REPRESENTATION THEORY OF THE SYMMETRIC GROUP

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Recommended reading: [8] G. D. James, *Representation theory of the symmetric groups*, Springer Lecture Notes in Mathematics 692, Springer (1980).

Acknowledgement. I thank Darij Grinberg for corrections to numerous errors in the original version of these notes, and also for supplying many useful clarifications and suggestions for extra content. I thank Bakhshinder Samra for further corrections. Of course I take full responsibility for all the remaining errors.

1. Background definitions

Let S_n denote the symmetric group of degree n. Let $\mathbf{N}$ denote the set of natural numbers $\{1,2,3,\ldots\}$ and let $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$.

In these notes all maps will be written on the right. For example, the composition of the cycles (12) and (123) in the symmetric group S_3 is $(12)(123) = (13)$, and the image of 1 under the permutation (12) is $1(12) = 2$. This convention agrees with James' lecture notes, and means that compositions of maps can be read from left to right.

ALGEBRAIC DEFINITIONS. Let G be a finite group and let R be a commutative ring. The *group ring* RG is defined to be the free R-module with basis $\{g : g \in G\}$ and multiplication defined by linear extension of the multiplication in G. So given $\sum_{h\in G} \alpha_h h \in RG$ and $\sum_{k \in G} \beta_k k \in RG$ we set

$$\Big(\sum_{h\in H} \alpha_h h\Big)\Big(\sum_{k\in G} \beta_k k\Big) = \sum_{\substack{h\in H\\ k\in K}} \alpha_h\beta_k hk = \sum_{g\in G}\Big(\sum_{h\in H}\alpha_h\beta_{h^{-1}g}\Big)g.$$

An *RG-module* is an abelian group V together with a bilinear map $V \times RG \to V$, which we shall write as $(v,r) \mapsto vr$, such that

(i) if $r, s \in RG$ then $v(rs) = (vr)s$ for all $v \in V$;
(ii) $v\,\mathrm{id}_G = v$ for all $v \in V$.

If V is an RG-module then a subgroup W of V is an RG-module if and only if the map $W \times RG \to V$ has image inside W. In this case we say that W is an *RG-submodule* of V.

Exercise: By looking at the action of $\alpha\,\mathrm{id}_G$ for $\alpha \in R$, show that V has the structure of an R-module.

Date: First term 2011/12 (revised October 2013).

Most of the time R will be a field. In this case the exercise above says that V is an F-vector space. We shall also sometimes use $\mathbf{Z}G$-modules: these will always be free as $\mathbf{Z}$-modules. All modules in these notes will have finite rank / dimension.

Exercise:

(a) Let F be a field and let V be an FG-module. Show that if $g\rho : V \to V$ is the map defined by $v(g\rho) = vg$ then ρ is a group homomorphism from G into the group $\mathrm{GL}(V)$ of invertible F-linear transformations of V.

(b) Show conversely that if V is an F-vector space and $\rho : G \to \mathrm{GL}(V)$ is a group homomorphism (so V is a representation of G) then setting $v(\alpha g) = \alpha v(g\rho)$ for $\alpha \in F$ and $g \in G$, and extending linearly to a general element of FG, gives V the structure of an FG-module.

In practice, it is usually most convenient to specify an FG-module V by defining V as an F-vector space, and then defining an action of G on V by invertible F-linear transformations. (There is then a unique way to extend the actions of F and G on V so that the map $V \times FG \to V$ is linear in its second component, and so bilinear.)

For example, the *trivial* FG-module is the F-vector space F on which each $g \in G$ acts as the identity map.

Remark 1.1. The definition of an FG-module is more technical than the definition of a representation of G, but, as the exercise shows, the two notions are equivalent. Module can be more convenient to work with, because there is less notation, and we can use results from ring theory without any translation. The language of representations is preferable if we want to have an explicit map $\rho : G \to \mathrm{GL}(V)$.

A similar situation arises when we have a group G acting on a set Ω. Here one can choose between the equivalent languages of G-actions (writing ωg for the image of $\omega \in \Omega$ under $g \in G$) and permutation representations (using a homomorphism from G into the symmetric group on Ω). Again, both have their advantages.

The easiest example of a non-trivial FS_n-module is given in the next example.

Example 1.2. Let F be a field and let $V = \langle e_1, \ldots, e_n \rangle_F$ be an n-dimensional vector space over F. We make V into an FS_n-module by defining

$$e_i g = e_{ig} \quad \text{for } i \in \{1, \ldots, n\} \text{ and } g \in S_n. \tag{1}$$

By the remark after the exercise above, this suffices to give V the structure of an FS_n-module. The corresponding representation $\rho : S_n \to$

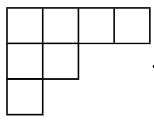
GL(V) represents each $g \in S_n$ as a permutation matrix. For example if $n = 3$, $g = (12)$ and $h = (123)$ then

$$g\rho = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad h\rho = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Note that these matrices act on the right, on row vectors.

COMBINATORIAL DEFINITIONS. A *partition* of $n \in \mathbf{N}_0$ is a non-increasing sequence of natural numbers whose sum is n. For example, the 7 partitions of 5 are (5), $(4,1)$, $(3,2)$, $(3,1,1)$, $(2,2,1)$, $(2,1,1,1)$ and $(1,1,1,1,1)$. The numbers making up a partition are its *parts*. We shall use powers to indicate multiplicities of parts: for example $(2,1,1,1) = (2,1^3)$.

It is often useful to represent a partition by a rectangular array of boxes, called its *Young diagram*[1]. For example, the Young diagram of $(4,2,1)$ is



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Remark 1.3. It is well-known that the partitions of n label the conjugacy classes of S_n *via* cycle types. For example, the conjugacy class of S_5 labelled by $(2,2,1)$ contains $(12)(34)$. We shall see later that the irreducible $\mathbf{C}S_n$-modules are also canonically labelled by the partitions of n.

Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition of $n \in \mathbf{N}$. A *λ-tableau* is an assignment of the numbers $\{1, \ldots, n\}$ to the boxes of the Young diagram of λ, so that each number is used exactly once. For example,

5	6	3	2
7	1		
4			

is a $(4,2,1)$-tableau. More formally, if we fix a numbering of the boxes of λ, then we may regard a λ-tableau t as a function

$$t : \{1, \ldots, n\} \to \{1, \ldots, n\}$$

such that $it = j$ if and only if box i contains j.

[1]Young diagrams are named after the Reverend Alfred Young: see `www-history.mcs.st-andrews.ac.uk/Biographies/Young_Alfred.html`. In a letter sent to Frobenius, Young wrote:

> I am delighted to find someone else really interested in the matter. The worst of modern mathematics is that it is now so extensive that one finds there is only about one person in the universe really interested in what you are ...

If s and t are λ-tableaux then s and t are said to be *row-equivalent* if for each $i \in \{1, \ldots, k\}$, the sets of numbers in the ith row of s and t are equal. For example,

$$\begin{array}{|c|c|c|c|}\hline 5&6&3&2\\\hline 7&1\\\cline{1-2} 4\\\cline{1-1}\end{array} \quad \text{and} \quad \begin{array}{|c|c|c|c|}\hline 2&3&5&6\\\hline 1&7\\\cline{1-2} 4\\\cline{1-1}\end{array}$$

are row-equivalent.

A λ-*tabloid* is a row-equivalence class of λ-tableaux. We denote the λ-tabloid corresponding to the λ-tableau t by $\{t\}$. Tabloids are drawn by omitting the vertical lines from a representative tableau. For example if t is the first tableau above then

$$\{t\} = \begin{array}{cccc}\hline 5&6&3&2\\\hline 7&1\\\cline{1-2} 4\\\cline{1-1}\end{array} = \begin{array}{cccc}\hline 2&3&5&6\\\hline 1&7\\\cline{1-2} 4\\\cline{1-1}\end{array} = \ldots .$$

A λ-tableau is said to be *row standard*, if its rows are increasing when read from left to right, *column standard*, if its columns are increasing when read from top to bottom, and *standard*, if it is both row standard and column standard.

Example 1.4. There are $6! = 720$ distinct $(4,2)$-tableaux, 15 distinct $(4,2)$-tabloids and 9 distinct standard $(4,2)$-tableaux.

A $(4,2)$-tabloid is determined by its second row, so the $(4,2)$-tabloids are in bijection with 2-subsets of $\{1,2,3,4,5,6\}$. For example, the 2-subset $\{2,4\}$ corresponds to the tabloid

$$\begin{array}{cccc}\hline 1&3&5&6\\\hline 2&4\\\cline{1-2}\end{array}.$$

The 9 standard $(4,2)$-tableaux,

$$\begin{array}{|c|c|c|c|}\hline 1&3&5&6\\\hline 2&4\\\cline{1-2}\end{array}, \ \begin{array}{|c|c|c|c|}\hline 1&3&4&6\\\hline 2&5\\\cline{1-2}\end{array}, \ \begin{array}{|c|c|c|c|}\hline 1&3&4&5\\\hline 2&6\\\cline{1-2}\end{array}, \ \begin{array}{|c|c|c|c|}\hline 1&2&5&6\\\hline 3&4\\\cline{1-2}\end{array}, \ \ldots$$

will turn out to index a basis of an irreducible 9-dimensional $\mathbf{C}S_6$-module.

2. Young permutation modules and Specht modules

Throughout this section let $n \in \mathbf{N}$ and let R be a commutative ring.

Let λ be a partition of n. The symmetric group S_n acts on λ-tableaux in an obvious way. For example, if $g = (235) \in S_5$ then

$$\begin{array}{|c|c|c|c|}\hline 1&2&5&6\\\hline 3&4\\\cline{1-2}\end{array}\, g = \begin{array}{|c|c|c|c|}\hline 1&3&2&6\\\hline 5&4\\\cline{1-2}\end{array}$$

since $1g = 1$, $2g = 3$, $5g = 2$, $6g = 6$, and so on. With the definition of tableau as function, if $t : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is a λ-tableau and

$i \in \{1, \ldots, n\}$ is the number of a box, then $i(tg) = (it)g$. Either way, it is clear that the action of S_n on λ-tableaux is regular, i.e. the stabiliser of any given tableau is the trivial group.

Lemma 2.1. *Let λ be a partition of n. There is a well-defined action of S_n on the set of λ-tabloids defined by $\{t\}g = \{tg\}$ where $\{t\}$ is a λ-tableau and $g \in S_n$.*

Definition 2.2. Let λ be a partition of n. The *Young permutation module* M_R^λ, defined over R, is the RS_n-permutation module with free R-basis $\{\{t\} : \{t\} \text{ a } \lambda\text{-tabloid}\}$.

Usually we will write M^λ rather than M_R^λ since the ground ring (or, more usually, field) will be clear from the context.

Example 2.3. We continue to work with RS_n-modules.

(1) If $\lambda = (n)$ then there is a unique λ-tabloid, namely $\overline{\underline{1 \; 2 \cdots n}}$ and so $M^{(n)} \cong R$, the trivial RS_n-module.

(2) If $\lambda = (n-1, 1)$ then there are n distinct $(n-1,1)$-tabloids, with representative tableaux

$$t_1 = \begin{array}{|c|c|c|c|}\hline 2 & 3 & \cdots & n \\ \hline 1 \\ \cline{1-1}\end{array}$$

$$t_2 = \begin{array}{|c|c|c|c|}\hline 1 & 3 & \cdots & n \\ \hline 2 \\ \cline{1-1}\end{array}$$

$$\vdots$$

$$t_n = \begin{array}{|c|c|c|c|}\hline 1 & 2 & \cdots & n-1 \\ \hline n \\ \cline{1-1}\end{array}$$

The top row of a two-row tabloid can be deduced from the second row so we may write

$$\{t_1\} = \overline{\underline{\;1\;}}, \quad \{t_2\} = \overline{\underline{\;2\;}}, \quad \ldots, \quad \{t_n\} = \overline{\underline{\;n\;}}.$$

It is clear that $\{t_i\}g = \{t_{ig}\}$ for each $g \in S_n$. Thus $M^{(n-1,1)}$ affords the natural permutation representation of S_n.

