Some answers to vacation questions.

If a question is pure bookwork then I have left you to look up the proof in notes or a book. Except for question 10 (which had a misprint I thought I should clear up) I haven't given answers to the optional questions; ask me if you want any help with them.

1. *True or false: (give brief proofs or counterexamples as appropriate)*

(i) *A convergent sequence is bounded.*

True. Let (a_n) be a convergent sequence. Take $\epsilon = 1$ in the definition of convergence to obtain

$$\exists a \in \mathbb{R}\ \exists N \in \mathbb{N}\ \forall n \geq N\ \ |a_n - a| < 1.$$

From this it follows that $|a_n| \leq |a| + 1$ if $n \geq N$. To get an overall upper bound just take $A = \max(a_1, a_2, \ldots, a_{N-1}, |a| + 1)$.

(ii) *A bounded sequence is convergent.*

False. For example $a_n = (-1)^n$.

(iii) *If the sequence (a_n) does not tend to infinity then there is a constant K such that $|a_n| < K$ for all $n \geq 1$.*

False. Remember we say that the sequence (a_n) tends to infinity if given any $K \in \mathbb{R}$ there exists an $N \in \mathbb{N}$ such that $|a_n| > K$ for all $n \geq N$. By this definition the sequence

$$a_n = \begin{cases} 0 & n \text{ is even} \\ n & n \text{ is odd} \end{cases}$$

does not tend to infinity. But $|a_{2m}| = 2m$ so the sequence is not bounded.

(iv) If X and Y are non-empty sets of real numbers and $x < y$ for all $x \in X$ and $y \in Y$ then $\sup X$ *and* $\inf Y$ *exist and* $\sup X \leq \inf Y$.

True. Take any $y \in Y$. By hypothesis $x \leq y$ for all $x \in X$ so X is bounded above. By the completeness property of $\mathbb{R}$, X has a supremum, $\sup X$. A similar argument shows that Y has an infimum. Now $\sup X < y$ for all $y \in Y$ so $\sup X < \inf Y$. Here we used the (often helpful) result that $A \leq y$ for all $y \in Y$ implies $A \leq \inf Y$.

Further exercise: Can the conclusion be strengthened to $\sup X < \inf Y$?

(v) *A subsequence of a convergent sequence is convergent, and has the same limit as the original sequence.*

True. Bookwork.

(vi) *If $a_n > 0$ for all n and $a_n \to 0$ as $n \to \infty$ then $1/a_n \to \infty$ as $n \to \infty$.*

True. We want to show that for all $K > 0$ there exists $N \in \mathbb{N}$ such that $|1/a_n| > K$ for all $n \in N$. All we have to play with is the hypothesis that $a_n \to 0$ as $n \to \infty$, i.e.

$$\forall \epsilon > 0\ \exists N \in \mathbb{N}\ \forall n \geq N\ \ |a_n| < \epsilon.$$

So if we put $\epsilon = 1/K$ we will obtain a suitable N.

2. (i) *Let (a_n) be a bounded monotone sequence of real numbers. Prove that (a_n) is convergent.*

Bookwork. By replacing a_n with $-a_n$ if necessary we may assume that (a_n) is increasing. Now show that a_n converges to $\sup\{a_n : n \in \mathbb{N}\}$ by using the approximation property.

(ii) *Let $b > 1$ be a fixed real number. We define a sequence (a_n) inductively by taking*

$$a_0 = b, \quad a_{n+1} = \frac{a_n}{2} + \frac{b}{2a_n} \quad \text{for } n \geq 0.$$

Prove that (a_n) converges. Show that if $\beta = \lim_{n\to\infty} a_n$ then $\beta > 0$ and $\beta^2 = b$.

Straightforward inductive arguments show that (a) a_n is positive, (b) $a_n^2 \geq b$ and (c) $a_{n+1} \leq a_n$ for all n. So the sequence (a_n) is decreasing and bounded below. By part (a) it must converge to some $\beta > 0$. As $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$, the limit β must satisfy $\beta = \beta/2 + b/2\beta$. It follows that $\beta^2 = b$.

(iii) ($\star$) *Show that if $a_n \geq 1$ and $|a_n - \beta| < \epsilon$ then $|a_{n+1} - \beta| < \epsilon^2/2$. Deduce that if for our initial guess a_0 we pick the natural number whose square is nearest to b then $|a_n - \beta| < 1/2^{2^n-1}$). Can $a_n = \beta$ for any n?*

The first part is just a calculation:

$$\begin{aligned} |a_{n+1} - \beta| &= \left| \frac{a_n}{2} + \frac{b}{2a_n} - \beta \right| \\ &= (a_n - \beta)^2/2a_n \\ &< \epsilon^2/2. \end{aligned}$$

Now $|a_0 - \beta| < 1$, as otherwise there would be a natural number nearer β than a_0, and its square would be nearer b than a_0^2. The error estimate given in the question now follows by induction on n.

3. (i) *State the Bolzano-Weierstrass Theorem concerning sequences of real numbers.*

The Bolzano-Weierstass Theorem states that any bounded sequence of real numbers has a convergent subsequence. (There is an analogous version for sequences of complex numbers.)

(ii) *What does it mean to say that a sequence is a* Cauchy sequence*? Prove that a sequence of real numbers is a Cauchy sequence if and only if it converges.*

Bookwork.

