \mathbb{N})(\forall x \in \mathbb{R})(n \geq x)$

Which are true are which are false? Justify your answers.

Using the recommended method on page 27 of the printed lecture notes, we find that the negation of (a) is

$$(\exists x \in \mathbb{R})\neg(\exists y \in \mathbb{R})(y^2 = x)$$

which is logically equivalent to

$$(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})\neg(y^2 = x).$$

Finally we replace $\neg(y^2 = x)$ with the easier to read $(y^2 \neq x)$. The other negations are found similarly:

$$\neg\text{(a)}\ (\exists x \in \mathbb{R})(\forall y \in \mathbb{R})(y^2 \neq x)$$
$$\neg\text{(b)}\ (\exists x \geq 0)(\forall y \in \mathbb{R})(y^2 \neq x)$$
$$\neg\text{(c)}\ (\exists x \in \mathbb{R})(\forall n \in \mathbb{N})(n < x)$$
$$\neg\text{(d)}\ (\forall n \in \mathbb{N})(\exists x \in \mathbb{R})(n < x)$$

- Since $(\forall y \in \mathbb{R})(y^2 \neq -1)$ is true, the negation of (a), namely $(\exists x \in \mathbb{R})(\forall y \in \mathbb{R})(y^2 \neq x)$, is true. Hence (a) is false.
- Given any $x \geq 0$ there exists $y \in \mathbb{R}$ such that $y^2 = x$, since we can take $y = \sqrt{x} \in \mathbb{R}$. Hence (b) is true.
- Given any $x \in \mathbb{R}$ there is a natural number n such that $n \geq x$. (For example, if $x = m + \theta$ where $0 \leq \theta < 1$ then take $n = m + 1$.) So (c) is true.
- The negation of (d) is true, since given any $n \in \mathbb{N}$, there exists $x \in \mathbb{R}$ such that $n < x$. (For example, take $x = n + 1$.) Hence (d) is false.

PROPOSITIONS AND TRUTH TABLES.

2. Recall that a proposition is a tautology if it is always true. Using truth tables, or by arguing directly (see the answer to Question 4 on Sheet 6 for an example of this approach) decide which of the following propositions are tautologies:

(i) $P \implies (Q \implies P)$,
(ii) $(P \implies Q) \implies P$,
(iii) $(P \implies Q) \implies \big((Q \implies R) \implies (P \implies R)\big)$.

(a) The truth table for $P \implies (Q \implies P)$ is shown below.

P	Q	$Q \implies P$	$P \implies (Q \implies P)$
T	T	T	T
T	F	T	T
F	T	F	T
F	F	T	T

Since all the entries in the column for $P \implies (Q \implies P)$ are T, $P \implies (Q \implies P)$ is a tautology.

(b) The truth table for $(P \implies Q) \implies P$ is shown below.

P	Q	$P \implies Q$	$(P \implies Q) \implies P$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	F

We see that $(P \implies Q) \implies P$ is false when P is false and Q is true, and also when P is false and Q is true. So $(P \implies Q) \implies P$ is not a tautology.

(c) Let A be the proposition

$$(P \implies Q) \implies \big((Q \implies R) \implies (P \implies R)\big)$$

One way to show that A is a tautology is to use a truth table with eight rows corresponding to the eight possible truth values for P, Q and R, as shown below.

P	Q	R	$P \Longrightarrow Q$	$Q \Longrightarrow R$	$P \Longrightarrow R$	$(Q \Longrightarrow R) \Longrightarrow (P \Longrightarrow R)$	A
T	T	T	T	T	T	T	T
T	T	F	T	F	F	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

It is also possible to argue directly. Suppose, for a contradiction that

$$(P \Longrightarrow Q) \Longrightarrow \big((Q \Longrightarrow R) \Longrightarrow (P \Longrightarrow R)\big)$$

is false. Then, since $A \Longrightarrow B$ is false if and only if A is true and B is false, we see that $(Q \Longrightarrow R) \Longrightarrow (P \Longrightarrow R)$ is false. Hence $(Q \Longrightarrow R)$ is true and $(P \Longrightarrow R)$ is false, so P is true, R is false and Q is false. But then $P \Longrightarrow Q$ is false, so the original proposition is true, a contradiction.

