MT181 Number Systems: Sheet 1

Attempt all numbered questions. Please staple your answers together and put your name and student number on the top sheet. Do not return the problem sheet.

The lecturer will be happy to discuss any of the questions in office hours.

To be handed it at the lecture on Tuesday 8th October.

1. Read the preface and Chapter 1 of *How to think like a mathematician* by Kevin Houston (Cambridge University Press, 2009).

2. Let $U = \{0, 1, 2, 3, 4, 5, 6, 7\}$ and let

$$X = \{1, 3, 5, 7\}, \quad Y = \{2, 3, 6, 7\}, \quad Z = \{4, 5, 6, 7\}.$$

(a) Draw a Venn diagram, like the one on page 8 of the lecture notes, showing X, Y, Z as subsets of U. (A suitable configuration of circles is overleaf.)

(b) Write down the members of $(X \cup Y) \cap Z$.

(c) Express each of the sets

$$\text{(i) } \{3, 7\}, \quad \text{(ii) } \{7\}, \quad \text{(iii) } \{3, 4, 5, 6, 7\}, \quad \text{(iv) } \{3, 5, 6, 7\}$$

in terms of X, Y and Z using only intersection $\cap$ and union $\cup$. (You may also use brackets, as in (b), to make clear the order of operations.)

(d) Write down the members of X', the complement of X in U.

(e) Express the set $\{1, 2, 4\}$ in terms of X, Y and Z using intersection $\cap$, union $\cup$, and complements in U.

3. Give 'Yes' or 'No' answers to the following questions. If the answer is 'Yes' give a short justification. If it is 'No', prove this by giving an appropriate example.

Let $X = \{\ldots, -3, 1, 1, 3, \ldots\}$ be the set of odd integers.

(a) Is X closed under addition?

(b) Is X closed under multiplication? [*Hint: for a rigorous proof, take* $x = 2m+1$ *and* $y = 2n + 1$, *where* m, $n \in \mathbb{Z}$, *and multiply out* xy ...]

Let $\mathbb{Q}_{>0}$ be the set of rational numbers x such that $x > 0$.

(c) Is $\mathbb{Q}_{>0}$ closed under addition? [*Hint: use that* $\mathbb{Q}$ *is closed under addition.*]

(d) Is $\mathbb{Q}_{>0}$ closed under subtraction?

(e) Is $\mathbb{Q}_{>0}$ is closed under multiplication?

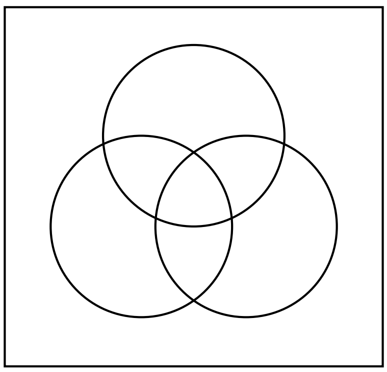
4. Prove the second of De Morgan's laws: if X and Y are subsets of a set U, and complements are taken in U, then $(X \cap Y)' = X' \cup Y'$.

Bonus question: the pond below Founder's Building is populated by 30 fish: 15 are red, 7 are blue and 8 are green. Whenever two fish of different colours meet they each change into fish of the third colour. Whenever two fish of the same colour meet, they change colour so that each has one of the the other two colours.

(For example, if a red and green fish meet, they both become blue, and if two red fish meet, then one becomes blue and the other green.)

It is possible that on some day, all the fish will be red?

Venn diagram for three sets



MT181 Number Systems: Sheet 2

Attempt all numbered questions. Please staple your answers together and put your name and student number on the top sheet. Do not return the problem sheet.

The lecturer will be happy to discuss any of the questions in office hours.

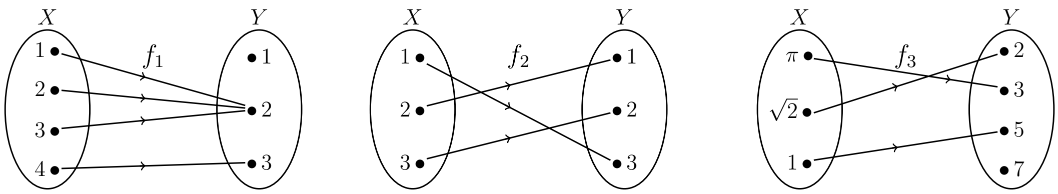
To be handed it at the lecture on Tuesday 15th October.