(iii) *Deduce that if the series $\sum_{r=1}^{\infty} a_r$ converges absolutely then it converges. Give an example to show that the converse of this result is false.*

The first part is bookwork. The series $\sum_{r=1}^{\infty} a_r$ where $a_r = (-1)^r/\sqrt{r}$ is convergent (by the alternating series test), but not absolutely convergent.

4. [Based on Q1 1999 Mods Analysis.] *Let (a_n) be a sequence of real numbers. What is meant by the statement that (a_n) is* convergent*?*

Let (a_n) and (b_n) be sequences converging to the limits l and m respectively. Show that:

(i) *The sequence $(a_n + b_n)$ converges to $l + m$.*
(ii) *The sequence $(a_n b_n)$ converges to lm.*
(iii) *If $a_n \leq b_n$ for all n then $l \leq m$.*

Yet more bookwork.

Give an example to show that if $a_n < b_n$ for all n then it is not neccesarily true that $l < m$.

One could take $a_n = 0$ for all n and $b_n = 1/n$. The moral of this example is that limits don't in general preserve 'sharp' inequalities, i.e. inequalities involving $<$ or $>$ signs.

Now suppose that $l = 0$. Define a new sequence (c_n) by $c_n = \frac{1}{n}\sum_{r=1}^{n} a_r$. Show that (c_n) also converges to 0.

Fix $\epsilon > 0$. Let $|a_r| < \epsilon$ for all $r \geq N$. If $n \geq N$ then

$$\left|\frac{1}{n}\sum_{r=1}^{n} a_r\right| \leq \frac{1}{n}\sum_{r=1}^{n}|a_r| = \frac{1}{n}\sum_{r=1}^{N}|a_r| + \frac{1}{n}\sum_{r=N+1}^{n}|a_r| \leq \frac{1}{n}\sum_{r=1}^{N}|a_r| + \frac{n-N}{n}\epsilon.$$

As n tends to ∞ the first summand tends to 0, and the second is bounded by ϵ. So $\lim_{n\to\infty} c_n = 0$.

5. [Based on Q4 2001 Mods Analysis] *State the comparison test and derive the integral test for a series of real numbers $\sum_{r=1}^{\infty} a_r$.*

It's important to include all the neccessary conditions in the integral test: let $f : \mathbb{R} \to \mathbb{R}$ be a *decreasing* function that takes values in the *positive* real numbers. The integral test states that the series with nth term $f(n)$ converges if and only if the integral

$$\int_1^{\infty} f(x)\mathrm{d}x$$

converges.

Prove that the series $\sum_{r=1}^{\infty} r^{-\alpha}$ converges if $\alpha > 1$ and diverges if $\alpha \leq 1$. Determine whether the following series converge or diverge:

$$\sum_{r=2}^{\infty}\frac{1}{r\log r}, \quad \sum_{r=1}^{\infty}\frac{1}{r}\sin\frac{1}{r}.$$

The first part follows from an application of the integral test with the function $f(x) = x^{-\alpha}$. For the next part apply the integral test with the function $f(x) = 1/x\log x$, which has integral

$$\int_2^t \frac{\mathrm{d}x}{x\log x} = \log\log t - \log\log 2.$$

For the last one use the inequality $\sin x \leq x$, which is valid for all $x \geq 0$ to get

$$\sum_{r=1}^{n}\frac{1}{r}\sin\frac{1}{r} \leq \sum_{r=1}^{n}\frac{1}{r^2}.$$

The comparison test now shows that the series converges.

9. ($\star$) *Let $\alpha > 0$ be an irrational number. Show that given $N > 0$ there exist natural numbers m and n such that $n \leq N$ and*

$$\left|\alpha - \frac{m}{n}\right| < \frac{1}{nN}$$

[Hint: let $\{x\}$ denote the fractional part of $x \in \mathbb{R}$. Apply the pigeonhole principle to the $N+1$ numbers $0, \{\alpha\}, \{2\alpha\}, \ldots, \{N\alpha\}$.] *Deduce that for any $\epsilon > 0$ there exist points $(m, n) \in \mathbb{N}\times\mathbb{N}$ lying within a distance ϵ of the line $y = \alpha x$.*

The numbers $0, \{\alpha\}, \{2\alpha\}, \ldots, \{N\alpha\}$ all lie between 0 and 1. So one of the disjoint intervals $(0, 1/N)$, $(1/N, 2/N)$, $\ldots$, $((N-1)/N, 1)$ must contain 2 of them. (We don't need to worry about endpoints as we supposed that α was irrational.) If both $\{r\alpha\}$ and $\{s\alpha\}$ appear in one such interval then there is an integer m such that

$$|r\alpha - s\alpha - m| < \frac{1}{N}.$$

Put $r - s = n$ and divide through to obtain

$$\left|\alpha - \frac{m}{n}\right| < \frac{1}{Nn}.$$

Fix $\epsilon > 0$. Choose $N > 1/\epsilon$. We have shown that there exist $m, n \in \mathbb{N}$ such that $|\alpha - m/n| < \epsilon/n$. Multiplying by n gives $|n\alpha - m| < \epsilon$, so the point (m, n) lies within ϵ of the line $y = m\alpha$.