Intuitive meaning: one interpretation of A is as follows: suppose we can prove that $P \Longrightarrow Q$. Then if we can also prove that $Q \Longrightarrow R$, we have that $P \Longrightarrow R$.

3. Let P, Q, R be propositions. Let M be the proposition below

$$(P \wedge Q) \vee (Q \wedge R) \vee (R \wedge P).$$

(a) Show that M is true if and only if at least two of P, Q and R are true.

(b) Show that M is logically equivalent to

$$(P \vee Q) \wedge (Q \vee R) \wedge (R \vee P).$$

(a) M is true if and only if at least one of $P \wedge Q$, $Q \wedge R$ and $R \wedge P$ is true. This is the case if and only if at least two of P, Q, R are true.

(b) Similarly, the proposition $(P \vee Q) \wedge (Q \vee R) \wedge (R \vee P)$ if and only if at least two of P, Q and R are true. So M is true if and only if $(P \vee Q) \wedge (Q \vee R) \wedge (R \vee P)$ is true.

Remark: this could also be done with an eight-row truth table, showing that the columns for M and $(P \vee Q) \wedge (Q \vee R) \wedge (R \vee P)$ are equal.

4. Show that the following propositions formed from propositions P, Q and R are logically equivalent:

(a) $(P \implies Q)$ and $(\neg Q \implies \neg P)$ [**corrected** Q **to** $\neg Q$ **on 29th November**]

(b) $\neg(P \vee Q \vee R)$ and $\neg P \wedge \neg Q \wedge \neg R$

(c) $\neg(P \vee Q) \vee R$ and $\neg\big((P \wedge Q) \wedge \neg R\big)$

The first is used in proof by the contrapositive: to show $P \implies Q$ you can instead show that $\neg Q \implies \neg P$ (see the bottom of page 25 of the printed notes).

(a) From the truth table below we see that $P \implies Q$ is true if and only if $\neg Q \implies \neg P$ is true.

P	Q	$P \implies Q$	$\neg Q$	$\neg P$	$\neg Q \to \neg P$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Alternatives: another good way to (a) is to use that $P \implies Q$ is logically equivalent to $\neg P \vee Q$. (Equivalently $P \implies Q$ is only false if P is true and Q is false.) So Hence

$$(P \implies Q) \iff (\neg P \vee Q) \iff (\neg(\neg Q) \vee \neg P) \iff (\neg Q \implies \neg P).$$

Or you use the argument given in lectures for Exercise 5.5.

Sets. Sets were introduced on page 4 and were the subject of §6.

5. Let X be the set $\big\{1, \pi, \{42, \sqrt{2}\}, \{\{1,3\}\}\big\}$. Decide which of the following statements are true and which are false.

(i) $\pi \in X$;

(ii) $\{\pi\} \notin X$;

(iii) $\{42, \sqrt{2}\} \in X$;

(iv) $\{1\} \subseteq X$;

(v) $\{1, \sqrt{2}\} \subseteq X$;

(vi) $\{1, \pi\} \subseteq X$;

(vii) $(\exists A \in X)(1 \in A)$;

(viii) $\{1, 3\} \subseteq X$;

(ix) $\{1, 3\} \in X$

(x) $(\exists A \in X)(\{1, 3\} \in A)$;

(i) True, since π is an element of X.

(ii) False, since $\{\pi\}$ is not an element of X. (It is true that $\{\pi\}$ is a subset of X, by (i), but this is not the same thing!)

(iii) True, since $\{42, \sqrt{2}\}$ is an element of X.

(iv) False, since $\{1\}$ is not an element of X.

(v) False, since $\{1, \sqrt{2}\}$ is not an element of X.

(vi) True, since $1 \in X$ and $\pi \in X$ so $\{1, \pi\} \subseteq X$.
(vii) False. There is no element of X which has 1 as an element. (Compare (x).)
(viii) False: since $3 \notin X$, $\{1, 3\} \nsubseteq X$.
(ix) False: $\{1, 3\}$ is not an element of X.
(x) True: take $A = \{\{1, 3\}\}$. Then $\{1, 3\} \in A$ and $A \in X$.