(3) More generally, let $1 \leq k < n$ and let $\lambda = (n-k, k)$. Ignoring the first row of a $(n-k,k)$-tabloid gives a bijection between λ-tabloids and k-subsets of $\{1, \ldots, n\}$. Hence $M^{(n-k,k)}$ is isomorphic to the RS_n-permutation module of S_n acting on k-subsets of $\{1, \ldots, n\}$.

(4) Let $0 \leq k < n$ and let $\lambda = (n-k, 1^k)$. Ignoring the first row of a $(n-k, 1^k)$-tabloid gives a bijection between λ-tabloids and k-tuples of distinct elements of S_n. Correspondingly $M^{(n-k,1^k)}$ is the permutation module of RS_n acting on k-tuples of distinct elements from $\{1, \ldots, n\}$. Taking $k = n-1$, it follows that $M^{(1^n)}$ is isomorphic to the group ring RS_n as a right RS_n-module.

Definition 2.4. Let λ be a partition of n and let t be a λ-tableau. The *column group* $C(t)$ consists of all permutations which preserve setwise the columns of t. The *signed column-sum* b_t is the element of the group algebra FS_n defined by

$$b_t = \sum_{h \in C(t)} h \operatorname{sgn}(h).$$

The *polytabloid* (also called a *λ-polytabloid*, if the partition needs to be emphasised) corresponding to t is defined by

$$e(t) = \{t\} b_t.$$

Note that while the polytabloid $e(t)$ is a linear combination of tabloids, it depends on the tableau t, and not just on the tabloid $\{t\}$. For example if $\lambda = (2,1)$ then

$$e\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\right) = \begin{array}{cc}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array} - \begin{array}{cc}\hline 3 & 2 \\ \hline 1 & \\ \cline{1-1}\end{array} = \begin{array}{cc}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array} - \begin{array}{cc}\hline 2 & 3 \\ \hline 1 & \\ \cline{1-1}\end{array}$$

$$e\left(\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}\right) = \begin{array}{cc}\hline 2 & 1 \\ \hline 3 & \\ \cline{1-1}\end{array} - \begin{array}{cc}\hline 3 & 1 \\ \hline 2 & \\ \cline{1-1}\end{array} = \begin{array}{cc}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array} - \begin{array}{cc}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}$$

In general the polytabloid $e(t)$ is a sum of $|C(t)|$ distinct tabloids. In particular, this shows that $e(t) \neq 0$, no matter what field we work over.

We observe that S_n permutes the set of all λ-polytabloids. If t is a λ-tableau and $g \in S_n$, then

$$e(t)g = \{t\} b_t g = \{t\} g g^{-1} b_t g = \{tg\} b_{tg} = e(tg). \tag{2}$$

where we have used that $C(tg) = C(t)^g$, and so $b_{tg} = g^{-1} b_t g$. (Note that if $h, g \in S_n$ then we set $h^g = g^{-1}hg$.) One special case of (2) worth noting is that

$$e(t)g = \operatorname{sgn}(g) e(t) \quad \text{if } g \in C(t). \tag{3}$$

Definition 2.5. Let λ be a partition of n. The Specht module S^λ_R, defined over R, is the submodule of M^λ_R spanned by all the λ-polytabloids.

Again, we shall usually write S^λ rather than S^λ_R. By (2) above and the previous remark, S^λ is a well-defined non-zero RS_n-module. It also follows from (2) that S^λ is cyclic, generated by any single polytabloid.

Example 2.6. Let $n \in \mathbf{N}$ and let F be a field. All modules in this example are modules for FS_n.

(A) Take $\lambda = (n)$. Then while there are $n!$ distinct (n)-tableaux, each gives the same polytabloid, namely

$$e\big(\,\begin{array}{|c|c|c|c|}\hline 1 & 2 & \;\cdots\; & n \\ \hline\end{array}\,\big) = \overline{\underline{1\ 2 \cdots n}}\,.$$

Hence $S^{(n)} = M^{(n)} \cong F$.

(B) Let $n \geq 2$, let $\lambda = (n-1, 1)$ and let $t_1, \ldots, t_n$ be the $(n-1,1)$-tableaux in Example 2.3 above. With the same notation (i.e. omitting the redundant top row in a tabloid), we have

$$e(t_1) = \overline{\underline{1}} - \overline{\underline{2}},\ e(t_2) = \overline{\underline{2}} - \overline{\underline{1}}, \ldots,\ e(t_n) = \overline{\underline{n}} - \overline{\underline{1}}\,.$$

Therefore

$$S^{(n-1,1)} = \big\langle \overline{\underline{i}} - \overline{\underline{1}} : 2 \leq i \leq n \big\rangle\,.$$

It is easy to see that these vectors are linearly independent, so $S^{(n-1,1)}$ is $(n-1)$-dimensional. Problem 5 on the first problem sheet shows that $S^{(n-1,1)}$ is irreducible if the characteristic of F is zero or coprime to n.

(C) Take $\lambda = (1^n)$ and let t be the (1^n)-tableau whose single column has entries $1, 2, \ldots, n$ from top to bottom. Then $C(t) = S_n$, and so by (3) above we have

$$e(tg) = e(t)g = e(t)\,\mathrm{sgn}(g)$$

for all $g \in S_n$. Hence if s is any (1^n)-tableau then $e(s) = \pm e(t)$ and $S^{(1^n)}$ is isomorphic to the 1-dimensional sign representation of S_n.

Example 2.7. Example 5.2 in James' notes on $S^{(3,2)}$ is well worth studying. He shows that $S^{(3,2)}$ is the subspace of $M^{(3,2)}$ given by imposing 'all reasonable conditions' on a general sum of tabloids. Omitting the redundant top row in a $(3,2)$-tabloid, the 'reasonable conditions' on

$$\sum_{1 \leq i < j \leq 5} \alpha_{ij}\, \overline{\underline{i\ \ j}} \in M^{(3,2)},$$

where $\alpha_{ij} \in F$, are

(i) $\sum_{1 \leq i < j \leq 5} \alpha_{ij} = 0$

(ii) $\sum_{1 \leq j \leq 5} \alpha_{ij} = 0$ for each $i \in \{1, \ldots, 5\}$, where we set $\alpha_{kk} = 0$ and $\alpha_{ij} = \alpha_{ji}$ if $i > j$.

Remark 2.8. There are at least three equivalent definitions of Specht modules (four, counting the determinantal version of polytabloids below):

(1) Definition 2.5 above, which defines S^λ as the subspace of M^λ spanned by all λ-polytabloids: in §6 below we will prove the Standard Basis Theorem which states that

$$\{e(t) : t \text{ is a standard } \lambda\text{-tableau}\}$$

is a basis for S^λ.

(2) The description by generators and Garnir relations, which defines S^λ as a quotient of FS_n. (See §6 below.)

(3) As the subspace of M^λ given by imposing all reasonable conditions on λ-tabloids: see Theorem 17.18 in James' notes.

Remark 2.9. Some historical comments may helpful to motivate the definition of Specht modules. First of all, it's worth mentioning that the irreducible characters of the symmetric groups were constructed by Frobenius in 1904, long before Specht's construction in 1935 of Specht modules. (See W. Specht, *Die irreduziblen Darstellungen der Symmetrischen Gruppe*, Math. Z. **39** (1935), no. 1, 696–711. Specht credits a 1907 paper by his supervisor I. Schur for many of the ideas.) Frobenius' character formula is naturally expressed in terms of symmetric polynomials, and Specht constructed his modules inside the polynomial ring in n commuting variables. This would also have been a familiar object from invariant theory.

To give a small example, let $R = F[x_1, x_2, x_3, x_4]$. By Specht's definition $S^{(3,1)}$ is the submodule of R spanned linearly by the 'Spechtian polytabloids'

$$c\left(\begin{array}{|c|c|c|}\hline a & b & c \\ \hline d \\ \cline{1-1}\end{array}\right) = \begin{vmatrix} x_a^0 & x_d^0 \\ x_a^1 & x_d^1 \end{vmatrix} \, |x_b^0| \, |x_c^0| = x_d - x_a.$$

(In general $c(t)$ is a product of Vandermonde determinants corresponding to the columns of the tableau t.) The map sending x_k to $\overline{\underline{\;k\;}}$ gives an isomorphism with the version of $S^{(3,1)}$ defined above. To give another example,

$$c\left(\begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array}\right) = \begin{vmatrix} x_a^0 & x_c^0 \\ x_a^1 & x_c^1 \end{vmatrix} \begin{vmatrix} x_b^0 & x_d^0 \\ x_b^1 & x_d^1 \end{vmatrix} = (x_c - x_a)(x_d - x_b),$$

is a FS_4-generator of $S^{(2,2)}$. Later we will see the Garnir relations which can express a general polytabloid as a sum of polytabloids $e(t)$ where t is standard. In Specht's setup, these express (specializations of) determinantal identities that were probably well known to algebraists of his time. For example, in $S^{(2,2)}$ we have, in James' notation,

$$e\left(\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 & 4 \\ \hline\end{array}\right) = e\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}\right) - e\left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}\right).$$

In Specht's language, this expresses the determinantal identity

$$\begin{vmatrix} 1 & 1 \\ x_2 & x_3 \end{vmatrix}\begin{vmatrix} 1 & 1 \\ x_1 & x_4 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ x_1 & x_3 \end{vmatrix}\begin{vmatrix} 1 & 1 \\ x_2 & x_4 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ x_1 & x_2 \end{vmatrix}\begin{vmatrix} 1 & 1 \\ x_3 & x_4 \end{vmatrix},$$

which is a specialization of the Plücker relation

$$\begin{vmatrix} \alpha & \gamma \\ \beta & \delta \end{vmatrix}\begin{vmatrix} A & C \\ B & D \end{vmatrix} = \begin{vmatrix} A & \gamma \\ B & \delta \end{vmatrix}\begin{vmatrix} \alpha & C \\ \beta & D \end{vmatrix} + \begin{vmatrix} \alpha & A \\ \beta & B \end{vmatrix}\begin{vmatrix} \gamma & C \\ \delta & D \end{vmatrix}.$$

One feature of Specht's construction is that it makes it very easy to prove that $\mathrm{End}_{FS_n} S^\lambda = F$, provided $\mathrm{char}\, F \neq 2$. (This result also holds in many cases when $\mathrm{char}\, F = 2$, but is harder to prove). For example, let $\phi : S^{(2,2)} \to S^{(2,2)}$ be an endomorphism and let

$$x = c\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}\right).$$

We must have $(x\phi)(13) = -x\phi$ and $x\phi(24) = -x\phi$, so $x\phi$ is divisible, in R, by $x_1 - x_3$ and $x_2 - x_4$. Hence $x\phi = \alpha x$ for some $\alpha \in F$. Over the complex numbers, a module is irreducible if and only if it has trivial endomorphism ring, and so Specht's construction leads to a quick proof of the irreducibility of Specht modules.

Finally, it is worth noting that while Specht worked only over the complex numbers, one of the nicest features of his construction is that it gives modules that are defined in a uniform way over all rings.