1. Read Chapters 2 and 3 of *How to think like a mathematician* by Kevin Houston (Cambridge University Press, 2009).

2. For each of the functions f_1, f_2, f_3 shown as a diagram below:

(i) State its domain, codomain and range.

(ii) State which combination of the properties injective, surjective, bijective it has. Give brief reasons for your answers.



3. The *floor function* $f : \mathbb{R} \to \mathbb{Z}$ is defined so that $f(x)$ is the greatest integer n such that $n \leq x$. For example, $f(\pi) = 3$ and $f(1) = 1$.

(a) Write down (i) $f(\sqrt{2})$, (ii) $f(-1/2)$. (Be careful!)

(b) Sketch the graph of the floor function.

(c) Is the floor function (i) injective, (ii) surjective? Give brief reasons.

(The standard notation for the floor of x is $\lfloor x \rfloor$. Please use this if you prefer.)

4. Let $X = \{x \in \mathbb{R} : x \neq 0 \text{ and } x \neq 1\}$. Let $f : X \to X$ be the function defined by

$$f(x) = \frac{1}{1-x}$$

(a) Show that f is bijective and find a formula for $f^{-1} : X \to X$.

(b) Let $g : X \to X$ be the function defined by $g(x) = 1/x$. Simplifying your answers as much possible, find

$$\text{(i) } f\big(f(x)\big), \quad \text{(ii) } f\big(g(x)\big) \quad \text{(iii) } g\big(f(g(x))\big), \quad \text{(iv) } f\big(f(f(x))\big).$$

(c) How many distinct functions can you make by composing f and g in any order?

5. Let X, Y and Z be sets and let $f : X \to Y$ and $g : Y \to Z$ be functions.

(a) Show that if f and g are surjective then gf is surjective.

(b) Show that if gf is surjective then g is surjective.

(c) Give an example where gf is surjective but f is not surjective.

Bonus question: A horizontal stick is one metre long. Fifty ants are placed in random positions on the stick, pointing in random directions. The ants crawl head first along the stick, moving at one metre per minute. If an ant reaches the end of the stick, it falls off. If two ants meet, they both change direction. How long do you have to wait to be sure that all the ants have fallen off the stick?

MT181 Number Systems: Sheet 3

Attempt all numbered questions. Please staple your answers together and put your name and student number on the top sheet. Do not return the problem sheet.

The lecturer will be happy to discuss any of the questions in office hours.

To be handed it at the lecture on Tuesday 22nd October.

1. Read Chapter 4 of *How to think like a mathematician* by Kevin Houston (Cambridge University Press, 2009). Try to put into practice the advice in Chapters 3 and 4 when you write your answers to this problem sheet.

2. Write the following complex numbers in the Cartesian form $a + bi$ and plot them on an Argand diagram.

(a) $z_1 = (4 + 3i) + (1 + i)$

(b) $z_2 = (4 + 3i) - (1 + i)$

(c) $z_3 = (4 + 3i)(1 + i)$

(d) $z_4 = (4 + 3i)/(1 + i)$.

3. Find, in Cartesian form, the solutions to the following equations:

(a) $2z + (3 - 3i) = 1 - i$

(b) $(1 + 3i)w + (1 + i) = 3 + 2i$.

Express your solution to (a) in polar form.

4. Let

$$S = \{a + bi\sqrt{3} : a,\, b \in \mathbb{Q}\}.$$

For example $\frac{1}{3} + 2i\sqrt{3}$ is an element of S, since $\frac{1}{3}$ and 2 are rational numbers.

(a) Show that S is closed under multiplication.

(b) Let $z = a + bi\sqrt{3}$ where $a,\, b \in \mathbb{Q}$. Suppose that $z \neq 0$. Show that $\mathrm{Re}(1/z) = a/(a^2 + 3b^2)$ and find $\mathrm{Im}(1/z)$. Hence show that $1/z \in S$.

(c) Using (a) and (b) show that S is closed under division.

5. Let φ be the angle such that $0 < \varphi < \pi/2$ and $\tan\varphi = 3/4$. For each of the following complex numbers, find $|z|$ and $\mathrm{Arg}(z)$, writing $\mathrm{Arg}(z)$ in terms of φ and π.

(a) $z = 4 + 3i$, (b) $z = -1 - 3i/4$, (c) $z = -3 + 4i$.

6. Find the complex numbers (a) 1, (b) i, (c) i^2, (d) i^3, (e) i^4 in Cartesian form. Hence find i^{2013} in Cartesian form.

