Character values and decomposition matrices of symmetric groups

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Abstract

The relationships between the values taken by ordinary characters of symmetric groups are exploited to prove two theorems in the modular representation theory of the symmetric group.

1. The decomposition matrices of symmetric groups in odd characteristic have distinct rows. In characteristic 2 the rows of a decomposition matrix labelled by the different partitions λ and μ are equal if and only if λ and μ are conjugate. An analogous result is proved for Hecke algebras.

2. A Specht module for the symmetric group S_n, defined over an algebraically closed field of odd characteristic, is decomposable on restriction to the alternating group A_n if and only if it is simple, and the labelling partition is self-conjugate. This result is generalised to an arbitrary field of odd characteristic.

Keywords: symmetric group, decomposition matrix, Specht module, alternating group, centre of group algebra

1 Introduction

In this paper we solve two problems in the modular representation theory of the symmetric group. The first asks for a necessary and sufficient condition for two rows of a decomposition matrix of a symmetric group to be equal. The second asks for a characterisation of the Specht modules which decompose on restriction from the symmetric group to the alternating group. Although these problems may seem quite different from one another, both can be solved by similar arguments using the ordinary characters of the symmetric group. In fact, both problems can be reduced to questions typified by the following:

Question 1.1. *Suppose that two ordinary irreducible characters of the symmetric group agree on all elements of order not divisible by* 3 *(that is,* $3'$*-elements) — must they be the same?*

We give a general strategy for answering questions such as this in §2. The idea is first of all to make a strategic choice of generators for $Z(\mathbf{Q}S_n)$, the centre of the rational group algebra of the symmetric group S_n, and then to use the central characters of symmetric groups to deduce algebraic relationships between the values taken by a fixed ordinary irreducible character on different conjugacy classes. The main results we prove concerning character values may be found below in Corollaries 2.2, 2.3, and 2.7.

To give a representative example, Corollary 2.3 implies that, given the values taken by an ordinary irreducible character of a symmetric group on $3'$-elements, one can determine all its remaining values. Thus the question posed above has an affirmative answer. As this example may suggest, our results on character values are of some independent interest. In §2.5 we give some questions they inspire.

Many results in the modular representation theory of the symmetric group have been proved by examining the centres of the *integral* group algebras $\mathbf{Z}S_n$. For example, the Nakayama Conjecture has been proved by both Farahat–Higman [2] and Murphy [16] by different variations on this technique. (For some more recent results on $Z(\mathbf{Z}S_n)$, see [17].) It is an important feature of the method used in this paper that we get all our results on modular representations by considering the centres of the more easily handled *rational* group algebras $\mathbf{Q}S_n$.

We now outline the problems that will be solved using the results on character values contained in §2.

1.1 Decomposition matrices

A *partition* of a number $n \in \mathbf{N}$ is a sequence $(\lambda_1, \ldots, \lambda_k)$ of positive integers such that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k \geq 1$ and $\lambda_1 + \ldots + \lambda_k = n$. To indicate that λ is a partition of n we write $\lambda \vdash n$.

Let F be a field and let S^λ be the Specht module for FS_n labelled by the partition λ of n. For the definition and some examples of these modules see Chapters 4 and 5 of [13]. We recall here that if F has characteristic zero then every Specht module is simple, and each simple FS_n-module is isomorphic to a unique Specht module. If F has prime characteristic p then this is no longer the case. However, if λ is p-regular — that is, λ has no more than $p-1$ parts of any given size — then S^λ has a simple top, denoted D^λ. The modules D^λ for λ a p-regular partition of n are pairwise non-isomorphic and give all the simple representations of FS_n. We record the composition factors of Specht modules in characteristic p in the *decomposition matrix* $D_p(n)$, defined by letting $D_p(n)_{\lambda\nu}$ be the number of composition factors of S^λ that are isomorphic to D^ν.

A fundamental problem in modular representation theory is to determine the decomposition matrices of symmetric groups. In §3 we prove the following theorem.

Theorem 1.2. *Let p be prime and let $n \in \mathbf{N}$.*

(i) *If $p > 2$ then the rows of $D_p(n)$ are mutually distinct.*

(ii) *If $p = 2$ then the rows labelled by λ and μ are the same if and only if $\lambda = \mu$ or $\lambda = \mu'$, the conjugate partition to μ.*

Thus in odd characteristic, a Specht module is determined by its set of composition factors. In characteristic 2, there are at most two Specht modules with any given set of composition factors. (Recall that if λ is a partition, then its conjugate λ' is the partition defined by $\lambda'_i = |\{j : \lambda_j \geq i\}|$. The diagram of λ' is obtained from that of λ by reflecting it in its main diagonal.)

Remarks on Theorem 1.2.

(1) It is well known (see [13, Corollary 12.3]) that when the partitions labelling the rows and columns of a decomposition matrix are ordered lexicographically, but with p-regular partitions placed before non-p-regular partitions, the matrix takes a 'wedge' shape, illustrated below by $D_3(5)$.