3. James' Submodule Theorem

Definition 3.1. Let F be a field and let λ be a partition of $n \in \mathbf{N}$. We define a symmetric bilinear form $\langle\, ,\, \rangle$ on M_F^λ by linear extension of

$$\langle \{s\}, \{t\} \rangle = \begin{cases} 1 & \text{if } \{s\} = \{t\} \\ 0 & \text{otherwise.} \end{cases}$$

Therefore given $x = \sum \alpha_s \{s\}$ and $y = \sum \beta_t \{t\} \in M^\lambda$, where the sums are over distinct tabloids, we have

$$\langle x, y \rangle = \sum_{s,t} \alpha_s \beta_t \, \langle \{s\}, \{t\} \rangle = \sum_u \alpha_u \beta_u.$$

When $F = \mathbf{C}$ it is most natural to define the form so that it is Hermitian, i.e. with the notation as above, $\langle x, y \rangle = \sum_t \alpha_t \overline{\beta_t}$.

If $U \subseteq M^\lambda$ we shall write $U^\perp$ for the orthogonal space to U, i.e.

$$U^\perp = \{v \in M^\lambda : \langle x, v \rangle = 0 \text{ for all } x \in U\}.$$

It is useful to note that the form $\langle\ ,\ \rangle$ is S_n-invariant, i.e.

$$\langle xg, yg \rangle = \langle x, y \rangle$$

for all $x, y \in M^\lambda$ and $g \in S_n$. (This follows easily from the case when x and y are tabloids.) One consequence of this invariance is that if $U \subseteq M^\lambda$ is an FS_n-submodule, and $v \in U^\perp$, $g \in S_n$ then

$$\langle vg, x \rangle = \langle v, xg^{-1} \rangle = 0$$

for all $x \in U$. Hence $U^\perp$ is also an FS_n-submodule of M^λ.

Example 3.2. Let F be a field of characteristic p. By definition $S_F^{(2,2)}$ is spanned linearly by the polytabloids $e(t)$ where t is a $(2,2)$-tableau. But by (3) just before Definition 2.5, $e(t)h = \mathrm{sgn}(h)e(t)$ for any $h \in C(t)$, so we need only take those t that are column standard. Hence if

$$x = e\left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}\right), \quad y = e\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}\right), \quad z = e\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & 3 \\ \hline\end{array}\right)$$

then $S^{(2,2)} = \langle x, y, z \rangle$. By writing out x, y and z as linear combinations of tabloids, one can check that $z = -x + y$, and that x and y are linearly independent. (This relation is a special case of the relations mentioned in Remark 2.9 above.) Therefore

$$S^{(2,2)} = \langle x, y \rangle_F \,.$$

Consider the restriction of $\langle\ ,\ \rangle$ to $S^{(2,2)}$. The Gram matrix of this form is, with respect to the basis x, y,

$$G = \begin{pmatrix} \langle x, x \rangle & \langle x, y \rangle \\ \langle y, x \rangle & \langle y, y \rangle \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}.$$

If $p = 2$ we see that the restriction of $\langle\ ,\ \rangle$ to $S^{(2,2)}$ is identically zero. If $p = 3$ then

$$G = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

and so

$$S^{(2,2)} \cap (S^{(2,2)})^\perp = \langle x + y \rangle \,.$$

Calculation shows that, when $p = 3$,

$$x + y = \frac{\overline{\ 1\ 2\ }}{\underline{\ 3\ 4\ }} + \frac{\overline{\ 1\ 3\ }}{\underline{\ 2\ 4\ }} + \frac{\overline{\ 1\ 4\ }}{\underline{\ 2\ 3\ }} + \frac{\overline{\ 2\ 3\ }}{\underline{\ 1\ 4\ }} + \frac{\overline{\ 2\ 4\ }}{\underline{\ 1\ 3\ }} + \frac{\overline{\ 3\ 4\ }}{\underline{\ 1\ 2\ }}$$

is the sum of all distinct $(2,2)$-tabloids. Therefore $\langle x + y \rangle$ affords the trivial representation of S_4.

Exercise:

(1) Let F be a field of characteristic 3. Show that $S_F^{(2,2)} / \langle x + y \rangle$ affords the sign representation of S_4, defined over F.

(2) Let $V_4 = \langle (12)(34), (13)(24) \rangle \leq S_4$. Show that, over any field, the kernel of the representation homomorphism $S_4 \to S^{(2,2)}$ is V_4, and that,

thought of as a representation of $S_4/V_4 \cong S_3$, there is an isomorphism $S^{(2,2)} \cong S^{(2,1)}$.

We now give the critical combinatorial lemma needed to prove James' Submodule Theorem. James states this in more general form (see his Lemma 4.6), but the version below suffices for this section.

Lemma 3.3. *Let λ be a partition of n. If u and t are λ-tableaux then either $ub_t = 0$, or there exists $h \in C(t)$ such that $\{uh\} = \{t\}$ and $ub_t = \pm e(t)$.*

Proof. Suppose that $ub_t \neq 0$. Let i and j be distinct numbers that appear in the same row of u. If these numbers also appear in the same column of t then $(i,j) \in C(t)$ and so

$$ub_t = u\big(1 - (i,j)\big) \sum_g g = 0$$

where the sum is over a set of right-coset representatives for the subgroup $\langle (i,j) \rangle$ of $C(t)$. Therefore:

> *Any two distinct numbers that appear in the same row of u must lie in different columns of t.*

Suppose that λ has k parts. Using the fact just proved, we may choose $h_1 \in C(t)$ such that u and th_1 have the same set of entries in their first rows. Now choose h_2 so that h_2 fixes the first rows of u and th_1, and u and th_1h_2 have the same set of entries in their second rows. Repeat until $h_1, h_2, \ldots, h_{k-1}$ have been chosen. Then u and $th_1 \ldots h_{k-1}$ have the same set of entries in all their rows. Hence if $h = h_1 \ldots h_{k-1}$ then $\{u\} = \{th\}$ and

$$\{u\}b_t = \{t\}hb_t = \pm\{t\}b_t = \pm e(t). \qquad \square$$

Observe that Lemma 3.3 immediately implies that if λ is a partition of n and t is a λ-tableau then

$$M^\lambda b_t = \langle e(t) \rangle. \tag{4}$$

We use this in the proof of the next theorem.

Theorem 3.4 (James' Submodule Theorem)**.** *Let F be a field and let λ be a partition of $n \in \mathbf{N}$. If U is an FS_n-submodule of M^λ then either $U \supseteq S^\lambda$ or $U \subseteq (S^\lambda)^\perp$.*

Proof. Suppose that U is not contained in $(S^\lambda)^\perp$. Since S^λ is spanned by the λ-polytabloids, there exists $v \in U$ and a λ-tableau t such that $\langle v, e(t) \rangle \neq 0$. Using the S_n-invariance of $\langle\ ,\ \rangle$ we have

$$\langle vb_t, \{t\} \rangle = \langle v, \{t\}b_t \rangle = \langle v, e(t) \rangle \neq 0.$$

Using the remark after Lemma 3.3, that $M^\lambda b_t = \langle e(t) \rangle$, it follows that $vb_t = \alpha e(t)$ for some non-zero $\alpha \in F$. Hence $e(t) \in U$. We saw after Definition 2.5 that S^λ is cyclic, generated by any single polytabloid. Hence $S^\lambda \subseteq U$, as required. $\square$

Corollary 3.5. *Let F be a field of characteristic zero and let λ be a partition of $n \in \mathbf{N}$. Then S_F^λ is irreducible.*

Proof. Let U be a non-zero submodule of S_F^λ. Since U is, in particular, a submodule of M_F^λ, it follows from James' Submodule Theorem that either $U \supseteq S^\lambda$ or $U \subseteq S^\lambda \cap (S_F^\lambda)^\perp$. Hence either $U = S^\lambda$, or $U \subseteq S^\lambda \cap (S_F^\lambda)^\perp$. It suffices to prove that $S^\lambda \cap (S_F^\lambda)^\perp = 0$.

If $F = \mathbf{Q}$ or $F = \mathbf{R}$, this is clear, because then the bilinear form $\langle\ ,\ \rangle$ is an inner product, and so if V is *any* subspace of M^λ then $V \cap V^\perp = 0$. If $F = \mathbf{C}$ the same argument holds, provided we define the form to be a Hermitian inner product.

To deal with the general case (or to deal with $F = \mathbf{C}$ while using a bilinear form rather than a Hermitian form), James argues as follows: since the polytabloids span S^λ, we may choose, as in Example 3.2, a basis of S^λ consisting of polytabloids. Let G be the Gram matrix of $\langle\ ,\ \rangle$ restricted to S^λ, with respect to this basis. The entries of G lie in $\mathbf{Z}$, so the rank of G does not depend on F.[2] Therefore $\langle\ ,\ \rangle$ is non-degenerate when restricted to S^λ, and so $S^\lambda \cap (S^\lambda)^\perp = 0$. $\square$

4. Homomorphisms

Definition 4.1. Let $n \in \mathbf{N}$ and let λ and μ be partitions of n, with k and ℓ parts, respectively. We say that λ *dominates* μ, and write $\lambda \trianglerighteq \mu$, if

$$\sum_{i=1}^{j} \lambda_i \geq \sum_{i=1}^{j} \mu_i$$

for all $j \leq \min(k, \ell)$.

Remarks:

(A) The relation $\trianglerighteq$ defines a partial order on the set of partitions of n. It is not a total order. For example, $(4,1,1)$ and $(3,3)$ are incomparable.

[2] Generally if A is a $d \times d$-matrix over a field F then there exist invertible $d \times d$ matrices P and Q over F, and $r \in \mathbf{N}_0$ such that PAQ is equal to the matrix

$$\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

where I_r is the $r \times r$-identity matrix and 0 indicates a zero matrix of the appropriate dimensions. It follows that if K is an extension field of F then the rank of A, thought of either as a matrix over F, or as a matrix over K, is r.

(B) Let $n \in \mathbf{N}$ and let F be a field. For each partition λ of n, let $J(\lambda) \in \mathrm{Mat}_n(F)$ be a nilpotent matrix of Jordan type λ. For instance, we may take $J(\lambda) = \mathrm{diag}(B_{\lambda_1}, \ldots, B_{\lambda_k})$ where

$$B_d = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{d \times d}.$$

Define $N(\lambda) = J(\lambda)^{\mathrm{GL}_n(F)}$. Then $J(\lambda)$ is the set of all $n \times n$ nilpotent matrices over F whose Jordan type is λ. Then

$$\overline{N(\lambda)} \supseteq N(\mu) \iff \lambda \trianglerighteq \mu.$$

Here $\overline{N(\lambda)}$ means the closure of $N(\lambda)$ in the Euclidean topology if $F = \mathbf{R}$ or $F = \mathbf{C}$, or in the Zariski topology otherwise. (The two closures agree in the cases $F = \mathbf{R}$ and $F = \mathbf{C}$.) For example, the closure of

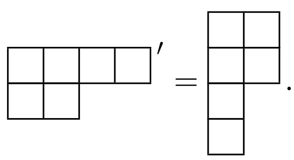
$$N(2) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} : a^2 + bc = 0 \right\} \setminus \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

contains the zero matrix, which is the unique element of $N(1,1)$.

(C) The dominance order appears in many other settings. The Gale–Ryser Theorem is an important example in algebraic combinatorics. To state this result we need some preliminary definitions. The *conjugate* partition μ' to a partition μ is defined by

$$\mu'_j = |\{i : \mu_i \geq j\}|.$$

The Young diagram of μ' is obtained by reflecting the Young diagram of μ. For example, $(4,2)' = (2,2,1,1)$, as seen below.



If B is a $k \times \ell$-matrix then the sequence of *row-sums* of B is

$$\Big(\sum_{j=1}^{\ell} B_{1j}, \ldots, \sum_{j=1}^{\ell} B_{kj}\Big)$$

and the sequence of *column-sums* of B is

$$\Big(\sum_{i=1}^{k} B_{i1}, \ldots, \sum_{i=1}^{k} B_{i\ell}\Big).$$

We say that B is zero-one if each entry of B is either 0 or 1.