Bonus question A: Consider this chain of claimed equalities:

$$-1 = \sqrt{-1}^{\,2} = \sqrt{-1}\sqrt{-1} = \sqrt{(-1)^2} = \sqrt{1} = 1.$$

Where is the mistake?

Bonus question B: Do there exist irrational real numbers x and y such that $x^y \in \mathbb{Q}$? [*Hint:* consider $x = \sqrt{2}^{\sqrt{2}}$ and $x^{\sqrt{2}}$.]

MT181 Number Systems: Sheet 4

Attempt all numbered questions. Please staple your answers together and put your name and student number on the top sheet. Do not return the problem sheet.

The lecturer will be happy to discuss any of the questions in office hours.

To be handed it at the lecture on Tuesday 29th October.

1. Read Chapter 5 of *How to think like a mathematician* by Kevin Houston (Cambridge University Press, 2009).

2. Write the complex numbers $z = 2 - 2i$ and $w = 4i$ in exponential form.

3. (a) Find all solutions $z \in \mathbb{C}$ to the equation $z^6 = 1$ in exponential form.

(b) Express each solution in Cartesian form.

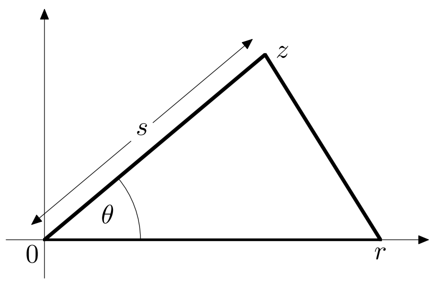
(c) Plot the solutions on an Argand diagram.

4. Let $C = \{z \in \mathbb{C} : |z - 1| = 2\}$. Let $D = \{z \in \mathbb{C} : |z + 1| = 2\}$.

(a) Plot C and D on an Argand diagram. [*Hint:* $|z - 1| = 2 \iff$ the distance between z and 1 is 2.]

(b) Express the elements of $C \cap D$ in Cartesian form.

5. The Argand diagram below shows a triangle with vertices at 0, $r \in \mathbb{R}$ and $z \in \mathbb{C}$. Let s be the length of the side with vertices at 0 and z and let θ be the marked angle.



(a) Express the lengths of the other two sides of the triangle in terms of r and z. [*Hint:* for the side from r to z, Question 4(a) has a relevant idea.]

(b) Show that $z + \overline{z} = 2s\cos\theta$. [*Hint:* what is z in polar form?]

(c) By expanding $|z - r|^2 = (z - r)\overline{(z - r)}$ prove the cosine rule.

6. Let $a, b, c, d \in \mathbb{R}$. Let

$$z = (a + bi)(a - bi)(c + di)(c - di).$$

(a) Show that $z = (a^2 + b^2)(c^2 + d^2)$.

(b) By reordering the factors in the product defining z and then multiplying out, show that $z = (ac - bd)^2 + (ad + bc)^2$.

(c) Given that $137 = 4^2 + 11^2$ and $149 = 7^2 + 10^2$, find *natural numbers* r and s such that

$$r^2 + s^2 = 137 \times 149.$$

Bonus question: You and two of your friends are on your way to a party. At the party, a white or black hat will be put on each person's head. You will be able to see your friends' hats, but not your own.

When the host says 'Go' you may stay silent, or say either 'White' or 'Black'. Then

- if at least one person speaks, and everyone who speaks says the colour of their own hat, you all get some cake;
- if anyone says the opposite colour to their own hat, or everyone stays silent, there is no cake.

You have a few minutes to agree a strategy with your friends. Find a good strategy.

MT181 Number Systems: Sheet 5

Attempt all numbered questions. Please staple your answers together and put your name and student number on the top sheet. Do not return the problem sheet.

The lecturer will be happy to discuss any of the questions in office hours.

To be handed it at the lecture on Tuesday 5th November.

1. Read Chapter 24 on induction of *How to think like a mathematician* by Kevin Houston (Cambridge University Press, 2009).

2. Find the complex numbers z that satisfy the equation $z^2 - (6+2i)z + 8 + 2i = 0$. Write your solutions in the form $a + bi$ where a, $b \in \mathbb{R}$.

3. Find all $z \in \mathbb{C}$ such that $\mathrm{e}^z = i$.

4. Calculate $\sum_{k=1}^{n} k^3$ for $n = 1, 2, 3, 4, 5$. Conjecture a formula for $\sum_{k=1}^{n} k^3$ and prove it by induction on n.