It is therefore easy to distinguish between the rows labelled by p-regular partitions. The force of Theorem 1.2 comes from the fact that, when n is large compared to p, most partitions are not p-regular. More precisely, if for $\ell \geq 2$ we let $r_\ell(n)$ be the proportion of ℓ-regular partitions of n (here ℓ is not necessarily prime) then

$$r_\ell(n) \sim A n^{1/4}\, \mathrm{e}^{-c\left(1-\sqrt{\frac{\ell-1}{\ell}}\right)\sqrt{n}} \quad \text{as } n \to \infty$$

where $c = 2\sqrt{\pi^2/6}$ and $A \in \mathbf{R}$ depends only on ℓ. The proportion of ℓ-regular partitions therefore tends rapidly to zero. This formula was proved by Hagis using the circle-method (see [10, Corollary 4.2]). It is interesting to see how close one can get to it by less sophisticated

	(5)	(4,1)	(3,2)	(3,1,1)	(2,2,1)
(5)	1				
(4, 1)	·	1			
(3, 2)	·	1	1		
(3, 1, 1)	·	·	·	1	
(2, 2, 1)	1	·	·	·	1
(2, 1, 1, 1)	·	·	·	·	1
(1^5)	·	·	1	·	·

Figure 1: The decomposition matrix of S_5 in characteristic 3.

methods. When $\ell = 2$, I have given an elementary proof (see [21, §5]), but when $\ell > 2$, the strongest result I have been able to obtain by elementary methods is

$$\log r_\ell(n) \sim -c\left(1 - \sqrt{\frac{\ell-1}{\ell}}\right)\sqrt{n} \quad \text{as } n \to \infty.$$

(2) Theorem 3.2 gives the analogue of Theorem 1.2 for the Hecke algebras of symmetric groups. I hope to report later on the situation for alternating groups; for a partial result see Theorem 3.3. As Schur algebras have lower-unitriangular decomposition matrices (see [9, Theorem 3.5a]), the rows of their decomposition matrices are always distinct.

(3) When defined over a field of characteristic 2, Specht modules labelled by different partitions may be isomorphic. In [13, Theorem 8.15] it is shown that whatever the characteristic of the ground field,

$$S^{\lambda'} \cong (S^\lambda)^\star \otimes \mathrm{sgn} \tag{1}$$

where sgn is the sign representation and $\star$ denotes duality. Hence, over a field of characteristic 2, $S^{\lambda'} \cong (S^\lambda)^\star$. Combining this with our Theorem 1.2(ii) gives the following result.

Theorem 1.3. *Let F have characteristic 2. The Specht modules S^λ and S^μ for FS_n are isomorphic if and only if either $\lambda = \mu$, or $\lambda = \mu'$ and S^λ is self-dual.* □

Unfortunately it does not seem easy to classify the self-dual Specht modules in characteristic 2. For example, in characteristic 2, $S^{(5,2)}$ is simple, and hence self-dual, but $S^{(5,1,1)}$ is also self-dual, and even decomposable (see [13, §23.10]).

1.2 The restriction of Specht modules to the alternating group

Our main result is the following theorem, which we prove in §4.

Theorem 1.4. *Let F be an algebraically closed field of odd characteristic. Let λ be a partition of $n \in \mathbf{N}$. The Specht module S^λ defined over F is decomposable on restriction to A_n if and only if S^λ is simple and $\lambda = \lambda'$.*

Remarks on Theorem 1.4.

(1) Theorem 1.4 also holds in characteristic zero, in which case the Specht modules S^λ are always simple. An alternative generalisation of this characteristic zero result is proved in [5], where Ford uses the Ford–Kleshchev proof [6] of the Mullineux conjecture to give a straightforward way of determining the simple FS_n-modules D^λ such that $D^\lambda \cong D^\lambda \otimes \mathrm{sgn}$, and hence (using some basic Clifford theory which we repeat in §4) those irreducible representations of FS_n which split on restriction to A_n.

(2) In [15, Conjecture 5.47] James and Mathas conjectured a necessary and sufficient condition for a Specht module to be simple. Their conjecture was subsequently proved by Fayers (see [3, 4]). His work makes it a simple matter to work with the criterion given in our theorem. In §4.1 we state the James–Mathas condition and use it to generalise Theorem 1.4 to fields of odd characteristic that are not algebraically closed.

2 Results on character values

Our approach uses the central characters of symmetric groups and some elementary properties of the sums of elements in conjugacy classes of symmetric groups. We work all the time over the field of rational numbers.

First we must introduce some notation. Fix $n \in \mathbf{N}$. For $i \in \mathbf{N}$ we let $s_i \in \mathbf{Q}S_n$ be the sum of all i-cycles in S_n, so in particular $s_1 = 1_{S_n}$, the identity element in $\mathbf{Q}S_n$. More generally, if μ is a partition of n, we let s_μ be the sum of all elements in S_n of cycle type μ. When writing the elements s_μ we shall simplify the notation by ignoring parts of size 1; thus $s_{(i)}$ is the same as s_i, and if, for example $n = 9$, then $s_{(3,2)} = s_{(3,2,1^4)}$. (This leads to no ambiguity, as the degree n is fixed throughout this section.) If μ has m parts of size 1 then we say that μ has *support* $n - m$, and write $\operatorname{supp} \mu = n - m$. Let K_μ be the number of elements of S_n of cycle type μ.

If λ is a partition of n, we write χ^λ for the the irreducible ordinary character of S_n afforded by the Specht module S^λ (now defined over a field of characteristic zero). Let $\chi^\lambda(\mu)$ be the value of χ^λ on elements of cycle type μ. The *central character* corresponding to λ is the algebra homomorphism $\omega_\lambda : Z(\mathbf{Q}S_n) \to \mathbf{Q}$ defined by mapping an element $z \in Z(\mathbf{Q}S_n)$ to the scalar by which it acts on the Specht module S^λ. As

$$\omega_\lambda(s_\mu) = \frac{\chi^\lambda(\mu)K_\mu}{\chi^\lambda(1)} \tag{2}$$

the values of χ^λ are determined by ω_λ. Since $\{s_\mu : \mu \vdash n\}$ is a linear basis for $Z(\mathbf{Q}S_n)$, the converse also holds.