Theorem (Gale–Ryser)**.** Let λ and μ be partitions of n. There is a zero-one matrix with row-sums λ and column-sums μ' if and only $\lambda \unlhd \mu$.

The necessity of the condition is fairly easily seen. Suppose that A is a zero-one matrix with row-sums λ and column-sums μ'. There are λ_1 1s in the first row of A so at least λ_1 of the column-sums are non-zero. Hence $\lambda_1 \leq \mu_1$. There are $\lambda_1 + \lambda_2$ 1s in the first two rows of A and there are $\mu_1 - \mu_2$ columns that can contain at most a single 1, and μ_2 that can contain two 1s. Hence

$$\lambda_1 + \lambda_2 \leq (\mu_1 - \mu_2) + 2\mu_2 = \mu_1 + \mu_2.$$

Repeating this argument shows that $\lambda \unlhd \mu$. See Problem Sheet 3 for a proof using representation theory that the condition is also sufficient.

The next lemma is a more general version of Lemma 3.3.

Lemma 4.2. *Let $n \in \mathbf{N}$ and let λ and μ be partitions of n where λ has k parts. Suppose that u is a μ-tableau and t is a λ-tableau such that $\{u\}b_t \neq 0$. Then $\lambda \unrhd \mu$ and there exists $h \in C(t)$ such that, for each $i \in \{1, \ldots, k\}$, all the numbers in the first i rows of u lie in the first i rows of th.*

Proof. Suppose that there exists $h \in C(t)$ with the claimed properties. Then, for each $i \in \{1, \ldots, k\}$, we see that there are $\mu_1 + \cdots + \mu_i$ numbers in the first i rows of u, and these must all appear as one of the $\lambda_1 + \cdots + \lambda_i$ numbers in the first i rows of th. Hence $\lambda \unrhd \mu$.

To construct such an $h \in C(t)$ we use the argument in Lemma 3.3, that any two numbers in the same row of u must appear in different columns of t.

Let $h_1 \in C(t)$ to be the permutation moving the numbers in the first row of u into the first row of t. Suppose inductively we have h_{i-1} such that all the numbers in the first $i-1$ rows of u lie in the first $i-1$ rows of th_{i-1}. Consider the numbers in row i of u. These all lie in different columns of th_{i-1}. Take $g_i \in C(th_{i-1}) = C(t)$ to be the permutation which moves these numbers as high as possible in th_{i-1}, while fixing the entries of the first $i-1$ rows of u. There is at most one entry in each column of t that has to be moved up by g_i, so the entries of row i of u lie in rows 1, ..., i in $th_{i-1}g_i$. So we may take $h_i = h_{i-1}g_i$ and $h = h_1 \ldots h_k$. $\square$

We are now ready to prove the key result on homomorphisms.

Theorem 4.3. *Let λ and μ be partitions of n. If there is a non-zero homomorphism $\theta : S^\lambda \to M^\mu$ of FS_n-modules that extends to a homomorphism $\widetilde{\theta} : M^\lambda \to M^\mu$ then $\lambda \unrhd \mu$.*

Proof. Since S^λ is spanned linearly by the polytabloids $e(t)$ where t is a λ-tableau, there exists a λ-tableau t such that $e(t)\theta \neq 0$. Suppose that

$$\{t\}\widetilde{\theta} = \sum_{\{u\}} \alpha_{\{u\}}\{u\}$$

where the sum is over all μ-tabloids $\{u\}$. Then, since $e(t) = \{t\}b_t$ and $\{t\}\widetilde{\theta}b_t \neq 0$, there exists a μ-tabloid $\{u\}$ such that $\{u\}b_t \neq 0$. Now apply Lemma 4.2. □

The technical condition in Theorem 4.3 on extensions of homomorphisms may fail to hold when F has characteristic two. The smallest example is the unique homomorphism

$$F \cong S^{(1,1)} \to M^{(2)}$$

which fails to extend to a homomorphism $M^{(1,1)} \to M^{(2)}$. (The unique such homomorphism has $S^{(1,1)} \cong F$ in its kernel.) When F has characteristic zero, there are no such problems.

Corollary 4.4. *Let F be a field of characteristic zero. If $S^\lambda_F \cong S^\mu_F$ then $\lambda = \mu$.*

Proof. Since F has characteristic zero, the FS_n-modules are completely reducible (by Maschke's Theorem). Hence $M^\lambda_F = S^\lambda_F \oplus C$ for some complementary FS_n-module C. If $S^\lambda_F \cong S^\mu_F$ then there is an injective FS_n-module homomorphism $\theta : S^\lambda_F \to M^\mu_F$. We extend θ to an FS_n-module homomorphism $\widetilde{\theta} : M^\lambda_F \to M^\mu_F$ by setting $v\widetilde{\theta} = 0$ for $v \in C$. By Theorem 4.3 we have $\lambda \trianglerighteq \mu$. By symmetry $\mu \trianglerighteq \lambda$, so $\lambda = \mu$. □

The following example gives a phenomenon related to the failure of the extension condition in Theorem 4.3 when F has characteristic two.

Example 4.5. Over $\mathbf{F}_2$ there is an isomorphism

$$S^{(5,1,1)} \cong S^{(5,2)} \oplus S^{(7)}.$$

Hence there is a non-zero homomorphism from $S^{(5,1,1)}$ to $S^{(5,2)}$. See Question 1 on Sheet 2 for a proof of this decomposition.

If G is any finite group then the number of complex irreducible representations of G is equal to the number of conjugacy classes of G. There are $|\mathrm{Par}(n)|$ conjugacy classes in S_n, so the Specht modules $S^\lambda_{\mathbf{C}}$ form a complete set of non-isomorphic irreducible $\mathbf{C}S_n$-modules, as λ varies over the partitions of n. (In fact this holds over any field of characteristic zero.)

Over $\mathbf{C}$ any module is completely reducible. So Theorem 4.3 implies that $M^\mu_{\mathbf{C}}$ is a direct sum of Specht modules labelled by partitions λ such that $\lambda \trianglerighteq \mu$. For example, it follows from Questions 2 and 3 on the second problem sheet that if $n \in \mathbf{N}$ then

$$M_{\mathbf{C}}^{(n-r,r)} \cong S_{\mathbf{C}}^{(n-r,r)} \oplus S_{\mathbf{C}}^{(n-r+1,r-1)} \oplus \cdots \oplus S_{\mathbf{C}}^{(n-1,1)} \oplus S_{\mathbf{C}}^{(n)}$$

for any r such that $2r \leq n$. To give another example, since $M_{\mathbf{C}}^{(1^n)}$ is isomorphic to the regular representation of $\mathbf{C}S_n$ we have

$$M_{\mathbf{C}}^{(1^n)} \cong \bigoplus_{\lambda \vdash n} (\dim S^\lambda) S_{\mathbf{C}}^\lambda.$$

5. Construction of simple modules in characteristic p

Overview: Let F be a field. By James' Submodule Theorem, if U is a proper FS_n-submodule of S^λ then $U \subseteq S^\lambda \cap (S^\lambda)^\perp$. Hence if

$$S^\lambda / (S^\lambda \cap (S^\lambda)^\perp)$$

is non-zero then it is a simple FS_n-module. When F has characteristic 0 we saw in Corollary 3.5 that $S^\lambda \cap (S^\lambda)^\perp = 0$ for all partitions λ. To describe the situation in prime characteristic p we need the following definition.

Definition 5.1. Let λ be a partition of n with exactly a_i parts of length i for each $i \in \{1, 2, \ldots, n\}$. Given $p \in \mathbf{N}$, we say that λ is p-regular if $a_i < p$ for all i.

In this section and the next we shall prove the following theorem, mainly following James' proof in §10 and §11 of [8].

Theorem. Let F be a field of prime characteristic p. If λ is a partition of n then $S^\lambda \not\subseteq (S^\lambda)^\perp$ if and only if λ is p-regular, and

$$\left\{ \frac{S^\lambda}{S^\lambda \cap (S^\lambda)^\perp} : \lambda \text{ a } p\text{-regular partition of } n \right\}$$

is a complete set of non-isomorphic simple FS_n-modules.

Remark: It is reasonable to ask for a heuristic reason for this theorem.

- If G is any finite group then the number of isomorphism classes of irreducible $\mathbf{C}G$-modules is equal to the number of conjugacy classes of G. A deeper result of Brauer states that if F is a splitting field for G (i.e. every FG module is absolutely irreducible, in the sense defined at the end of this section), then the number of isomorphism classes of irreducible FG-modules is equal to the number of conjugacy classes of G of elements of order coprime to p. By Proposition 5.11 below, the number of p-regular partitions is equal to the number of irreducible FG-modules. Thus the set of modules specified in the theorem above does at least have the right size (assuming they are all non-zero).

- One consequence of the theorem is that every simple FS_n-module appears as the unique top composition of some Specht module. A similar (but nicer) situation occurs for modules for a quasi-hereditary algebra: If A is a quasi-hereditary algebra then there is a canonical set of *standard* modules for A such that any projective A-module is filtered by standard modules, and the top composition factors of the standard modules form a complete set of non-isomorphic simple A-modules. See [3] for an introduction to this area.

- An important example of quasi-hereditary algebras are the Schur algebras. The module category of $S(n,r)$, defined over an infinite field F, is equivalent the category of polynomial modules of degree r for the general linear group $\mathrm{GL}_n(F)$. (See [7] for an introduction.) For any $n \in \mathbf{N}$ there is a Schur algebra A and an idempotent $e \in A$ such that $FS_n = eAe$. This means that the symmetric group algebra FS_n inherits some of the desirable properties of quasi-hereditary algebras.

- Cellular algebras generalize quasi-hereditary algebras in a way modelled on symmetric group algebras. For an introduction, see [11]. An analogous version of the theorem above holds for the simple modules of any cellular algebra.

Example 5.2. Let F have prime characteristic p.

(1) In Example 3.2 we saw that the restriction of $\langle\ ,\ \rangle$ to $S_F^{(2,2)}$ is zero if and only if F has characteristic 2. Correspondingly, $(2,2)$ is p-regular if and only if $p > 2$.

(2) Let $\lambda = (2,1,1)$. Let

$$t_1 = \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}\ , \quad t_2 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}\ , \quad t_3 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}\ , \quad t_4 = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}$$

be the four column standard $(2,1,1)$-tableaux. Reasoning as in Example 3.2 we see that if F is any field then

$$S_F^{(2,1,1)} = \langle e(t_1), e(t_2), e(t_3), e(t_4) \rangle .$$

These polytabloids are not linearly independent: one can check that

$$e(t_1) = e(t_2) - e(t_3) + e(t_4).$$

However $e(t_2), e(t_3), e(t_4)$ are linearly independent, because each involves a tabloid not appearing in the other two. For instance, since 1 and 2 appear in the first column of both t_3 and t_4, the polytabloid $e(t_2)$ is the unique polytabloid having

$$\begin{array}{l}\hline 1\ 2\\ \hline 3\\ \hline 4\\ \hline\end{array}$$

as a summand. Each polytabloid $e(t_i)$ has 6 summands, so $\langle e(t_i), e(t_i)\rangle = 6$ for each i. To calculate $\langle e(t_2), e(t_3)\rangle$ we argue that any common tabloid must have 2 and 3 in its first row. Calculation shows that

$$e(t_2) = -\begin{array}{l}\hline 3\ 2\\ \hline 1\\ \hline 4\\ \hline\end{array} + \begin{array}{l}\hline 3\ 2\\ \hline 4\\ \hline 1\\ \hline\end{array} + \cdots,$$

$$e(t_3) = -\begin{array}{l}\hline 2\ 3\\ \hline 1\\ \hline 4\\ \hline\end{array} + \begin{array}{l}\hline 2\ 3\\ \hline 4\\ \hline 1\\ \hline\end{array} + \cdots$$

so $\langle e(t_2), e(t_3)\rangle = 2$. Similar calculations show that $\langle e(t_3), e(t_4)\rangle = 2$ and $\langle e(t_2), e(t_4)\rangle = -2$. Hence the matrix for $\langle\ ,\ \rangle$ on $S_F^{(2,1,1)}$ is

$$\begin{pmatrix} 6 & 2 & -2 \\ 2 & 6 & 2 \\ -2 & 2 & 6 \end{pmatrix}.$$

It is clear this matrix is zero if and only if $p = 2$. Correspondingly $(2,1,1)$ is p-regular if and only if $p > 2$. In fact the determinant of the matrix is 128, and so when $p > 2$ the matrix always has full rank. Hence $S^{(2,1,1)} \cap (S^{(2,1,1)})^\perp = 0$ and $S_F^{(2,1,1)}$ is irreducible in these cases.