[*Hint:* to spot the pattern, try taking square-roots of the numbers you get. You may assume the result in Example 4.3. You might also find `www.oeis.org` useful.]

5. Use the Principle of Mathematical Induction to show that

(a) $4^n + 5$ is a multiple of 3 for all $n \in \mathbb{N}$.

(b) $2^n \geq 6n$ for all integers n such that $n \geq 5$.

6. Let $n \in \mathbb{N}$ and let $z \in \mathbb{C}$.

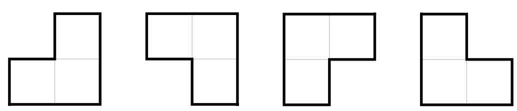
(a) Express $1 + 2z + 3z^2 + 4z^3 + \cdots + (n+1)z^n$ using Sigma notation.

(b) Write $\sum_{k=0}^{n} 3^k$ using the $\cdots$ notation.

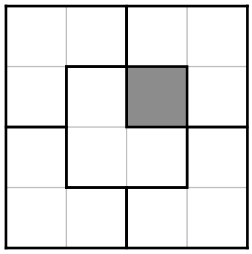
(c) Simplify $\sum_{j=0}^{n-1} (z+j)^j - \sum_{k=1}^{n} (z+k)^k$.

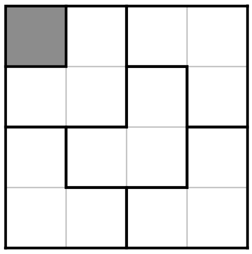
(d) Simplify $\sum_{j=1}^{n} z^j - \sum_{j=1}^{n+1} z^{j-1}$.

Bonus question: You are given a $2^n \times 2^n$ board with one square missing. Show that, no matter which square is missing, it is always possible to tile the board with L-shaped pieces, as shown below.



Two example tilings of 4×4 boards are shown overleaf.





MT181 Number Systems: Sheet 6

Attempt questions 1 to 6. Question 7 is optional. Please staple your answers together and put your name and student number on the top sheet. Do not return the problem sheet.

The lecturer will be happy to discuss any of the questions in office hours.

To be handed it at the lecture on Tuesday 12th November.

1. Read Chapters 15 and 16 of *How to think like a mathematician* by Kevin Houston (Cambridge University Press, 2009).

2. Find the quotient and remainder when n is divided by m in each of these cases:

(i) $n = 42$, $m = 8$ (ii) $n = 43$, $m = 8$ (iii) $n = 8$, $m = 43$ (iv) $n = -43$, $m = 8$.

3. Suppose that when $m, n \in \mathbb{N}$ are divided by 4 they both leave remainder 3. What is the remainder when mn is divided by 4?

4. (a) By adapting the proof of Claim 5.10, prove that $\sqrt[3]{5}$ is irrational.

(b) Write down a proposition that generalizes the result proved in (a). [*Hint:* the end of Chapter 16 of *How to think like a mathematician* may be helpful. There are many correct generalizations.]

5. (a) Let n, r and s be natural numbers all greater than 1 such that

$$n = rs \quad \text{and} \quad r \leq s.$$

By supposing that $r > \sqrt{n}$ and deriving a contradiction, show that $r \leq \sqrt{n}$.

(b) Deduce that if n is a composite number then n is divisible by a prime p such that $p \leq \sqrt{n}$.

(c) Find the prime factorizations of 1327 and 2662.

6. Define a function $d : \mathbb{N} \to \mathbb{N}$ so that $d(n)$ is the number of natural numbers m such that n is divisible by m. For example, 12 is divisible by 1, 2, 3, 4, 6 and 12, so $d(12) = 6$.

(a) Find $d(26)$ and $d(27)$.

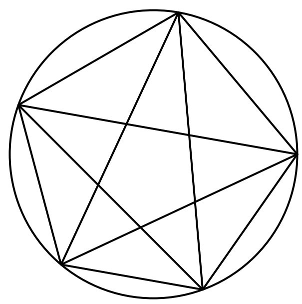
(b) Make a table showing $d(n)$ for each n between 1 and 18.

(c) When is $d(n) = 2$? (Make a general statement that applies to all natural numbers, not just those between 1 and 18.)

(d) Describe, in terms of their prime factorizations, the natural numbers n such that (i) $d(n) = 3$ and (ii) $d(n) = 4$.