2.1 Generating sets for $Z(\mathbf{Q}S_n)$

It appears to have first been proved by Kramer [14] that the cycle sums $s_1, \ldots, s_n$ generate $Z(\mathbf{Q}S_n)$ as a $\mathbf{Q}$-algebra. We shall give a short proof of this result in Proposition 2.1 below. First though we mention an immediate consequence, namely that for each partition μ of n, there is a polynomial $P_\mu(X_1, \ldots, X_n) \in \mathbf{Q}[X_1, \ldots, X_n]$ such that

$$\omega^\lambda(s_\mu) = P_\mu\big(\omega^\lambda(1_{S_n}), \omega^\lambda(s_2), \ldots, \omega^\lambda(s_n)\big).$$

The important feature of these polynomials is that they are entirely independent of the partition λ.

As an example, we determine $P_{(2,2,1^{n-4})}$. A short calculation shows that in $\mathbf{Q}S_n$, $s_2^2 = 2s_{(2,2)} + 3s_3 + n(n-1)/2$, and hence

$$P_{(2,2,1^{n-4})} = -\frac{n(n-1)}{4}X_1 + \frac{1}{2}X_2^2 - \frac{3}{2}X_3.$$

It now follows from (2) that if λ is a partition of n, $\chi^\lambda((1,2,3))$ is determined by the values of χ^λ at the $3'$-elements 1_{S_n}, $(1,2)$, and $(1,2)(3,4)$. A straightforward generalisation of this fact, given in part (ii) of the proposition below, shows that Question 1.1 has an affirmative answer.

In connection with these polynomials, it is worth mentioning that the values taken by ordinary characters on cycles may easily be calculated using the Murnagham–Nakayama rule (see [13, Ch. 21]). Another way to find these values, of more theoretical interest, is to use the combinatorial interpretation of Frumkin, James and Roichman [7]. For cycles of small length there are also some interesting explicit formulae (see for instance [11]).

Proposition 2.1. *Let μ be a partition of n and let $\ell \in \mathbf{N}$.*

(i) *The $\mathbf{Q}$-algebra generated by*

$$\{s_i : i \leq \operatorname{supp} \mu\}$$

contains s_μ. Moreover, if μ labels a conjugacy class of permutations of sign -1, then s_μ may be expressed as a polynomial in the s_i in such a way that in every monomial term in the expression, at least one class sum s_{2j} appears.

(ii) *If $\ell > 2$ then the conjugacy class sums*

$$X_\ell(n) = \{s_i : 1 \leq i \leq n,\ \ell \not| \ i\} \cup \{s_{(\ell j-1,2)} : 1 < \ell j < n\}$$

generate $Z(\mathbf{Q}S_n)$ as a $\mathbf{Q}$-algebra.

(iii) *If ℓ is odd and $\ell > 2$ then the conjugacy class sums*

$$Y_\ell(n) = \{s_i : 1 \leq i \leq n,\ \textit{if } i \textit{ is even then } \ell \not| \ i\} \cup \{s_{(2\ell j-1,2)} : 1 < 2\ell j < n\}$$

generate $Z(\mathbf{Q}S_n)$ as a $\mathbf{Q}$-algebra. Moreover, if μ labels a conjugacy class of permutations of sign -1 then s_μ may be expressed as a polynomial in the elements of $Y_\ell(n)$ in such a way that in every monomial term in the expression, at least one class sum s_{2j} or $s_{(2\ell j-1,2)}$ appears.

Proof. Suppose that $\mu = (n^{a_n}, \ldots, 2^{a_2}, 1^{a_1})$.

(i) We work by induction on $\operatorname{supp}\mu$. We have

$$s_1^{a_1} s_2^{a_2} \ldots s_n^{a_n} = \alpha s_\mu + y$$

where α is a strictly positive integer and y is an integral linear combination of conjugacy class sums s_λ for partitions λ of support at most $\operatorname{supp}\mu - 1$. Moreover, if μ labels a conjugacy class of permutations of sign -1 then so does every λ which appears in y. Hence, by induction, y may be written as a polynomial in the s_i of the required form.

(ii) Given (i), it is sufficient to prove that the conjugacy class sums $s_{\ell j}$ are in the $\mathbf{Q}$-algebra generated by $X_\ell(n)$. For this, it is sufficient to prove by induction on j that if $1 < \ell j \leq n$ then the conjugacy class sum $s_{\ell j}$ is in the $\mathbf{Q}$-algebra generated by

$$\{s_i : 1 \leq i < \ell j\} \cup \{s_{(\ell j-1,2)}\}.$$

(The last term above should be disregarded if $\ell j = n$.) If $\ell j < n$ then

$$s_2 s_{\ell j-1} = \alpha s_{\ell j} + \beta s_{(\ell j-1,2)} + \sum_{1 \leq i \leq \ell j/2-1} \gamma_i s_{(\ell j-1-i,i)} \tag{3}$$

for some coefficients α, β, γ_i, about which all we need to know is that $\alpha > 0$. Each element of S_n appearing in the conjugacy class sums $s_{(\ell j-1-i,i)}$ fixes one more point than a ℓj-cycle, so by (i), each $s_{(\ell j-1-i,i)}$ is in the $\mathbf{Q}$-algebra generated by $\{s_i : i < \ell j\}$. Hence $s_{\ell j}$ is in the $\mathbf{Q}$-algebra generated by $\{s_i : 1 \leq i < \ell j\} \cup \{s_{(\ell j-1,2)}\}$, as required.