It is an instructive exercise to generalize Example 5.2(2) to the partitions $(2, 1^{n-2})$. (Question 6 on Sheet 1 is relevant.)

Remark on integral Specht modules: Work over $\mathbf{F}_2$. We saw above that the bilinear form $\langle\ ,\ \rangle$ is identically zero. If we divide each entry in the matrix by 2 we obtain a symmetric, S_4-invariant bilinear form on $S_{\mathbf{F}_2}^{(2,1,1)}$ with matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

The radical $S^{(2,1,1)} \cap (S^{(2,1,1)})^\perp$ of this new form is

$$V = \langle e(t_1) - e(t_2), e(t_1) - e(t_3)\rangle.$$

One can check that V is a simple $\mathbf{F}_2 S_4$-module: in fact $V \cong S^{(3,1)}/S^{(3,1)} \cap (S^{(3,1)})^\perp$. There is also an isomorphism $V \cong S^{(2,2)}$.

This idea is usually applied to integral Specht modules. Given an integral Specht module $S_{\mathbf{Z}}^\lambda$ and a prime p we can define a filtration:

$$S_{\mathbf{Z}}^\lambda = U_0 \supseteq U_1 \supseteq U_2 \supseteq \ldots$$

where

$$U_i = \{u \in S^\lambda_{\mathbf{Z}} : p^i \mid \langle u, v\rangle \text{ for all } v \in S^\lambda_{\mathbf{Z}}\}.$$

If we then reduce mod p, by quotienting out by $pS^\lambda_{\mathbf{Z}}$, we get a filtration of the Specht module S^λ_F. In this example we have

$$U_1 = \big\langle e(t_1) - e(t_2), e(t_1) - e(t_3), 2e(t_1)\big\rangle_{\mathbf{Z}}$$

and

$$\frac{U_1}{2S^{(2,1,1)}_{\mathbf{Z}}} \cong V$$

where V is the simple $\mathbf{F}_2S_4$-module defined above.

The Jantzen–Schaper formula gives some information about the simple modules that appear. See [1] for one account. In [4] Fayers used the Jantzen–Schaper formula to determine all the Schaper layers for Specht modules labelled by hook partitions $(n-r, 1^r)$ and two-row partitions.

The following two lemmas will imply that $S^\lambda \not\subseteq (S^\lambda)^\perp$ if and only if λ is p-regular. The lemmas make sense for Specht modules defined over any field but are strongest for Specht modules defined over $\mathbf{Z}$.

Lemma 5.3. *Let λ be a partition with exactly a_j parts of size j for each $j \in \{1, \ldots, n\}$. If t and t' are λ-tableaux then $\langle e(t), e(t')\rangle$ is a multiple of $\prod_{j=1}^n a_j!$.*

Proof. We shall say that two λ-tabloids $\{u\}$ and $\{v\}$ are *Foulkes equivalent* if it is possible to obtain $\{v\}$ from $\{u\}$ by reordering the rows of u. (This term is not standard.) Let

$$T = \big\{\{u\} : \{u\} \text{ is a summand in both } e(t) \text{ and } e(t')\big\}.$$

If $\{u\} \in T$ and $\{v\}$ is Foulkes equivalent to $\{u\}$ then there exist permutations $h \in C(t)$ and $h' \in C(t')$ which permute the entries in the rows of $\{u\}$ as blocks for their action, such that

(i) $\{u\}h = \{u\}h' = \{v\}$,
(ii) $\operatorname{sgn} h = \operatorname{sgn} h'$.

(Here (ii) holds because we can swap any two rows of length r in $\{u\}$ by a product of r disjoint transpositions in either $C(t)$ or $C(t')$.) Hence T is a union of Foulkes equivalence classes. Moreover, if $\{u\} \in T$ then the contribution from $\{u\}$ to the inner product $\langle e(t), e(t')\rangle$, namely

$$\langle \{u\}, e(t)\rangle\langle\{u\}, e(t')\rangle,$$

depends only on the Foulkes class containing u. Since all classes have size $\prod_{j=1}^n a_j!$, it follows that $\langle e(t), e(t')\rangle$ is a multiple of $\prod_{j=1}^n a_j!$. $\square$

Example 5.4. Let

$$t = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \hline \end{array} \quad t' = \begin{array}{|c|c|c|}\hline 1 & 4 & 7 \\ \hline 2 & 5 & 8 \\ \hline 3 & 6 & 9 \\ \hline \end{array} \quad u = \begin{array}{|c|c|c|}\hline 1 & 5 & 9 \\ \hline 2 & 6 & 7 \\ \hline 3 & 4 & 8 \\ \hline \end{array} \quad v = \begin{array}{|c|c|c|}\hline 1 & 6 & 8 \\ \hline 2 & 4 & 9 \\ \hline 3 & 5 & 7 \\ \hline \end{array}.$$

One Foulkes class of tabloids common to both $e(t)$ and $e(t')$ has representative $\{u\}$ since

$$\{t\}(47)(25)(39) = \{u\} = \{t'\}(456)(798).$$

The Foulkes equivalence class of $\{u\}$ consists of the six tabloids

$$\left\{ \begin{array}{c}\hline 1\ 5\ 9 \\ \hline 2\ 6\ 7 \\ \hline 3\ 4\ 8 \\ \hline \end{array}, \begin{array}{c}\hline 1\ 5\ 9 \\ \hline 3\ 4\ 8 \\ \hline 2\ 6\ 7 \\ \hline \end{array}, \begin{array}{c}\hline 2\ 6\ 7 \\ \hline 1\ 5\ 9 \\ \hline 3\ 4\ 8 \\ \hline \end{array}, \begin{array}{c}\hline 2\ 6\ 7 \\ \hline 3\ 4\ 8 \\ \hline 1\ 5\ 9 \\ \hline \end{array}, \begin{array}{c}\hline 3\ 4\ 8 \\ \hline 1\ 5\ 9 \\ \hline 2\ 6\ 7 \\ \hline \end{array}, \begin{array}{c}\hline 3\ 4\ 8 \\ \hline 2\ 6\ 7 \\ \hline 1\ 5\ 9 \\ \hline \end{array} \right\}.$$

For example, to swap the first two rows in $\{u\}$ we may either apply $(17)(25)(69) \in C(t)$ or $(12)(56)(79) \in C(t')$. As claimed in the proof, these permutations both have sign -1. Since $\{u\}$ appears with sign -1 in $e(t)$ and sign $+1$ in $e(t')$, and the contribution from the Foulkes class of $\{u\}$ to $\langle e(t), e(t')\rangle$ is -6.

Since

$$\{t\}(285)(369) = \{v\} = \{t'\}(465)(789)$$

there is another Foulkes equivalence class of tabloids common to both $e(t)$ and $e(t')$, namely

$$\left\{ \begin{array}{c}\hline 1\ 6\ 8 \\ \hline 2\ 4\ 9 \\ \hline 3\ 5\ 7 \\ \hline \end{array}, \begin{array}{c}\hline 1\ 6\ 8 \\ \hline 3\ 5\ 7 \\ \hline 2\ 4\ 9 \\ \hline \end{array}, \begin{array}{c}\hline 2\ 4\ 9 \\ \hline 1\ 6\ 8 \\ \hline 3\ 5\ 7 \\ \hline \end{array}, \begin{array}{c}\hline 2\ 4\ 9 \\ \hline 3\ 5\ 7 \\ \hline 1\ 6\ 8 \\ \hline \end{array}, \begin{array}{c}\hline 3\ 5\ 7 \\ \hline 1\ 6\ 8 \\ \hline 2\ 4\ 9 \\ \hline \end{array}, \begin{array}{c}\hline 3\ 5\ 7 \\ \hline 2\ 4\ 9 \\ \hline 1\ 6\ 8 \\ \hline \end{array} \right\}.$$

giving a contribution to $\langle e(t), e(t')\rangle$ of $+6$.

If $\{w\}$ is a tabloid appearing in both $e(t)$ and $e(t')$ then the entries in each row w appear in different columns of t, and in different columns of t'. Since the columns of t' are the rows of t, an equivalent condition is that the rows of w form three disjoint transversals for the rows and columns of t. There are six such transversals, namely $\{1,5,9\}$, $\{4,8,3\}$, $\{7,2,6\}$, $\{1,8,6\}$, $\{4,2,9\}$ and $\{7,5,3\}$; these transversals admit a unique partition into two parts each containing three disjoint sets, namely

$$\Big\{\big\{\{1,5,9\},\{4,8,3\},\{7,2,6\}\big\},\big\{\{1,8,6\},\{4,2,9\},\{7,5,3\}\big\}\Big\}.$$

The two parts correspond to the tabloids $\{u\}$ and $\{v\}$ already seen. Hence $\langle e(t), e(t')\rangle = -6+6 = 0$.

Lemma 5.5. *Let λ be a partition of n having exactly a_j parts of size j for each $j \in \{1, \ldots, n\}$. Let t be a λ-tableau and let $t^\star$ be the λ-tableau obtained from t by reversing each row. Then*

$$\langle e(t), e(t^\star) \rangle = \prod_{j=1}^{n} (a_j!)^j.$$

Note that $\{t\} = \{t^\star\}$. The following example should clarify the relationship between $e(t)$ and $e(t^\star)$, and will show some of the key ideas in the proof of the lemma. (This example is a trivial variation on the one given by James in his proof: see [8, Lemma 10.4].)

Example 5.6. Let $\lambda = (4, 3, 3, 2)$ and let t and $t^\star$ be as shown below.

$$t = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 & \multicolumn{1}{c}{} \\ \cline{1-3} 8 & 9 & 10 & \multicolumn{1}{c}{} \\ \cline{1-3} 11 & 12 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} \qquad t^\star = \begin{array}{|c|c|c|c|} \hline 4 & 3 & 2 & 1 \\ \hline 7 & 6 & 5 & \multicolumn{1}{c}{} \\ \cline{1-3} 10 & 9 & 8 & \multicolumn{1}{c}{} \\ \cline{1-3} 12 & 11 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}$$

Suppose that $\{u\}$ is a summand in both $e(t)$ and $e(t^\star)$. So $\{u\} = \{t\}h = \{t^\star\}h^\star$ for some $h \in C(t)$ and $h^\star \in C(t^\star)$. Working down the first column of t we see that

(1) $1h^\star$ is in row 1 of $t^\star h^\star$. Hence $1h^\star = 1h = 1$;
(2) $5h^\star$ and $8h^\star$ are in rows 1, 2 or 3 of $t^\star h^\star$, and in rows 2, 3 or 4 of th. Hence $\{5, 8\}h = \{5, 8\}h^\star = \{5, 8\}$;
(3) $11h$ is in row 4 of th. Hence $11h = 11h^\star = 11$.