6. Define subsets X, Y and Z of the natural numbers as follows:

$$X = \{n \in \mathbb{N} : 6 \mid (n-1)\}$$
$$Y = \{n \in \mathbb{N} : 3 \mid (n-1)\}$$
$$Z = \{n \in \mathbb{N} : 3 \mid (n^2-1)\}$$

Show that $X \subseteq Y$ and $Y \subseteq Z$. Deduce that $X \subseteq Z$.

To show that $X \subseteq Y$ we must show that if $n \in X$ then $n \in Y$. A chain of implications is often a good way to do this:

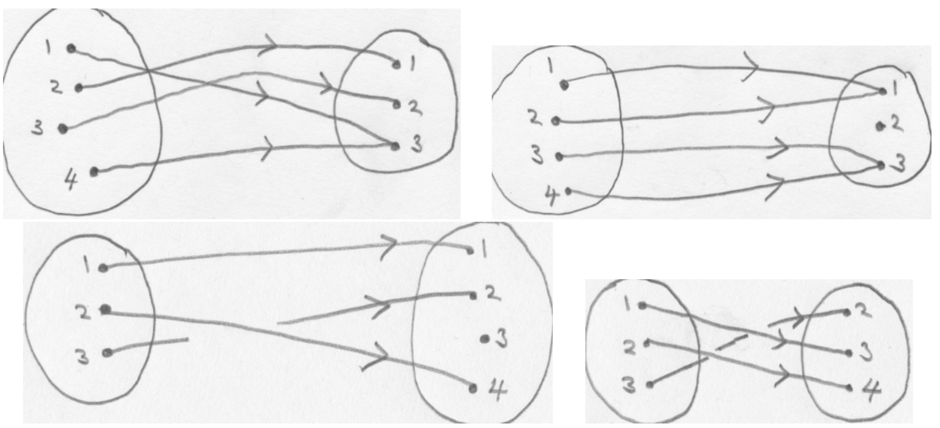
$$n \in X \implies 6 \mid (n-1) \implies n-1 = 6k \text{ for some } k \in \mathbb{Z}$$
$$\implies n-1 = 3(2k) \text{ for some } k \in \mathbb{Z} \implies 3 \mid (n-1) \implies n \in Y.$$

Similarly, if $n \in Y$ then $3 \mid (n-1)$ so $n-1 = 3k$ for some $k \in \mathbb{Z}$. But $n^2-1 = (n-1)(n+1)$ so $n^2-1 = 3k(n+1)$ and so $3 \mid n^2-1$ and hence $n \in Z$. Hence $Y \subseteq Z$.

We have $X \subseteq Y \subseteq Z$ and so $X \subseteq Z$. (The relation $\subseteq$ is transitive.)

FUNCTIONS.

7. For each of the diagrams below decide whether the function it represents is (1) injective, (2) surjective, (3) bijective.



For example, the top left diagram shows the function

$$f : \{1, 2, 3, 4\} \to \{1, 2, 3\}$$

defined by $f(1) = 3$, $f(2) = 1$, $f(3) = 2$, $f(4) = 3$.

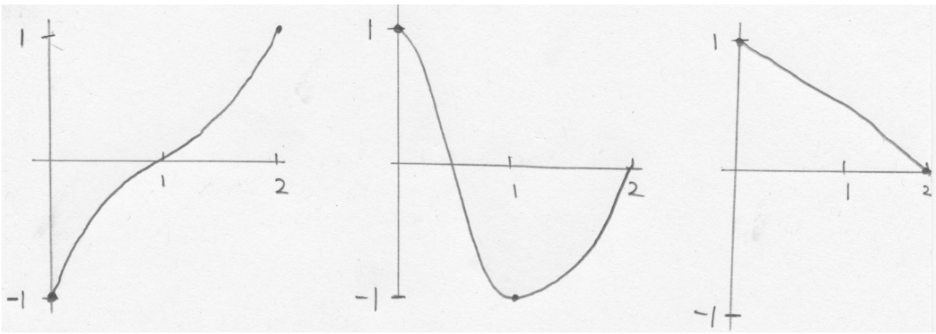
Top left: (1) not injective since $f(1) = f(4) = 3$; (2) surjective since for all $y \in \{1, 2, 3\}$ there exists $x \in \{1, 2, 3, 4\}$ such that $f(x) = y$ (specifically: $f(2) = 1$, $f(3) = 2$, $f(4) = 3$); (3) not bijective, since not injective.