If $\ell j = n$ then

$$s_2 s_{n-1} = \alpha' s_n + \sum_{1 \leq i \leq n/2-1} \gamma'_i s_{(n-1-i,i)} \tag{4}$$

for some further coefficients α', γ'_i. Again it is clear that $\alpha' > 0$, so the result follows in the same way as before.

(iii) We use the same strategy as in (ii). By (i) it suffices to prove that if $1 < 2\ell j \leq n$ then $s_{2\ell j}$ may be expressed as a polynomial in the elements

$$\left\{s_i : 1 \leq i < 2\ell j\right\} \cup \left\{s_{(2\ell j-1,2)}\right\}$$

in such a way that in every monomial in the expression, at least one class sum of permutations of sign -1 appears. As in (ii), this follows by induction on j, using (3) and (4) above. □

In order to move from conjugacy class sums to individual elements of S_n we introduce a carefully chosen family of group elements. For $\ell \in \mathbf{N}$, let

$$Z_\ell(n) = \{(1,2,\ldots,2j) : 1 \leq j \leq n/2,\ \ell \nmid j\} \\ \cup \{(1,2,\ldots,2k\ell-1)(2k\ell,2k\ell+1) : 1 < 2k\ell < n\}\,.$$

Note that $Z_\ell(n)$ consists of odd ℓ'-elements. We can now apply Proposition 2.1 to give results about ordinary characters.

Corollary 2.2. *Let λ be a partition of n and let $\ell > 2$ be an odd natural number. Each of the following conditions implies that $\lambda = \lambda'$:*

(i) *χ^λ vanishes on all cycles of even length in S_n;*

(ii) *χ^λ vanishes on every element of $Z_\ell(n)$;*

(iii) *χ^λ vanishes on every ℓ'-element in S_n of sign -1.*

Proof. (i) By hypothesis, $\omega_\lambda(s_{2i}) = 0$ for all i such that $1 \leq i \leq n/2$. Hence, by Proposition 2.1(i), $\omega_\lambda(s_\mu) = 0$ whenever μ labels a conjugacy class of permutations of sign -1. This implies that χ^λ vanishes on every element in S_n of sign -1, and so $\chi^\lambda = \chi^\lambda \times \mathrm{sgn}$. Since $\chi^\lambda \times \mathrm{sgn} = \chi^{\lambda'}$ (see for instance [13, §6.6]) we may deduce that $\lambda = \lambda'$.

(ii) This follows in the same way as (i), this time using Proposition 2.1(iii).

(iii) This is merely a weaker version of the previous part. □

Corollary 2.3. *Let λ and μ be partitions of n. Let $\ell > 2$ be a natural number. If χ^λ and χ^μ agree on all ℓ'-elements of S_n then $\lambda = \mu$.*

Proof. By hypothesis the central characters ω_λ and ω_μ agree on the set $X_\ell(n)$ of generators of $Z(\mathbf{Q}S_n)$ given in Proposition 2.1(ii). Hence $\omega_\lambda = \omega_\mu$ and so $\lambda = \mu$. □

2.2 Generating sets for $Z(\mathbf{Q}A_n)$

Recall that the only conjugacy classes of S_n which split in A_n are those labelled by partitions of n whose parts are odd and mutually distinct. If μ is such a partition, let $s_\mu^+ \in Z(\mathbf{Q}A_n)$ and $s_\mu^- \in Z(\mathbf{Q}A_n)$ be the sums of the elements in the two associated conjugacy classes of A_n. (The signs $+$ and $-$ may be assigned arbitrarily.)

For part (ii) of Theorem 1.2 we need a generating set for $Z(\mathbf{Q}A_n)$ involving only conjugacy class sums labelled by $2'$-permutations (that is, permutations of odd order). Fortunately for us, the split classes consist of $2'$-elements, so they do not create any additional difficulties. Let

$$X(n) = \{s_\lambda : \lambda \vdash n, \text{ all parts of } \lambda \text{ are odd}\} \\ \cup \{s_\lambda^+ : \lambda \vdash n, \lambda \text{ has odd distinct parts}\}.$$

We shall prove that $X(n)$ is a generating set for $Z(\mathbf{Q}A_n)$ as a $\mathbf{Q}$-algebra. To do this we need the following lemma.

Lemma 2.4. *Let $k, l \geq 3$ be natural numbers such that $k+l \leq n+2$. Define coefficients c_μ for $\mu \vdash n$ by*

$$s_k s_l = \sum_{\mu \vdash n} c_\mu s_\mu.$$

If $a, b \geq 2$ are natural numbers such that $a + b = k + l - 2$ then

$$c_{(a,b,1^{n-a-b})} = ab \min(k-1, l-1, a, b).$$

The only other conjugacy class sums s_μ with $\operatorname{supp} \mu \geq k + l - 2$ which may appear as summands of $s_k s_l$ are $s_{(k+l-1)}$, which appears only if $k+l-1 \leq n$, and $s_{(k,l)}$, which appears only if $k + l \leq n$.

The proof of this lemma is postponed to §2.3.