The diagram below shows the positions of the elements in the sets $\{1\}$, $\{5, 8\}$, $\{11\}$ that are known to be permuted by h and $h^\star$ in the tableaux t and $t^\star$.

$$\begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 & \multicolumn{1}{c}{} \\ \cline{1-3} 8 & 9 & 10 & \multicolumn{1}{c}{} \\ \cline{1-3} 11 & 12 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} \qquad \begin{array}{|c|c|c|c|} \hline 4 & 3 & 2 & 1 \\ \hline 7 & 6 & 5 & \multicolumn{1}{c}{} \\ \cline{1-3} 10 & 9 & 8 & \multicolumn{1}{c}{} \\ \cline{1-3} 12 & 11 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}$$

Observe that if we remove the first column of t and the final entries in the rows of $t^\star$ we can repeat this argument, showing that h and $h^\star$ also permute the elements in the sets $\{2\}$, $\{6, 9\}$, $\{12\}$, and $\{3\}$, $\{7, 10\}$, and finally $\{4\}$. It is now clear that $h = h^\star$.

Conversely since $t = t^\star$, we clearly have $\{t\}k = \{t^\star\}k$ for all

$$k \in C(t) \cap C(t^\star) = \langle (5, 8), (6, 9), (7, 10) \rangle.$$

Hence the tabloids that appear as summands in both $e(t)$ and $e(t^\star)$ are precisely the tabloids $\{t\}k$ for $k \in C(t) \cap C(t^\star)$ and so $\langle e(t), e(t^\star) \rangle = 2!^3$, as stated by Lemma 5.5.

We now prove Lemma 5.5. Suppose that the tabloid $\{th\} = \{t^\star h^\star\}$ is a summand of both

$$\begin{aligned} e(t) &= \sum_{h \in C(t)} \{t\}h \,\mathrm{sgn}(h), \\ e(t^\star) &= \sum_{h^\star \in C(t^\star)} \{t^\star\}h^\star \,\mathrm{sgn}(h^\star). \end{aligned}$$

Claim: $h = h^\star$ and if $i \in \{1, \ldots, n\}$ and $ih = ih^\star = j$ then i and j lie in rows of equal length of λ.

Proof of claim. We shall prove the claim by induction on the number of columns of the Young diagram of λ.

Let $\alpha_1, \ldots, \alpha_\ell$ be the sequence of distinct part sizes in λ in decreasing order of size. Suppose that there are m_i parts of length α_i. For each $i \in \{1, \ldots, \ell\}$ let X_i be the set of entries in the first column of t that lie in rows of length α_i. Working down the first column of t (or equivalently, the ends of the rows of $t^\star$), we see that $X_i h^\star = X_i$ and $X_i h = X_i$ for each i. Hence the restrictions of h and $h^\star$ to $X_1 \cup \cdots \cup X_\ell$ agree. Moreover, the restricted permutations permute entries within rows of equal length as required by the claim.

Let s be the tableau obtained by removing the first column from t and let $s^\star$ be the tableau obtained by removing the final entry in each row of $t^\star$. By induction the claim holds for the restrictions of h and $h^\star$ to s and $s^\star$. Hence the claim holds for h and $h^\star$ in their action on $\{1, \ldots, n\}$. $\square$

It follows from the first statement in the claim that the set of tabloids $\{u\}$ such that $\{u\}$ is a summand in both $e(t)$ and $e(t^\star)$ is exactly

$$\big\{\{t\}k : k \in C(t) \cap C(t^\star)\big\}.$$

Hence

$$\begin{aligned} \langle e(t), e(t^\star)\rangle &= \sum_{k \in C(t) \cap C(t^\star)} \langle \{t\}k \,\mathrm{sgn}(k), \{t^\star\}k \,\mathrm{sgn}(k)\rangle \\ &= |C(t) \cap C(t^\star)|. \end{aligned}$$

If $k \in C(t)$ then $k \in C(t^\star)$ if and only if for each $i \in \{1, \ldots, n\}$, the rows of t (or equivalent, the rows of $t^\star$) containing i and ik have the same length. Therefore

$$C(t) \cap C(t^\star) = \prod_{j=1}^{n} H_j$$

where H_j is the subgroup of $C(t) \cap C(t^\star)$ that permutes the entries within the rows of t of length j. Since λ has a_j parts of length j, the subgroup H_j has order $(a_j!)^j$. Therefore

$$|C(t) \cap C(t^\star)| = \prod_{j=1}^{n} (a_j!)^j.$$

This completes the proof of Lemma 5.5.

Theorem 5.7. *Let F be a field of prime characteristic p and let λ be a partition. Then $S^\lambda_F \not\subseteq (S^\lambda_F)^\perp$ if and only if λ is p-regular.*

Proof. As usual, we suppose that λ has exactly a_j parts of length j for each $j \in \{1, 2, \ldots, n\}$.

By Lemma 5.3, if t and t' are any λ-tableaux, then we have

$$\langle e(t), e(t') \rangle = \alpha \prod_{j=1}^{n} a_j!$$

for some $\alpha \in F$. By definition, S^λ_F is the F-linear span of all λ-polytabloids, so if λ is not p-regular then $S^\lambda \subseteq (S^\lambda)^\perp$.

Conversely, suppose that λ is p-regular. Let t be any λ-tableau and let $t^\star$ be the tableau obtained by reversing the rows of t. By Lemma 5.5 we have

$$\langle e(t), e(t^\star) \rangle = \prod_{j=1}^{n} (a_j)!^j \neq 0$$

Hence the restriction of $\langle\ ,\ \rangle$ to S^λ is non-zero, and so $S^\lambda \not\subseteq (S^\lambda)^\perp$. $\square$

To show that the irreducible modules coming from the previous theorem are non-isomorphic we need the following result. In its proof, we shall on two occasions use the corollary of Lemma 3.3, that if t is any λ-tableau, then

$$M^\lambda b_t = \langle e(t) \rangle .$$

Theorem 5.8. *Let F be a field of prime characteristic p. Suppose that λ and μ are partitions of n and that λ is p-regular. Let V be a submodule of M^μ. If there exists a non-zero FS_n-module homomorphism $\theta : S^\lambda \to M^\mu / V$ then $\lambda \trianglerighteq \mu$. Moreover, if $\lambda = \mu$ and t is a λ-tableau, then $e(t)\theta \in \langle e(t) + V \rangle$.*

Proof. As usual, we suppose that λ has exactly a_j parts of size j for each $j \in \{1, 2, \ldots, n\}$. Let t be a λ-tableau and let $t^\star$ denote the tableau obtained from t by reversing each of its rows. Since

$$e(t^\star) b_t \in M^\lambda b_t = \langle e(t) \rangle ,$$

we have $e(t^\star) b_t = \alpha e(t)$ for some $\alpha \in F$. Using that $\langle e(t), \{t\} \rangle = 1$ we can determine α as follows:

$$\alpha = \langle e(t^\star) b_t, \{t\} \rangle = \langle e(t^\star), \{t\} b_t \rangle = \langle e(t^\star), e(t) \rangle = \prod_{j=1}^{n} (a_j!)^j$$

where the first equality uses the S_n-invariance of $\langle\ ,\ \rangle$, and the final step uses Lemma 5.5. In particular, $\alpha \neq 0$. Since $e(t)$ generates S^λ and θ is non-zero we have $e(t)\theta \neq 0$. Therefore

$$e(t^\star)\theta b_t = e(t^\star) b_t \theta = \alpha e(t)\theta \neq 0.$$

Note that

$$e(t^\star)\theta b_t \in (M^\mu/V)b_t.$$

Hence $M^\mu b_t \neq 0$, and so there exists a μ-tabloid $\{u\}$ such that $\{u\}b_t \neq 0$. By Lemma 3.3 we get $\lambda \trianglerighteq \mu$. Moreover if $\lambda = \mu$ then we have $M^\mu b_t = \langle e(t) \rangle$ and so

$$e(t^\star)\theta b_t \in \langle e(t) + V \rangle .$$

Since $\alpha e(t)\theta = e(t^\star)\theta b_t$, it follows that $e(t)\theta \in \langle e(t) + V \rangle$. □

Remark: it is worth comparing this theorem with Theorem 4.3. In Theorem 5.8 we assume that λ is p-regular, whereas in Theorem 4.3 we make no assumption on λ, but instead assume that the FS_n-module homomorphism $\theta : S^\lambda \to M^\mu$ extends to a non-zero homomorphism $\hat{\theta} : M^\lambda \to M^\mu$. In either case, the conclusion is the same, that $\lambda \trianglerighteq \mu$.

Corollary 5.9. *Let F be a field of prime characteristic p and let λ and μ be p-regular partitions of $n \in \mathbf{N}$. If*

$$\frac{S^\lambda}{S^\lambda \cap (S^\lambda)^\perp} \cong \frac{S^\mu}{S^\mu \cap (S^\mu)^\perp}$$

then $\lambda = \mu$. Moreover,

$$\mathrm{End}_{FS_n}\Big(\frac{S^\lambda}{S^\lambda \cap (S^\lambda)^\perp}\Big) \cong F.$$

Proof. Let

$$\phi : \frac{S^\lambda}{S^\lambda \cap (S^\lambda)^\perp} \to \frac{S^\mu}{S^\mu \cap (S^\mu)^\perp}$$

be a non-zero FS_n-module homomorphism. Since both modules involved are simple, ϕ is an isomorphism. We may lift ϕ to a non-zero map $\theta : S^\lambda \to M^\mu/V$ where $V = S^\mu \cap (S^\mu)^\perp$ by taking the composition of the maps

$$S^\lambda \to \frac{S^\lambda}{S^\lambda \cap (S^\lambda)^\perp} \xrightarrow{\phi} \frac{S^\mu}{S^\mu \cap (S^\mu)^\perp} \to M^\mu/V.$$

Therefore, by Theorem 5.8, we have $\lambda \trianglerighteq \mu$. By symmetry we also have $\mu \trianglerighteq \lambda$ and so $\lambda = \mu$. Hence, by Theorem 5.8 every λ-tableau t satisfies

$$e(t)\theta = \gamma(e(t) + V)$$

for some $\gamma \in F$. Since $e(t)$ generates $S^\lambda/S^\lambda \cap (S^\lambda)^\perp$, this implies that $\phi = \gamma\,\mathrm{id}$. □

Let F be a field of prime characteristic p. For λ a p-regular partition, let $D^\lambda = S^\lambda/S^\lambda \cap (S^\lambda)^\perp$. By the remark using the submodule theorem made at the start of this section, Theorem 5.7, and Corollary 5.9, the D^λ are non-isomorphic simple FS_n-modules.

It remains to show that any simple FS_n-module is isomorphic to one of the D^λ. This can be proved using the first remark following the statement of the main theorem at the start of this section, but it takes some work to show that F is a splitting field for S_n. So instead we shall follow a remark of James indicating an alternative approach using another theorem of Brauer (taken from [2, 82.6]).

To state this theorem we need two definitions. Recall that an FG-module U is said to be *absolutely irreducible* if $U \otimes_F K$ is irreducible for every extension field $K : F$. A conjugacy class is said to be *p-regular* if its elements have order coprime to p.

Theorem (Brauer)**.** Let F be a field and let G be a finite group. The number of isomorphism classes of absolutely irreducible FG-modules is *at most* the number of conjugacy classes of G of p-regular elements.

To prove that every absolutely irreducible FS_n-module is isomorphic to one of the D^λ, it suffices to prove:

(a) each D^λ is absolutely irreducible,
(b) the number of p-regular partitions of n is equal to the number of partitions of n with no part divisible by p.

Part (a) follows from Corollary 5.9 and the next lemma.