7. Prove that there are infinitely many primes of the form $4m+3$ with $m \in \mathbb{N}_0$. [*Hint:* show that any number of the form $4m+3$ with $m \in \mathbb{N}_0$ is divisible by a prime also of this form. Then use a variation on Euclid's argument where N is constructed so that it is of the form $4m+3$.]

Bonus question: Let a_n be the maximum number of regions that can be formed by taking n points on the circumference of a circle, and joining them all up by straight lines. (Do not count the region outside the circle.) For example, the diagram below shows that $a_5 = 16$.



Find a_1, a_2, a_3 and a_4. Make a conjecture about the general pattern. Now test your conjecture by finding a_6.

Make a table showing the differences $a_n - a_{n-1}$, then the second differences, $(a_n - a_{n-1}) - (a_{n-1} - a_{n-2})$, and so on. Use your table to guess a_7. It might help to include the value $a_0 = 1$.

This problem is like Example 4.1: both show the danger of jumping to conclusions from small cases.

MT181 Number Systems: Sheet 7

Attempt all the numbered questions. Please staple your answers together and put your name and student number on the top sheet. Do not return the problem sheet.

The lecturer will be happy to discuss any of the questions in office hours.

The first six problem sheets and Sheets 8 and 9 will be used for the 10% mark awarded for reasonable attempts at the problem sheets. You should however hand in this sheet as usual at the lecture on 19th November.

1. Read Chapters 6 and 7 of *How to think like a mathematician* by Kevin Houston (Cambridge University Press, 2009).

2. Let P, Q and R be propositions. Show using truth tables that

(a) $P \implies Q$ and $\neg P \vee Q$ are logically equivalent;

(b) $\neg(P \wedge Q)$ and $\neg P \vee \neg Q$ are logically equivalent;

(c) $(P \vee Q) \wedge R$ and $(P \wedge R) \vee (Q \wedge R)$ are logically equivalent.

3. A proposition is said to be a *tautology* if it is always true. For example, if P and Q are propositions then $P \vee (\neg P)$ is a tautology, because either P is true or $\neg P$ is true. But $P \Longrightarrow Q$ is not a tautology, because it is false when P is true and Q is false. Which of the following are tautologies?

(i) $P \wedge (P \vee Q) \iff P$,

(ii) $\big((P \Longrightarrow Q) \wedge (Q \Longrightarrow R)\big) \Longrightarrow (P \Longrightarrow R)$,

(iii) $\big(P \Longrightarrow (Q \Longrightarrow R)\big) \Longrightarrow \big((P \Longrightarrow Q) \Longrightarrow R\big)$,

(iv) $\big((P \Longrightarrow Q) \Longrightarrow R\big) \Longrightarrow \big(P \Longrightarrow (Q \Longrightarrow R)\big)$

[*Hint:* You can use truth tables, or argue directly, as you prefer. The direct proofs are usually shorter. If you use truth tables you will need eight rows in some cases.]

4. Let X, Y and Z be sets and let $f : X \to Y$ and $g : Y \to Z$ be functions. Decide whether the following propositions are true or false. Justify your answers with proofs or counterexamples as appropriate.

(a) f injective and g injective $\implies$ gf injective

(b) gf injective $\implies$ f injective

(c) gf injective $\implies$ f injective and g injective

5. There are eight truth tables of the form below, where each of the starred entries is either true or false. (For example, one has entries T, T, T, F read from top to bottom, another has entries F, T, T, F, and so on.)

P	Q	?
T	T	$\star$
T	F	$\star$
F	T	$\star$
F	F	F

For each such table, write down a proposition having that truth table. Use only parentheses and the symbols $P, Q, \vee, \wedge$ and $\neg$. (You need not use both propositions. You need not use all symbols.)

Bonus question A: One day you meet three people, A, B and C. You know that one is a *knight*, who always tell the truth, one is a *knave*, who always lies, and the other is a *spy*, who answers as he sees fit. You hear the following.

A says to B: 'I have heard you tell the truth today'

B says to C: 'I have heard you lie today'

C says to A: 'You are a knight'

State the identity of each person.

Bonus question B: There is an island whose population consists entirely of *humans* and *vampires*. The two species are indistinguishable to the eye, but humans always tell the truth and vampires always lie.

To complicate matters, humans and vampires may be sane or insane. Sane inhabitants believe statements if and only if they are true, while insane inhabitants believe statements if and only if they are false.

Find a single question which you can ask an inhabitant of the island to determine whether they are a vampire.