Proposition 2.5. *The elements of $X(n)$ generate $Z(\mathbf{Q}A_n)$ as a $\mathbf{Q}$-algebra.*

Proof. It is sufficient to prove that if μ is a partition of n with evenly many even parts and at least two even parts, then s_μ is in the $\mathbf{Q}$-algebra generated by $X(n)$. Suppose that $\operatorname{supp} \mu = m$. By induction we may assume that all conjugacy class sums labelled by partitions with support at most $m-1$ can be written as polynomials in elements of $X(n)$.

The hardest case occurs when μ has just two even parts and all its other parts are of size 1. Suppose that $m = 2r$. Let $t = \lfloor r/2 \rfloor$ be the number of partitions of $2r$ into two even parts. By Lemma 2.4, if $1 \leq i \leq t$, then

$$s_{2r-2i+1} s_{2i+1} = \sum_{j=1}^{i} 8(r-j)j^2 s_{(2r-2j,2j)} + 8i \sum_{j=i+1}^{t} (r-j)j s_{(2r-2j,2j)} + y$$

where y is a rational linear combination of conjugacy class sums s_ν such that either

$$\nu \in \{(2r-2j+1, 2j-1) : 1 \leq j \leq t\} \cup \{(2r+1), (2r-2i+1, 2i+1)\}$$

or $\operatorname{supp}\nu \leq 2r-1$. The partitions in the sets above all have only odd parts, so our inductive hypothesis implies that for $1 \leq i \leq t$,

$$u_i = \sum_{j=1}^{i}(r-j)j^2 s_{(2r-2j,2j)} + i\sum_{j=i+1}^{t}(r-j)js_{(2r-2j,2j)}$$

is in the $\mathbf{Q}$-algebra generated by $X(n)$. Now

$$u_t - u_{t-1} = (r-t)ts_{(2r-2t,2t)}$$

and if $i < t$ then

$$u_i - u_{i-1} = (r-i)is_{(2r-2i,2i)} + \sum_{j=i+1}^{t}(r-j)js_{(2r-2j,2j)}.$$

Hence, by starting at $i = t$ and working down to $i = 1$, we may express each conjugacy class sum $s_{(2r-2i,2i)}$ as a polynomial in the elements of $X(n)$. In particular, this shows that s_μ lies in the $\mathbf{Q}$-algebra generated by $X(n)$.

The other possibility is that μ has two even parts, $2u$ and $2v$ say, and some further parts, not all of size 1. Let ν be the partition of $n-2u-2v$ obtained by removing the parts of size $2u$ and $2v$ from μ. By induction s_ν and $s_{(2u,2v)}$ are in the $\mathbf{Q}$-algebra generated by $X(n)$ and evidently

$$s_\nu s_{(2u,2v)} = s_\mu + y$$

where y is a rational linear combination of conjugacy class sums s_λ for partitions λ of support at most $m-1$. Hence s_μ is in the $\mathbf{Q}$-algebra generated by $X(n)$. This completes the inductive step. □

To obtain the expected corollary concerning ordinary characters of S_n we need a small result about how characters of S_n restrict to A_n.

Lemma 2.6. *Let λ and μ be partitions of n. The restricted characters $\chi^\lambda\!\downarrow_{A_n}$ and $\chi^\mu\!\downarrow_{A_n}$ agree if and only if $\lambda = \mu$ or $\lambda = \mu'$.*

Proof. This may be proved using the Clifford theory developed at the start of §4. Alternatively see [8, §5.1] for a proof using only the language of basic character theory. □

Corollary 2.7. *Let λ and μ be partitions of n. If χ^λ and χ^μ agree on all $2'$-elements of S_n then either $\lambda = \mu$ or $\lambda = \mu'$.*

Proof. By Proposition 2.5 and our usual argument with central characters, the hypothesis implies that $\chi^\lambda(g) = \chi^\mu(g)$ for all $g \in A_n$. Now apply the previous lemma. □

2.3 Proof of Lemma 2.4

We may assume that $k \geq l$. To find $s_k s_l$ we shall first calculate the product $(12\ldots k)s_l$. As we are mainly interested in terms in this product whose support is exactly $k+l-2$, we start by looking at those l-cycles which move exactly two members of $\{1,2,\ldots,k\}$. Let $i,j \in \{1,2,\ldots,k\}$ be distinct numbers with $i<j$ and let

$$\tau = (i\, p_1 \ldots p_r\, j\, q_1 \ldots q_{l-2-r})$$

where $0 \leq r \leq l-2$ and $p_1,\ldots,p_r,q_1,\ldots,q_{l-2-r} \in \{k+1,\ldots,n\}$ are distinct. Computation shows that if $i>1$ then

$$(1\,2\ldots k)\tau = (1\ldots i-1\, p_1 \ldots p_r\, j \ldots k)(i\, i+1 \ldots j-1\, q_1 \ldots q_{l-2-r}),$$

which is a product of cycles of lengths $(k+r)-(j-i)$ and $(j-i)+(l-2-r)$. If $i=1$ then we have

$$(1\,2\ldots k)\tau = (1\ldots j-1\, q_1 \ldots q_{l-2-r})(j\, j+1 \ldots k\, p_1 \ldots p_r),$$

so while there are some small differences, the cycle structure of the product is unaltered. It follows that if $2 \leq a < (k+l-2)/2$ then the number of ways to choose i and j so that $(1\,2\ldots k)\tau$ has cycle type $(k+l-2-a,a)$ is

$$\begin{cases} a-r & : \ a>r \\ 0 & : \ a \leq r \end{cases} + \begin{cases} k-a+(l-2)-r & : \ a>l-2-r \\ 0 & : \ a \leq l-2-r \end{cases}.$$

Here the first term comes from the case $(j-i)+(l-2-r)=k+l-2-a$ and the second from the case $(j-i)+(l-2-r)=a$. The remaining possibility is that $2a=k+l-2$. Then these two cases coincide, and the correct number of choices for i and j is $a-r=(k+l-2)/2-r$.