Lemma 5.10. *Let G be a finite group, let F be a field and let U be an irreducible FG-module such that* $\mathrm{End}_F(U) \cong F$*. Then U is absolutely irreducible.*[3]

Proof. Let $K : F$ be an extension field of F. We must show that $U \otimes_F K$ is an irreducible KG-module. Since $\mathrm{End}_{FG}(U) \cong F$ it follows from Jacobson's Density Theorem that the image of the action map $FG \to \mathrm{End}_F(U)$ is all of $\mathrm{End}_F(U)$. Hence the image of the map

$$KG \to \mathrm{End}_K(U \otimes_F K)$$

is also all of $\mathrm{End}_K(U \otimes_F K)$. Therefore KG acts transitively on the non-zero vectors in $U \otimes_F K$ and so $U \otimes_F K$ is an irreducible KG-module. □

Part (b) is proved in the next proposition. Bijective proofs are also known, the first due to Glaisher [6]: a convenient source is [9, page 278].

Proposition 5.11. *Let $p \geq 2$. The number of p-regular partitions of n is equal to the number of partitions of n with no part divisible by p.*

[3]The converse result, that if U is absolutely irreducible then $\mathrm{End}_{FG}(U) \cong F$ also holds, but we do not need this here.

Proof. The generating function for p-regular partitions is

$$F(x) = \prod_{i \geq 1} (1 + x^i + x^{2i} + \cdots + x^{(p-1)i});$$

a partition with a_i parts of size i corresponds to multiplying out the product by choosing ia from the ith term. to a partition with a parts of size i. Hence

$$F(x) = \prod_{i \geq 1} \frac{1 - x^{ip}}{1 - x^i} = \prod_{\substack{i \geq 1 \\ p \nmid i}} \frac{1}{1 - x^i}$$

which is the generating function for partitions with no part divisible by p. □

We note that this argument does not rule out the possibility that there are further irreducible FS_n-modules that are not absolutely irreducible.

6. Standard Basis Theorem

Definition 6.1. If t is a standard tableau then we say that the corresponding polytabloid $e(t)$ is *standard*.

The object of this section is to prove the following theorem giving a basis for S^λ_F over any field F.

Theorem 6.2 (Standard Basis Theorem)**.** *Let λ be a partition of n. If F is a field then the standard λ-polytabloids form a basis for S^λ.*

Remarks:

(1) In Example 2.6(B) the basis we found of $S^{(n-1,1)}$ is in fact the basis of standard $(n-1,1)$-polytabloids. In Example 3.2 we saw that the Standard Basis Theorem holds for $S^{(2,2)}$.

(2) An important consequence of the Standard Basis Theorem is that if λ is a partition and F is a field then the dimension of S^λ_F is equal to the number of standard λ-tableaux, and so is independent of the field F. The Hook-Formula (see [8, Chapter 20] or the reference in (3) in the suggestions for further reading below) is a remarkable combinatorial formula for this number.

(3) Another corollary of the Standard Basis Theorem is the isomorphism $\bigwedge^r S_F^{(n-1,1)} \cong S_F^{(n-r,1^r)}$ over any field F: see Question 6 on Problem Sheet 3.

6.1. **Linear independence.** To show that the standard polytabloids are linearly independent we need the following order on tabloids.

Definition 6.3. Let λ be a partition of n. Given distinct λ-tabloids $\{t\}$ and $\{s\}$, we define

$$\{t\} > \{s\} \iff \begin{array}{l}\text{the greatest number appearing in a different row}\\ \text{in } \{t\} \text{ than } \{s\} \text{ lies in a lower row of } \{t\} \text{ than } \{s\}\end{array}$$

It is straightforward to show that $>$ is a total order on the set of all λ-tabloids for any partition λ.

Working downwards from n down to 1, one sees if $\lambda = (\lambda_1, \ldots, \lambda_k)$ then the greatest λ-tabloid has $n, n-1, \ldots, n-\lambda_k+1$ in its bottom row, $n-\lambda_k, \ldots, n-\lambda_k-\lambda_{k-1}+1$ in its second from bottom row, and so on. So the elements in its ith row are precisely

$$\{\lambda_1 + \cdots + \lambda_{i-1} + 1, \ldots, \lambda_1 + \cdots + \lambda_i\}.$$

For example, the greatest $(4,3,2)$-tabloid is

$$\begin{array}{l}\hline 1\ 2\ 3\ 4 \\ \hline 5\ 6\ 7 \\ \hline 8\ 9 \\ \hline \end{array}.$$

Exercise: Show that the 10 distinct $(3,2)$-tabloids are, in increasing order

$$\begin{array}{l}\hline 3\ 4\ 5 \\ \hline 1\ 2 \\ \hline \end{array} < \begin{array}{l}\hline 2\ 4\ 5 \\ \hline 1\ 3 \\ \hline \end{array} < \begin{array}{l}\hline 1\ 4\ 5 \\ \hline 2\ 3 \\ \hline \end{array} < \begin{array}{l}\hline 2\ 3\ 5 \\ \hline 1\ 4 \\ \hline \end{array} < \begin{array}{l}\hline 1\ 3\ 5 \\ \hline 2\ 4 \\ \hline \end{array}$$
$$< \begin{array}{l}\hline 1\ 2\ 5 \\ \hline 3\ 4 \\ \hline \end{array} < \begin{array}{l}\hline 2\ 3\ 4 \\ \hline 1\ 5 \\ \hline \end{array} < \begin{array}{l}\hline 1\ 3\ 4 \\ \hline 2\ 5 \\ \hline \end{array} < \begin{array}{l}\hline 1\ 2\ 4 \\ \hline 3\ 5 \\ \hline \end{array} < \begin{array}{l}\hline 1\ 2\ 3 \\ \hline 4\ 5 \\ \hline \end{array}.$$

See Problem Sheet 3, Question 9 for a connection with the colexicographic order on subsets of $\mathbf{N}$.

Lemma 6.4. *Let λ be a partition of n and let s be a column standard λ-tableau. If h is a non-identity element of $C(s)$ then $\{s\} > \{sh\}$.*

Proof. Let $m = \max\{j \in \{1, 2, \ldots, n\} : jh \neq j\}$. Suppose that $\ell h = m$. Note that since $h \in C(s)$, we have that ℓ and m lie in the same column of s. Suppose that ℓ lies in row $r(\ell)$ of s and m lies in row $r(m)$ of s. Since s is column standard, $r(\ell) < r(m)$. Therefore the greatest number that appears in a different row in $\{s\}$ and $\{sh\}$, namely m, appears in a lower row in $\{s\}$ than in $\{sh\}$. $\square$

Proposition 6.5. *Let λ be a partition of n. Working over a field F, the set*

$$\{e(s) : s \text{ a standard } \lambda\text{-tableau}\}$$

is F-linearly independent.

Proof. Suppose, for a contradiction, that

$$\sum_t \alpha_t e(t) = 0$$

where the sum is over all standard λ-tableaux t in a non-trivial linear dependency. (So $\alpha_t \neq 0$ for all t in this sum.) Among these λ-tableaux, let $t_{\max}$ be the one such that $\{t\}$ is greatest in the order $>$. Now suppose that s is a standard λ-tableau such that $\alpha_s \neq 0$. Since s and $t_{\max}$ are row standard, either $s = t_{\max}$ or $s \neq t_{\max}$ and so $\{t_{\max}\} > \{s\}$. Moreover, if h is a non-identity element of $C(s)$ then, by Lemma 6.4, we have

$$\{t_{\max}\} \geq \{s\} > \{sh\}.$$

Hence

$$0 = \sum_t \alpha_t e(t) = \alpha_{t_{\max}} \{t_{\max}\} + y$$

where $y \in M_F^\lambda$ is an F-linear combination of tabloids $\{u\}$ such that $\{t_{\max}\} > \{u\}$. This contradicts the linear independence of the distinct λ-tabloids. $\square$

6.2. **Garnir relations.** The 'spanning' part of the proof of Theorem 6.2 is best presented as a result about integral Specht modules. Recall from Definition 2.2 that $M_{\mathbf{Z}}^\lambda$ is the $\mathbf{Z}S_n$-permutation module with free $\mathbf{Z}$-basis given by the set of λ-tabloids. By definition $S_{\mathbf{Z}}^\lambda$ is the $\mathbf{Z}S_n$-submodule of $M_{\mathbf{Z}}^\lambda$ spanned by all the λ-polytabloids.

Definition 6.6. Let λ be a partition of n and let $i \in \mathbf{N}$. Let t be a λ-tableau. Let X be a subset of the entries in column i of t and let Y be a subset of the entries in column $i+1$ of t. Let S_X, S_Y and $S_{X\cup Y}$ denote the full symmetric groups on X, Y and $X \cup Y$, respectively. Let

$$S_{X\cup Y} = \bigcup_{j=1}^{c} (S_X \times S_Y) g_j$$

where the union is disjoint. We say that the element

$$\sum_{j=1}^{c} g_j \operatorname{sgn}(g_j) \in \mathbf{Z}S_n$$

is a *Garnir element* for X and Y.

Note that there is an arbitrary choice of coset representatives involved in this definition. We will see that the choice made is irrelevant in all our applications of Garnir elements.

Example 6.7. Let $\lambda = (2,1)$ and let $t = \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}$. Taking $X = \{2,3\}$ and $Y = \{1\}$ we see that a set of coset representatives for $S_X \times S_Y$ in $S_{X\cup Y}$ is 1, (12) and (13). Hence

$$G_{X,Y} = 1 - (12) - (13)$$

is a Garnir element for X and Y. Observe that

$$\begin{aligned} e(t)G_{X,Y} &= \left(\begin{array}{cc}\hline 2 & 1 \\ \hline 3 & \\ \cline{1-1}\end{array} - \begin{array}{cc}\hline 3 & 1 \\ \hline 2 & \\ \cline{1-1}\end{array}\right)\big(1-(12)-(13)\big) \\ &= \begin{array}{cc}\hline 2 & 1 \\ \hline 3 & \\ \cline{1-1}\end{array} - \begin{array}{cc}\hline 3 & 1 \\ \hline 2 & \\ \cline{1-1}\end{array} - \begin{array}{cc}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array} + \begin{array}{cc}\hline 3 & 2 \\ \hline 1 & \\ \cline{1-1}\end{array} - \begin{array}{cc}\hline 2 & 3 \\ \hline 1 & \\ \cline{1-1}\end{array} + \begin{array}{cc}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array} \\ &= 0. \end{aligned}$$

This example is a special case of a more general result that leads to an algorithm for writing an arbitrary polytabloid as an integral linear combination of standard polytabloids. In the following theorem, λ' denotes the conjugate of the partition λ, as defined in Remark C on page 13.

Theorem 6.8. *Let λ be a partition of n and let t be a λ-tableau. Let $i \in \mathbf{N}$, let X be a subset of the entries in column i of t, and let Y be a subset of the entries in column $i+1$ of t. Let $G_{X,Y} \in \mathbf{Z}S_n$ be a Garnir element for X and Y. If $|X| + |Y| > \lambda'_i$ then*

$$e(t)G_{X,Y} = 0.$$

Proof. Let

$$G_{X\cup Y} = \sum_{g \in S_{X\cup Y}} g \,\mathrm{sgn}(g).$$

It is clear that

$$G_{X\cup Y} = \Big(\sum_{h\in S_X} h\,\mathrm{sgn}(h)\Big)\Big(\sum_{k\in S_Y} k\,\mathrm{sgn}(k)\Big)G_{X,Y}.$$

Moreover, since $e(t)h\,\mathrm{sgn}(h) = e(t)$ and $e(t)k\,\mathrm{sgn}(k) = e(t)$ for all $h \in S_X$ and $k \in S_Y$ we have

$$e(t)G_{X\cup Y} = |X|!|Y|!e(t)G_{X,Y}.$$

Therefore, since we work over $\mathbf{Z}$ and $e(t) = \sum_{h\in C(t)}\{th\}\,\mathrm{sgn}(h)$, it will suffice to show that

$$\{th\}G_{X\cup Y} = 0$$

for each $h \in C(t)$. If $h \in C(t)$ then, since $|X| + |Y| > \lambda'_i$, there exist $x \in X$ and $y \in Y$ such that x and y lie in the same row of th. Now

$$\{th\}G_{X\cup Y} = \{th\}(1-(xy))\sum_{j=1}^{d} k_j\,\mathrm{sgn}(k_j) = 0$$

where $k_1, \dots, k_d$ is a set of coset representatives for $\langle (xy)\rangle$ inside $S_{X\cup Y}$. The theorem follows. $\square$

6.3. **The standard polytabloids span $S^\lambda_{\mathbf{Z}}$.** Let λ be a partition. If t is a λ-tableau and $h \in C(t)$ then $e(th) = e(t)\,\mathrm{sgn}(h) = \pm e(t)$. So it is clear that

$$S^\lambda_{\mathbf{Z}} = \langle e(t) : t \text{ a column standard } \lambda\text{-tableau} \rangle .$$

To show that in fact $S^\lambda_{\mathbf{Z}}$ is spanned by the standard tableaux we need an analogous definition to Definition 6.3.