[For many more questions like these see *What is the name of this book? The riddle of Dracula and other logical problems* by Raymond Smullyan (Prentice-Hall, 1978).]

MT181 Number Systems: Sheet 8

Attempt questions 1 to 6. Question 7 is optional. Please staple your answers together and put your name and student number on the top sheet. Do not return the problem sheet.

The lecturer will be happy to discuss any of the questions in office hours.

To be handed it at the lecture on Tuesday 26th November.

1. Read Chapters 10 and 11 on quantifiers of *How to think like a mathematician* by Kevin Houston (Cambridge University Press, 2009).

2. State the truth value (true or false) of each of the following propositions.

(i) $\{1,2\}$ is an element of $\big\{\{1,2\},2,3\big\}$,

(ii) $\{1,2\}$ is a subset of $\big\{\{1,2\},2,3\big\}$,

(iii) $\{1,2\}$ is a subset of $\big\{\{1,2\},1,2,3\big\}$

(iv) $\big|\big\{\{1,2\},2,3\big\}\big| = 4$,

(v) $\big|\big\{\{1,2\},1,2,3\big\}\big| = 4$.

3. Show that the proposition below is false:

$$a, b \in \mathbb{R} \text{ and } \mathrm{e}^{a+bi} = 2 \implies a = \ln 2 \text{ and } b = 0.$$

Correct this proposition by rewriting part of the right-hand side using an existential quantifier.

4. (a) Let Q be the proposition $(\forall m \in \mathbb{N})(\exists n \in \mathbb{N})(n \text{ is divisible by } m)$.

(i) Find a proposition logically equivalent to $\neg Q$ that starts $(\exists m \in \mathbb{N})$.

(ii) Is Q true? Explain your answer.

(b) Let R be the proposition $(\exists n \in \mathbb{N})(\forall m \in \mathbb{N})(n \text{ is divisible by } m)$.

(i) Find a proposition logically equivalent to $\neg R$ that starts $(\forall n \in \mathbb{N})$.

(ii) Is R true? Explain your answer.

5. Let $U = \{1, 2, \ldots, 2013\}$. Define subsets X, Y, Z of U by

$$\begin{aligned} X &= \{n \in U : n \text{ is even}\} \\ Y &= \{n \in U : n \text{ is divisible by } 3\} \\ Z &= \{n \in U : n \text{ is divisible by } 5\} \end{aligned}$$

(a) Show that if $m \in \mathbb{N}$ then the number of elements of U that are divisible by m is $\lfloor 2013/m \rfloor$, where $\lfloor\ \rfloor$ is the floor function first seen in Question 3 of Sheet 2.

(b) By applying the Principle of Inclusion and Exclusion to the sets X, Y and Z, find the number of elements of U that are *not* divisible by *any* of 2, 3 or 5.

6. In the land of Erewhon everyone has a different number of hairs on their head. Moreover,

(1) No-one is completely bald.

(2) No citizen has exactly 1729 hairs.

(3) There are at least as many citizens as there are hairs on the head of any citizen.

Suppose that exactly M people live in Erewhon. Let $X = \{1, 2, \ldots, M\}$ and let $f : X \to \mathbb{N}$ be the function which sends $m \in X$ to the number of hairs on the head of person m.

(a) Explain why f is injective.

(b) Show that the range of f is a subset of $\{m \in \mathbb{N} : 1 \leq m \leq M \text{ and } m \neq 1729\}$.

(c) What is the largest possible number of citizens of Erewhon? [*Hint:* the start of the next question has a helpful idea.]

7. The *pigeonhole principle* states that if $m < n$ and n objects are put in m pigeonholes then some pigeonhole must contain two objects.

More formally, if $m, n \in \mathbb{N}$ and $m < n$ then there is no injective function

$$f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, m\}.$$

(a) Show that if $X \subseteq \{1, 2, \ldots, 2m\}$ and $|X| \geq m + 1$ then X contains two consecutive numbers. [*Hint:* make the pigeonholes the subsets $\{1, 2\}$, ..., $\{2m - 1, 2m\}$, and show that one pigeonhole must contain two elements of X.]

(b) Making any reasonable assumptions, prove that there are two students at British universities whose bank balances agree to the nearest penny.

(c) Show that if five points are inside an equilateral triangle of size 1 then two of the points are at distance 1/2 or less.

(d) Show that if $X \subseteq \{1, 2, \ldots, 2m\}$ and $|X| \geq m+1$ then X contains elements r and s such that r divides s.