We now let r vary between 0 and $l-2$ and add up the total number of choices for i and j so that $(1\,2\ldots k)\tau$ has cycle type $(k+l-2-a,a)$.

(1) If $a \leq l-2$ then the total number of choices for i and j is

$$\sum_{0 \leq r < a} (a-r) + \sum_{l-2-a<r\leq l-2} (k-a+(l-2)-r) = ak.$$

(2) If $l-2<a<(k+l-2)/2$ then the conditions needed to get a positive contribution always hold in both cases, and the total number of choices is

$$\sum_{0\leq r\leq m} \big((a-r)+k-a+(l-2)-r\big) = (l-1)k.$$

(3) If $2a=k+l-2$ then the sum in the previous case double counts every choice and so the total number of choices is $(l-1)k/2$.

We must also choose $p_1, \dots, p_r$ and $q_1, \dots, q_{m-2-r}$. Whatever the value of r, this can always be done in exactly $(n-k)^{\underline{l-2}}$ ways; here $x^{\underline{a}}$ stands for $x(x-1)\dots(x-a+1)$ for $x, a \in \mathbf{N}$. Hence the total number of elements of cycle type $(k+l-2-a, a)$ in the product $(12\dots k)s_l$ is

$$(n-k)^{\underline{l-2}} \times \begin{cases} ak & : \ 2 \le a \le l-2 \\ (l-1)k & : \ l-2 < a < (k+l-2)/2 \\ (l-1)k/2 & : \ a = (k+l-2)/2. \end{cases}$$

To obtain the coefficient $c_{(k+l-2-a,a,1^{n-k-l+2})}$ we must multiply by $K_{(k,1^{n-k})}$ and then divide by $K_{(k+l-2-a,a,1^{n-k-l+2})}$. This gives

$$\begin{aligned} c_{(k+l-2,a,1^{n-k-l+2})} &= \min(a, l-1) \frac{k(n-k)^{\underline{l-2}} n^{\underline{k}}/k}{n^{\underline{k+l-2}}/(k+l-2-a)a} \\ &= \min(a, l-1)(k+l-2-a)a \end{aligned}$$

as required.

It is easy to see that if τ is an l-cycle such that $(1\,2\dots k)\tau$ has support strictly more than $k+l-2$ then either τ moves just one element of $\{1, 2, \dots, k\}$, in which case $(1\,2\dots k)\tau$ has cycle type $(k+l-1, 1^{n-k-l+1})$, or τ moves no elements of $\{1, 2, \dots, k\}$, in which case $(1\,2\dots k)\tau$ has cycle type $(k, l, 1^{n-k-l})$. This gives the final statement in the lemma. □

2.4 Some related problems

We now pose some problems suggested by Corollaries 2.2, 2.3 and 2.7. The reader keen to see the applications to the modular theory should skip to §3.

Problem 2.8. *Let $n \in \mathbf{N}$. What is the size c_n of the smallest set $X \subseteq S_n$ such that for all partitions $\lambda \vdash n$, if χ^λ vanishes on all the elements of X then λ is self-conjugate?*

It follows from Corollary 2.2(i) that $c_n \le n/2$. However, as Suzuki points out in [20], for $n \le 14$, it suffices to take $X = \{(1,2)\}$, so this result is not always the best possible. An exhaustive search using the computer algebra package MAGMA shows that for $n \le 59$ one may take $X = \{(1,2), (1,2,3,4)\}$, hence $c_n = 1$ if $n \le 14$ and $c_n \le 2$ if $n \le 59$. (For $n = 60$ there is a non-self-conjugate partition λ such that $\chi^\lambda((1,2)) = \chi^\lambda((1,2,3,4)) = 0$, so it seems likely that $c_{60} = 3$.) We would ask for an asymptotic formula for c_n, as its precise behaviour may be too erratic to be easily described.

This problem may of course be posed with other properties in place of the condition that λ be self-conjugate. For example, one might ask instead that λ be a p-core for a given prime p. Also one may restrict the possible set X, for example by insisting that X consist of p'-elements for a given prime p.

Problem 2.9. *Let $n \in \mathbf{N}$. What is the size b_n of the smallest set $X \subseteq S_n$ such that for all partitions $\lambda \vdash n$, if χ^λ and χ^μ agree on all elements of X then $\lambda = \mu$?*

Here Kramer's result shows that $b_n \leq n$, but again this is not always the best possible.

3 Consequences for decomposition matrices

We are ready to prove Theorem 1.2. Let ϕ_ν be the Brauer character of the irreducible module D^ν defined over a field of prime characteristic p (for an introduction to Brauer characters see [18, §2]). Suppose that in the decomposition matrix $D_p(n)$ of S_n modulo p, the rows labelled by partitions λ and μ are equal. Adding up irreducible Brauer characters we find that if g is a p'-element of S_n then

$$\chi^\lambda(g) = \sum_\nu D_p(n)_{\lambda\nu}\phi_\nu(g) = \sum_\nu D_p(n)_{\mu\nu}\phi_\nu(g) = \chi^\mu(g).$$

Thus χ^λ and χ^μ agree on all p'-elements of S_n. If p is odd then Corollary 2.3 implies that $\lambda = \mu$. If $p = 2$ then Corollary 2.7 implies that either $\lambda = \mu$ or $\lambda = \mu'$. This completes the proof.