Definition 6.9. Let λ be a partition. Given distinct column standard λ-tableaux t and s, we define

$$t \succ s \iff \begin{array}{l}\text{the greatest number appearing in a different}\\ \text{column in } t \text{ than } s \text{ lies in a column further to}\\ \text{the right in } t \text{ than } s.\end{array}$$

Note that this is the opposite order to that used by James on page 30 of [8]. See Question 7 on Sheet 3 for one reason for preferring the order as defined above.

The column standard $(2,2)$-tableaux are, in increasing order

$$\begin{array}{|c|c|}\hline 3 & 1 \\ \hline 4 & 2 \\ \hline\end{array} \prec \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 4 & 3 \\ \hline\end{array} \prec \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & 3 \\ \hline\end{array} \prec \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 & 4 \\ \hline\end{array} \prec \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array} \prec \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}$$

For instance, since we compare on the largest number in a different column, any tableau with 4 in the first column comes before any tableau with 4 in the second column.

Lemma 6.10. *Let λ be a partition and let s be a column standard λ-tableau. Then $e(s) \in S^\lambda_{\mathbf{Z}}$ is a $\mathbf{Z}$-linear combination of standard λ-polytabloids.*

Proof. We may suppose that s is not standard. By induction we may assume that if t is column standard and $t \succ s$ then $e(t)$ is a $\mathbf{Z}$-linear combination of standard λ-polytabloids.

Since s is not standard there is some row, say row q, that is not increasing. Thus there exists $i < \lambda_1$ such that the entries in column i of s in rows q up to λ'_i are

$$x_q < x_{q+1} < \cdots < x_{\lambda'_i},$$

the entries in rows 1 up to q of column $i+1$ of s are

$$y_1 < y_2 < \cdots < y_q,$$

and $x_q > y_q$. Set $X = \{x_q, x_{q+1}, \ldots, x_{\lambda'_i}\}$ and $Y = \{y_1, y_2, \ldots, y_q\}$. Let $G_{X,Y}$ be a Garnir element corresponding to the sets X and Y. We may suppose that $G_{X,Y} = \sum_{j=1}^k g_j \,\mathrm{sgn}(g_j)$ where g_j is a product of transpositions swapping a subset of Y with a subset of X. Let g_1 be the identity, corresponding to the case where both subsets are empty.

By Theorem 6.8 we have $e(s)G_{X,Y} = 0$, so

$$e(s) = -\sum_{j=2}^{d} \operatorname{sgn}(g_j)e(sg_j).$$

It is possible that some of the sg_j are not column standard. (We can assume that the entries of sg_j in the positions occupied by X and Y are increasing, but not anything stronger than this: see Example 6.11 below.) For each $j \in \{2,\ldots,d\}$ let u_j be the column standard tableau whose columns agree setwise with sg_j. Then

$$e(s) = \sum_{j=2}^{k} \pm e(u_j)$$

for some appropriate choice of signs. It therefore suffices to show that $u_j \succ s$ for each $j \in \{2,\ldots,d\}$.

Since $(S_X \times S_Y)g_j \neq S_X \times S_Y$ we have $X \cap Yg_j \neq \emptyset$. By choice of the g_j, the effect of applying g_j to s is to move the elements in $X \cap Yg_j$ from X to Y. Let $x_{\max}$ be the greatest element of $X \cap Yg_j$. Since g_j fixes all elements not in $X \cup Y$, and if $x \in X$ and $y \in Y$ then $x > y$, we see that $x_{\max}$ is the largest element that appears in a different column in s and sg_j, and that in sg_j, $x_{\max}$ lies in column $j+1$. It follows that $u_j \succ s$, as required. $\square$

Example 6.11. Let $t = \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 4 & 3 \\ \hline\end{array}$. Following the proof of Lemma 6.10 we might take $X = \{4\}$ and $Y = \{1,3\}$. A Garnir element for these sets is

$$G_{X,Y} = 1 - (14) - (34),$$

so by Theorem 6.8 we have

$$\begin{aligned} e\left(\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 4 & 3 \\ \hline\end{array}\right) &= e\left(\begin{array}{|c|c|}\hline 2 & 4 \\ \hline 1 & 3 \\ \hline\end{array}\right) + e\left(\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 & 4 \\ \hline\end{array}\right) \\ &= e\left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}\right) + e\left(\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 & 4 \\ \hline\end{array}\right). \end{aligned}$$

Note that by replacing (14) with the alternative coset representative

$$(134) \in (S_X \times S_Y)(14) = \{(14), (13)(14)\}$$

we would get $e\left(\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 4 & 3 \\ \hline\end{array}\right)(134) = e\left(\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 & 4 \\ \hline\end{array}\right)$ which has its second column in increasing order. But in both cases the first column must have 1 in its second row, so we cannot immediately obtain a column standard tableau.

The second tableau in the sum is still not standard, so we repeat the argument with $X = \{2,3\}$ and $Y = \{1\}$. This leads to

$$e\left(\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 4 & 3 \\ \hline\end{array}\right) = e\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}\right).$$

This can be seen (in an easier way) working directly from the definition of polytabloids. (See Question 8 on Sheet 3.)

Remarks:

(1) There are other relations that can be used to write an arbitrary λ-polytabloid as a linear combination of standard polytabloids. Fulton's quadratic relations (see [5, §7.4]) are often easier to use in practice than the Garnir relations.

(2) It follows from [10, Proposition 4.1] that if λ is a partition and t is any column standard λ-tableau then

$$e(t) = e(\bar{t}) + x$$

where $\bar{t}$ is the row standard λ-tableau obtained from t by rearranging its rows into increasing order, and x is a $\mathbf{Z}$-linear combination of polytabloids $e(s)$ such that s is standard and $t \succ s$. (By Question 4 on Sheet 1, the tableau $\bar{t}$ is in fact standard.)

We are now ready to prove the Standard Basis Theorem. Let F be a field and let λ be a partition of n. By Proposition 6.5 the standard λ-polytabloids are linearly independent over F. By the argument at the start of §6.3, it suffices to show that if t is a column-standard λ-tableau then $e(t)$ is an F-linear combination of standard λ-polytabloids.

Thinking of $e(t)$ as an element in $S^\lambda_{\mathbf{Z}}$, Lemma 6.10 implies that

$$e(t) = \sum_s c_s e(s)$$

where $c_s \in \mathbf{Z}$ and the sum is over all standard λ-tableaux s. But the same equation holds if we think of $e(s)$ as an element of S^λ_F, and regard the c_s as elements of F, so we are done.

We remark that the Standard Basis Theorem also holds for integral Specht modules: linear independence over $\mathbf{Z}$ is clear from the argument in Proposition 6.5, and the harder part, that the standard λ-polytabloids span $S^\lambda_{\mathbf{Z}}$, follows from Lemma 6.10.

This remark has the following technical consequence. Let p be prime. The tensor product sends the inclusion $S^\lambda_{\mathbf{Z}} \to M^\lambda_{\mathbf{Z}}$ to a homomorphism of $\mathbf{F}_p S_n$-modules

$$S^\lambda_{\mathbf{Z}} \otimes_{\mathbf{Z}} \mathbf{F}_p \to M^\lambda_{\mathbf{Z}} \otimes_{\mathbf{Z}} \mathbf{F}_p.$$

It is clear that the codomain is isomorphic to $M^\lambda_{\mathbf{F}_p}$ as an $\mathbf{F}_p S_n$-module. There is an obvious homomorphism of $\mathbf{F}_p S_n$-modules

$$S^\lambda_{\mathbf{Z}} \otimes_{\mathbf{F}_p} \mathbf{Z} \to S^\lambda_{\mathbf{F}_p}$$

defined by $e(t) \otimes \alpha \mapsto \alpha e(t)$. It follows from the Standard Basis Theorem that this map is an isomorphism. So the induced map is simply the expected inclusion of $S^\lambda_F \to M^\lambda_F$. Therefore, there is an isomorphism of $\mathbf{Z}S_n$-modules:

$$\frac{S^\lambda_{\mathbf{Z}}}{pS^\lambda_{\mathbf{Z}}} \cong S^\lambda_{\mathbf{Z}} \otimes \mathbf{F}_p \cong S^\lambda_{\mathbf{F}_p}.$$

Example 6.12. To see that there is something non-trivial going on here, consider the $\mathbf{Z}S_2$-permutation module $M_{\mathbf{Z}}$ with $\mathbf{Z}$-basis e_1, e_2 and its $\mathbf{Z}$-submodule $U_{\mathbf{Z}} = \langle e_1 + 2e_2, e_2 + 2e_1 \rangle$. Let $M_{\mathbf{F}_3}$ be the $\mathbf{F}_3 S_2$-module with $\mathbf{F}_3$-basis v_1, v_2. The surjective map

$$M_{\mathbf{Z}} \otimes_{\mathbf{Z}} \mathbf{F}_3 \to M_{\mathbf{F}_3}$$

defines by $e_i \otimes \alpha \mapsto \alpha v_i$ restricts to a map

$$U_{\mathbf{Z}} \otimes_{\mathbf{Z}} \mathbf{F}_3 \to M_{\mathbf{F}_3}$$

with image $\langle e_1 + 2e_2, e_2 + 2e_1 \rangle = \langle e_1 + 2e_2 \rangle$. So in this case the map induced by the inclusion $U_{\mathbf{Z}} \to M_{\mathbf{Z}}$ is not injective.

We end with an immediate corollary of the Standard Basis Theorem.

Corollary 6.13. *Let λ be a partition of $n \in \mathbf{N}$. The dimension of S^λ_F is equal to the number of standard λ-tableaux. It is therefore independent of the field F.*

7. Some suggestions for further reading

(1) James' proof of the branching rule for representations of symmetric groups: see §9 of his book.

(2) Semistandard homomorphisms between Young permutation modules: see §13 of his book. (This leads to a combinatorial description of the multiplicity of $S^\lambda_{\mathbf{C}}$ as a summand of $M^\mu_{\mathbf{C}}$ for $\lambda \trianglerighteq \mu$.)

(3) Hook formula for the degreees of the characters of S_n: one elegant proof was given by Greene, Nijenhius and Wilf in *Adv. in Math.* **31** (1979) 104–109.

(4) Symmetric functions are closely related to the representation theory of the symmetric group. Sagan's book: *The Symmetric Group*, Graduate Texts in Mathematics 203, Springer 2001 has an accessible introduction.

(5) The representations of symmetric groups labelled by hook partitions $(n-r, 1^r)$ are often useful as a source of examples. A nice account of their properties (including the result in Question 6) was given by Hamernik in *Specht modules and the radical of the group ring over the symmetric group γ_p*, Comm. Algebra. **4** (1976) 435–476.

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