Bonus question: Six pirates have secured their treasure chest with padlocks, labelled A, B, C, and so on. The chest can only be opened when every single padlock has been unlocked. Each pirate has keys to a subset of the padlocks; for example, one might have keys to padlocks A, C, D, another might have keys to padlocks B, D, E, and so on. The distribution of keys is arranged so that the box can be opened if and only if at least four pirates are present.

For reasons of fiscal prudence, the pirates were keen to arrive at this situation using as few padlocks as possible. How many padlocks are there on their treasure chest?

MT181 Number Systems: Sheet 9

Attempt questions 1 to 6. Please staple your answers together and put your name and student number on the top sheet. Do not return the problem sheet.

The lecturer will be happy to discuss any of the questions in office hours.

To be handed it at the lecture on Tuesday 3rd December.

1. Read Chapter 29 on modular arithmetic of *How to think like a mathematician* by Kevin Houston (Cambridge University Press, 2009).

2. (a) Find an integer n such that $0 \le n < 5$ and $n \equiv 2013 \bmod 5$.

(b) Find an integer n such that $-5 \le n < 0$ and $n \equiv 2013 \bmod 5$.

(c) Find all integers k such that $2k \equiv 1 \bmod 5$.

(d) Does the congruence $3x \equiv 7 \bmod 9$ have a solution? Justify your answer.

3. Find $2^n \bmod 11$ for $n = 0, 1, 2, \ldots, 10$. Hence, or otherwise, find all solutions $n \in \mathbb{N}$ to the equation $2^n \equiv 5 \bmod 11$.

4. (a) Use Euclid's algorithm to find the greatest common divisor of 170 and 2921.

(b) Determine $s, t \in \mathbb{Z}$ such that $170s + 2921t = 1$.

(c) Find all $x \in \mathbb{Z}$ such that $170x \equiv 15 \bmod 2921$.

5. An *ISBN* is any sequence $u_1u_2 \ldots u_{10}$ of the symbols $\{0, 1, \ldots, 9, \mathrm{X}\}$, where the roman numeral X stands for 10, such that

$$\sum_{j=1}^{10} (11 - j)u_j \equiv 0 \bmod 11.$$

The sum on the left-hand side is called the *check sum* for u.

(a) Which of the following are valid ISBNs

0-521-71978-X, 0-521-71987-X, 1-84628-400-0?

(b) Suppose that u is an ISBN and that u is written down as

$$v = u_1u_2 \ldots u_{k-1}\, s\, u_{k+1} \ldots u_{10}$$

where $s \in \{0, 1, 2, 3, \ldots, 9, \mathrm{X}\}$ and $s \neq u_k$. Compute the difference between the check sums for u and v. Hence show that v is not an ISBN.

(c) Prove that if u is an ISBN and the digits of u in positions k and $k + 1$ are swapped then the new sequence is not an ISBN. [**Correction: assume that the digits in these positions are different.**]

6. Set a question on any part of the MT181 course, in the style of previous compulsory problem sheet questions.

Please give the answers to any numerical parts. A full model solution is not required, but you may supply one if you wish. State *briefly* how hard you think your question is.

Bonus question A: find the number of zeros at the end of 2013!

Bonus question B: a sequence $a_1, a_2, a_3, \ldots$ satisfies the following conditions: (i) $a_n \in \mathbb{N}$ for all $n \in \mathbb{N}$, (ii) $a_m < a_n$ if $m < n$ and (iii) $a_{a_n} = 3n$ for all $n \in \mathbb{N}$. Find a_{2013}.

MT181 Number Systems: Sheet 10

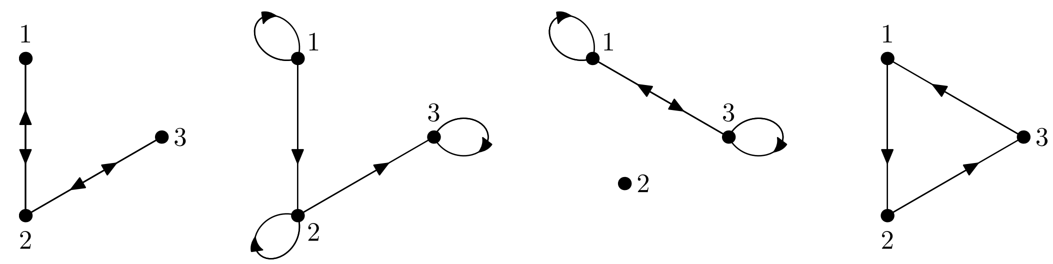
Attempt all numbered questions. Please staple your answers together and put your name and student number on the top sheet. Do not return the problem sheet.