3.1 Hecke algebras

We now generalise Theorem 1.2 to Hecke algebras of symmetric groups. Let F be any field (maybe of characteristic zero), and let $q \in F\backslash\{0\}$. Let $\mathcal{H}_{F,q}(S_n)$ be the corresponding Hecke algebra, as defined in [15, §1.2] by deforming the group algebra FS_n. We may assume that q is a root of unity, for if it is not, $\mathcal{H}_{F,q}(S_n)$ is semisimple, and so the analogue of Theorem 1.2 is trivial. Moreover, if $q = 1$ then $\mathcal{H}_{F,q}(S_n) = FS_n$, so we may also exclude this case. From now on we write $\mathcal{H}$ for $\mathcal{H}_{F,q}(S_n)$.

Let ℓ be minimal such that $1 + q + \ldots + q^{\ell-1} = 0$. The simple $\mathcal{H}$-modules are indexed by the ℓ-regular partitions of n, and there are the expected analogues of Specht modules, so the decomposition matrix of $\mathcal{H}$ has rows labelled by all partitions of n, and columns labelled by the ℓ-regular partitions of n. We shall need the following lemma, which follows from the remarks just after Proposition 2.6 in [19].

Lemma 3.1. *The $\mathbf{Z}$-span of the columns of the decomposition matrix of $\mathcal{H}$ is equal to the $\mathbf{Z}$-span of the columns of the ordinary character table of S_n labelled by the ℓ-regular partitions of n.* □

We can now prove the following analogue of Theorem 1.2.

Theorem 3.2. *Let $\mathcal{H}$ and ℓ be as above.*

(i) *If $\ell > 2$ then the rows of the decomposition matrix of $\mathcal{H}$ are distinct.*

(ii) *If $\ell = 2$ then the rows labelled by λ and μ are the same if and only if $\lambda = \mu$ or $\lambda = \mu'$.*

Proof. The previous lemma implies that the rows of the decomposition matrix of $\mathcal{H}$ labelled by partitions λ and μ are equal if and only if $\chi^\lambda(g) = \chi^\mu(g)$ for all ℓ'-elements $g \in S_n$. The result now follows from Corollary 2.3 and Corollary 2.7 in the same way as Theorem 1.2. □

3.2 Alternating groups

For alternating groups, the situation in odd characteristic appears to be quite difficult, and the obvious analogue of Theorem 1.2 is false. There is however one result we can prove without doing any further work.

Theorem 3.3. *Let $n \in \mathbf{N}$. The rows of the decomposition matrix of A_n in characteristic 2 are distinct.*

Proof. Suppose that the rows labelled by the ordinary characters χ and ψ are equal. Then $\chi(g) = \psi(g)$ for all $2'$-elements of A_n and so by Proposition 2.5, $\chi(g) = \psi(g)$ for all $g \in A_n$. Hence $\chi = \psi$. □

4 Specht modules and the alternating group

In this section we prove Theorem 1.4. Recall that this theorem states that if λ is a partition of n and F is an algebraically closed field of odd characteristic, then the Specht module S^λ is decomposable on restriction to A_n if and only if S^λ is simple and $\lambda = \lambda'$.

We begin with some Clifford theory. Let F have characteristic p. As $p \neq 2$ the Specht module S^λ is indecomposable (see [13, Corollary 13.18]). Also, when $p \neq 2$, the Sylow p-subgroups of S_n are contained in the alternating group A_n, so by Higman's criterion (see [1, Proposition 3.6.4]), S^λ is relatively A_n-projective. Thus there exists an indecomposable FA_n-module U such that S^λ is a direct summand of the induced module $U{\uparrow}_{A_n}^{S_n}$. We denote this by writing

$$S^\lambda \mid U{\uparrow}_{A_n}^{S_n}.$$

By Mackey's Lemma (see [1, Theorem 3.3.4]),

$$S^\lambda{\downarrow}_{A_n} \mid U \oplus U^t \tag{5}$$

where t is any odd element in S_n and U^t is the A_n-module with the same underlying vector space as U, but with the action defined by $u \cdot g = ug^t$ for $u \in U^t$, $g \in A_n$.

We can now prove the 'if' part of Theorem 1.4. As λ is self-conjugate, the restricted ordinary character $\chi^\lambda{\downarrow}_{A_n}$ splits as a sum of two ordinary irreducible characters of A_n. Hence the Brauer character of $S^\lambda\downarrow_{A_n}$ has at least two irreducible summands. (Notice that we have used that F is sufficiently large here.) Furthermore, by Clifford's theorem on the restriction of simple modules to normal subgroups, $S^\lambda\downarrow_{A_n}$ is semisimple. Hence the restriction of S^λ to A_n has at least two non-trivial direct summands. This is sufficient to prove the result.

It is however not hard to give a little more information. By (5), $S^\lambda\downarrow_{A_n}$ has at most two non-trivial direct summands. Therefore

$$S^\lambda\big\downarrow_{A_n} = U \oplus U^t$$

and the Brauer characters of the simple summands U and U^t are the reduction modulo p of the two ordinary irreducible A_n characters associated to λ.

