Some analysis questions to motivate vacation revision.

You should regard any question marked ($\star$) or ($\star\star$) as optional. These are harder questions that are included in the hope that they may be found interesting; you should not worry if you find them difficult.

1. True or false: give brief proofs or counterexamples as appropriate.

(i) If $\lim_{x \to a} f(x)$ exists then f is continuous at a.

(ii) If $f : \mathbb{R} \to \mathbb{R}$ is continuous and (x_n) is a sequence converging to x as $n \to \infty$ then $f(x_n) \to f(x)$ as $n \to \infty$.

(iii) If $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing then $f(x) \to \infty$ as $x \to \infty$.

(iv) If $f : [a, b] \to \mathbb{R}$ is a continuous strictly increasing function with range $[c, d]$ then the inverse function $f^{-1} : [c, d] \to [a, b]$ is continuous.

(v) If f and g are real valued functions, both differentiable at $a \in \mathbb{R}$, then the product function $(fg)(x) = f(x)g(x)$ is differentiable at a.

2. (i) Prove that if $f : [a, b] \to \mathbb{R}$ is a continuous function then f is bounded and achieves its bounds.

(ii) Suppose that $g : \mathbb{R} \to \mathbb{R}$ is differentiable. Let $c < d$ and suppose that $g'(c) < r < g'(d)$. Show that there exists $\zeta \in (c, d)$ such that $f'(\zeta) = r$. [*Hint: Consider the function* $f(x) = g(x) - rx$ *on the interval* $[c, d]$.]

(iii) Deduce Rolle's theorem from part (i).

3. Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ are both solutions to the differential equation $y'' + y = 0$ satisfying the initial conditions $y(0) = 0$, $y'(0) = 1$.

Show that $h(x) = f(x) - g(x)$ is infinitely differentiable and that $h^{(m)}(0) = 0$ for all $m \geq 0$. By bounding the error term in Taylor's theorem, prove that $h(x) = 0$ for all $x \in \mathbb{R}$. Conclude that $f(x) = g(x) = \sin x$.

4. (i) What does it mean to say that the power series $f(x) = \sum_{m=0}^{\infty} a_m x^m$ has *radius of convergence* R?

(ii) Suppose that $R > 0$. Fix $r \in \mathbb{R}$ such that $0 < r < R$. Show that the partial sums $\sum_{m=0}^{n} a_m x^m$ converge uniformly to f on $[-r, r]$ as $n \to \infty$. Deduce that f is continuous for all $x \in (-R, R)$.

5. [Taken from Q2 2002 analysis]. (i) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable at a and $g : \mathbb{R} \to \mathbb{R}$ is differentiable at $f(a)$. Prove that $g(f(x))$ is differentiable at a with derivative $g'(f(a))f'(a)$.

(ii) Consider the function f defined by

$$f(x) = \begin{cases} x^2 \cos(1/x) & : x \neq 0 \\ 0 & : x = 0 \end{cases} .$$

Prove that f is differentiable for all x but f' is not continuous at $x = 0$. (Note that by 2(ii), f' does satisfy the intermediate value property.)

6. ($\star$) Draw a map of the analysis course.

For revision of last term's work on sequences and series I recommend some past exam questions, for example questions 3 and 4 from 2002 Mods Paper 2. (Exam papers going back to 1999 are available from the Institute website.)

Further questions for enthusiasts.

7. Let $f : \mathbb{R} \to \mathbb{R}$ map the interval $[a, b]$ into $[c, d]$. Suppose also that $f(a) = c$ and $f(b) = d$. Prove that any two of the following properties imply the third:

(a) f is strictly increasing,

(b) f is continuous,

(c) f is bijective.

8. (i) Define $f : \mathbb{R} \to \mathbb{R}$ as follows:

$$f(x) = \begin{cases} \exp(-1/x^2) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Show that f is everywhere differentiable. Sketch the graph of f.

(ii) Show that in fact f has derivatives of every order and $f^{(n)}(0) = 0$ for all n. Why does this not contradict Taylor's theorem?

9. Show that $f : [0, 1] \to [0, 1]$ has a fixed point (that is, there exists $x \in [0, 1]$ such that $f(x) = x$) if either of the following conditions hold: (i) f is continuous, (ii) ($\star$) f is increasing.

10. Two questions connecting countability and analysis.

(i) ($\star$) A closed interval is a set of the form $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$. Is $[0, 1]$ a countably infinite union of disjoint closed intervals?

(ii) ($\star\star$) Can uncountably many Y shapes be arranged in the plane, with no two intersecting?

11. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Suppose that for every $a \in \mathbb{R}$, the limit $\lim_{x \to a} f(x)$ exists. Show that the function g defined by $g(a) = \lim_{x \to a} g(x)$ is continuous. Need f be continuous?

12. By considering $\int_1^n \log x \, \mathrm{d}x$, prove that $n^n \mathrm{e}^{1-n} \leq n! \leq (n+1)^{n+1} \mathrm{e}^{-n}$ for $n \geq 1$.

13. (i) ($\star$) Before the 10 pirates (see the Christmas vacation sheet) are able to divide up their bar of gold, it must first be retrieved from their safety deposit box. This box is secured by a number of padlocks, labelled A, B, C, and so on, and can only be opened when every single padlock has been unlocked.

Each pirate has keys to a subset of the locks; for example, the leader might have keys to locks A, C, D, number two might have keys to locks A, B, E, and so on. The distribution of keys is arranged so that the box can be opened if and only if the pirates are quorate; that is, any 6 of them are present[1].

Since good quality padlocks are quite expensive, the pirates were keen to arrive at this situation using as few padlocks as possible. How many padlocks are there on their safety deposit box?

(ii) ($\star$) For $\alpha \in (0, 1)$ let $f_\alpha(n)$ be the minimum number of locks needed when there are in total n pirates, and αn pirates are required to be present for the box to be opened. Discuss the limiting behaviour of $f_\alpha(n)$ as n tends to infinity.

[1]A figure only arrived at after much heated debate. Since the pirates do not trust one another, each pirate would prefer it if the only way the box can be opened is if he is present; that is, the quorum is 10. It was quickly realised that in this scheme, were a single one of them to fall in battle, the contents of the box would be lost forever. This was felt to be a compelling argument in favour of a smaller quorum.