The lecturer will be happy to discuss any of the questions in office hours.

Answers to the first four questions will appear on Moodle on December 10th, and the rest will appear on December 17th. You can get help with this problem sheet in workshops and office hours as usual.

1. Read Chapter 31 on equivalence relations of *How to think like a mathematician* by Kevin Houston (Cambridge University Press, 2009).

2. The four diagrams below show relations on the set $\{1, 2, 3\}$: an arrow is drawn from x to y if and only if $x \sim y$. A loop is drawn on x if and only if $x \sim x$.



(a) For each relation state whether it is (i) reflexive, (ii) symmetric, and (iii) transitive. Justify your answers briefly.

(b) Define four further relations on the set $\{1, 2, 3\}$ that have each of the four remaining combinations of these properties.

3. Define a relation $\sim$ on $\mathbb{C}$ by

$$z \sim w \iff |z| = |w|.$$

(a) Prove that $\sim$ is an equivalence relation.

(b) Draw the equivalence classes $[2i]$, $[e^{i\pi/3}]$ and $[0]$ on an Argand diagram.

4. The following argument claims to show that if $\sim$ is a relation on a set X that is symmetric and transitive then $\sim$ must be reflexive.

> 'Given $x \in X$ choose $y \in X$ such that $x \sim y$. By symmetry $y \sim x$. Hence $x \sim y$ and $y \sim x$, so by transitivity $x \sim x$. Thus $\sim$ is reflexive.'

Where is the flaw in this argument? [*Hint:* question 2(a) is relevant.]

5. Let $n \in \mathbb{N}$. Suppose that $n = d_k d_{k-1} \dots d_1 d_0$ in base 10.

(a) Prove that $n \equiv d_0 + d_1 + \cdots + d_k \bmod 9$. Hence show that 9 divides n if and only if 9 divides $d_0 + d_1 + d_2 + \cdots + d_k$.

(b) Prove that 11 divides n if and only if 11 divides $d_0 - d_1 + d_2 - \cdots + (-1)^k d_k$.

(c) Let $n = 123\,456\,789\,123\,456\,789$. Find the remainder when n is divided by 9. Find the remainder when n is divided by 11.

6. (a) Write down addition and multiplication tables for $\mathbb{Z}_6$.

(b) Find all $x \in \mathbb{Z}_6$ such that $[2] \times x = [4]$.

(c) Does $[2]$ have an inverse (see Definition 10.2) in $\mathbb{Z}_6$? [**reference to definition corrected on 7 December**]

7. Find the inverses of $[10]$ and $[14]$ in $\mathbb{Z}_{37}$.

8. Let R be a ring. Prove the following parts of Lemma 10.5.

(ii) The one element in R is unique. [*Hint:* suppose that $u, u' \in R$ are such that $ur = u'r = r$ for all $r \in R$. Adapt the proof of Lemma 10.5(i).]

(vii) If $x \in R$ then $-(-x) = x$.

(viii) If $x, y \in R$ then $-(xy) = (-x)y = x(-y)$ and $(-x)(-y) = xy$.

(ix) $0 = 1$ if and only if $R = \{0\}$.

Bear in mind that $-x$ means the element of R given by property (3) which satisfies $-x + x = 0$. Other parts of Lemma 10.5 proved in lectures will be useful.

9. Let R be a finite commutative ring such that for all x, $y \in R$,

$$xy = 0 \implies x = 0 \text{ or } y = 0.$$

Show that for each non-zero $x \in R$, the map $f_x : R \to R$ defined by $f_x(y) = xy$ is injective. Hence show that R is a field.

Bonus question: There are 10 pirates who have recently acquired a bag containing 100 gold coins. The leader, number 1, must propose a way to divide up the loot. For instance he might say 'I'll take 55 coins and the rest of you can have five each'. A vote is then taken. If the leader gets *half or more* of the votes (the leader getting one vote himself), the loot is so divided. Otherwise he is made to walk the plank by his dissatisfied subordinates, and number 2 takes over, with the same responsibility to propose an acceptable division.

Assuming that the pirates are all greedy, untrustworthy, and capable mathematicians, what happens? [*Hint:* try thinking about a smaller 2 or 3 pirate problem to get started.]