We now turn to the 'only if' part of Theorem 1.4. Suppose that $S^\lambda\downarrow_{A_n}$ is decomposable. Using (5), we have

$$S^\lambda \otimes \mathrm{sgn} = U\Big\uparrow_{A_n}^{S_n} \otimes \mathrm{sgn} \cong \Big(U \otimes \mathrm{sgn}\big\downarrow_{A_n}\Big)\Big\uparrow_{A_n}^{S_n} \cong U\Big\uparrow_{A_n}^{S_n} \cong S^\lambda.$$

It follows that the ordinary character of S^λ vanishes on all odd p'-elements of S_n. By Corollary 2.2(iii) this implies that $\lambda = \lambda'$. Finally, since by (1) we have $S^\lambda \otimes \mathrm{sgn} \cong (S^{\lambda'})^\star$, it follows that S^λ is self-dual, and hence simple by Theorem 23.1 of [13]. This completes the proof of Theorem 1.4.

4.1 Theorem 1.4 for non-algebraically closed fields

We now consider Specht modules defined over an arbitrary field F of odd characteristic p. As explained in Remark 2 of §1.2, we shall use a result due to Fayers on the irreducibility of Specht modules. To state it we need two final pieces of notation. If λ is a partition, and α is a node of the diagram of λ, let h_α be the hook-length of the hook on α. (See [13, Ch. 18] for the definition of hooks in partitions.) Given $n \in \mathbf{N}$, we set $(n)_p = p^a$ if p^a is the highest power of p which divides $n \in \mathbf{N}$.

Theorem 4.1 (Fayers)**.** *Let F be a field of odd characteristic. The Specht module S^λ defined over F is reducible if and only if λ contains a node α, a node β in the same row as α, and a node γ in the same column as α, such that $p \mid h_\alpha$, $(h_\alpha)_p \neq (h_\beta)_p$ and $(h_\alpha)_p \neq (h_\gamma)_p$.* □

Using the 'if' direction of this theorem (proved in [3]) we may prove the following lemma.

Lemma 4.2. *Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a self-conjugate partition of n. Suppose that there is a node on the main diagonal of λ whose hook-length is divisible by p. Then S^λ is reducible.*

Proof. For $1 \le i \le k$ and $1 \le j \le \lambda_i$ let h_{ij} be the hook-length of the node in position (i,j) of λ. Suppose that $(h_{ss})_p = p^c$ where $c \ge 1$. As λ is self-conjugate, the sequence of hook lengths of nodes in row s is the same as the sequence of hook lengths of nodes in column s. Hence, by Theorem 4.1, if S^λ is irreducible then every node in row s and every node in column s has hook length exactly divisible by p^c. By considering the subdiagram of λ obtained by taking all nodes in positions (a,b) for $a,b \ge s$ we obtain a partition μ whose hook lengths satisfy

$$(h_\alpha)_p = p^c$$

for all nodes α in the first row and column of μ. Let $\mu_1 = l$. As $p^c \mid h_{1l}$ we must have $\mu'_l > 1$. Hence $\mu_1 = \mu_2$, and so $h_{21} = h_{11} - 1$. But both these hook lengths are supposed to be divisible by p^c, so we have reached a contradiction. □

We are now ready to prove the following generalisation of Theorem 1.4.

Theorem 4.3. *Let F be a field of odd characteristic. Let λ be a partition of $n \in \mathbf{N}$ with main diagonal hook lengths $q_1, \ldots, q_r$. The FS_n-Specht module S^λ is decomposable on restriction to A_n if and only if all of the following conditions hold:*

(i) *λ is self-conjugate,*

(ii) *$(-1)^{(n-r)/2} q_1 \ldots q_r$ has a square root in F,*

(iii) *S^λ is simple (we shall see this implies that p does not divide any of the q_i).*

Proof. By the proof of Theorem 1.4, if $S^\lambda{\downarrow_{A_n}}$ is decomposable then (i) and (iii) hold. Let χ_1 and χ_2 be the two ordinary characters of A_n associated to λ: at p' elements these are the Brauer characters of the summands of $S^\lambda{\downarrow_{A_n}}$. Proposition 5.3 of [8] tells us that

$$\tfrac{1}{2}(-1)^{(n-r)/2} \pm \tfrac{1}{2}\sqrt{(-1)^{(n-r)/2} q_1 \ldots q_r}$$

are the values of χ_1 and χ_2 on elements of the conjugacy class labelled by $(q_1, \ldots, q_r)$. If any of the q_i are divisible by p then Lemma 4.2 implies that S^λ is reducible, a contradiction. Hence this conjugacy class is p-regular. Therefore

$$(-1)^{(n-r)/2} q_1 \ldots q_r$$

has a square root in F, which gives (ii).

Conversely if all the conditions hold then the proof of the 'if' part of Theorem 1.4 shows that $S^\lambda{\downarrow_{A_n}}$ decomposes. Where before we used that F was sufficiently large, now we merely use the fact that if χ is the character, in the naïve sense, of an irreducible representation of a group G over a field E of prime characteristic, then the representation can be defined over a subfield F of E if and only if the values of χ lie in F. (For a proof of this statement see [12, Theorem 9.14].) □

We conclude by noting that the last result has an especially nice form for Specht modules labelled by hook partitions.

Corollary 4.4. *Let F be a field of odd prime characteristic, and let $1 < r < n-1$. The Specht module $S^{(n-r,1^r)}$ decomposes on restriction to A_n if and only if $n = 2r+1$, p does not divide n, and $(-1)^{(n-1)/2}\,n$ has a square root in F.* □

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