Top right: (1) not injective since $f(1) = f(2) = 1$; (2) not surjective since there is no $x \in \{1, 2, 3, 4\}$ such that $f(x) = 2$; (3) not bijective since not injective.

Bottom left: (1) injective; (2) not surjective since there is no $x \in \{1, 2, 3, 4\}$ such that $f(x) = 3$; not bijective since not surjective.

Bottom right: (1) injective; (2) surjective; (3) hence bijective.

8. The graphs below show functions $f : [0, 2] \to [-1, 1]$. Decide for each each graph whether the function it shows is (1) injective, (2) surjective, (3) bijective.



Left function: (1) injective, (2) surjective and so (3) bijective.

Middle function: (1) not injective: for example there exists $x < 1$ such that $f(x) = f(2) = 0$; (2) surjective; (3) not bijective since not injective.

Right function: (1) injective; (2) not surjective, for instance the horizontal line through $-1/2$ does not meet the graph so there is no $x \in [0, 2]$ such that $f(x) = -1/2$; (3) not bijective since not surjective.

9. Let $f : X \to Y$ be a function. In symbols, the condition that f is surjective is

$$(\forall y \in Y)(\exists x \in X)(f(x) = y).$$

Write down the negation of this proposition.

Following the method on page 27 of the printed lecture notes we get

$$(\exists y \in Y) \not(\exists xinX)(f(x) = y)$$

which is logically equivalent to

$$(\exists y \in Y)(\forall x \in X) \not(f(x) = y)$$

which we can rewrite as

$$(\exists y \in Y)(\forall x \in X)(f(x) \neq y).$$

10. Let $f : [0, \infty) \to [1, \infty)$ be the function defined by $f(x) = (x + 1)^3$.

(a) What is the domain of f? What is the codomain of f?

(b) Show that f is injective. Start your answer:
'suppose that $x, x' \in [0, \infty)$ and $f(x) = f(x')$. Then ...'

(c) Show that f is surjective.

(d) What are the domain and codomain of the inverse function f^{-1}?

(e) Find a formula for $f^{-1}(y)$ where y is in the domain of f^{-1}.

(a) The domain of f is $[0, \infty)$ and the codomain of f is $[1, \infty)$.

(b) Suppose that $x, x \in [0\infty)$ and $f(x) = f(x')$. Then $(x + 1)^3 = (x' + 1)^3$ and since each real number has a unique real cube root, $x + 1 = x' + 1$. Hence $x = x'$. Therefore f is injective.

(c) Let $y \in [1, \infty)$. We want to find $x \in [0, \infty)$ such that $f(x) = y$. Now

$$(x + 1)^3 = y \iff x + 1 = \sqrt{y} \iff x = \sqrt[3]{y} - 1$$

so we can take $x = \sqrt[3]{y} - 1$.

(d) The inverse function has domain $[1, \infty)$ (the codomain of f) and codomain $[0, \infty)$ (the domain of f).

(e) By (c) we have $f^{-1}(y) = \sqrt[3]{y} - 1$.

11. Let $X = \{x \in \mathbb{R} : x \neq -1\}$ and let $Y = \{y \in \mathbb{R} : y \neq 0\}$. Let $g : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$g(x) = \frac{1}{x + 1}$$

Show that $g : X \to Y$ is bijective and find a formula for $g^{-1} : Y \to X$.

We first show that g is injective. Suppose that $x, x' \in X$ and $g(x) = g(x')$. Then $1/(x + 1) = 1/(x' + 1)$, so inverting we get $x + 1 = x' + 1$. Hence $x = x'$. Now let $y \in Y$. We have

$$g(x) = y \iff \frac{1}{x + 1} = y \iff x + 1 = \frac{1}{y} \iff x = \frac{1}{y} - 1.$$

(Note that since $y \neq 0$, it is okay to invert $1/(x + 1) = y$.) Hence $g(1/y - 1) = y$ and g is surjective. This also shows that the inverse function is

$$g^{-1}(t) = \frac{1}{y} - 1.$$