Two theorems on the vertices of Specht modules

by

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Abstract. Two theorems about the vertices of indecomposable Specht modules for the symmetric group, defined over a field of prime characteristic p, are proved:

1. The indecomposable Specht module S^λ has non-trivial cyclic vertex if and only if λ has p-weight 1.

2. If p does not divide n and $S^{(n-r,1^r)}$ is indecomposable then its vertex is a p-Sylow subgroup of $S_{n-r-1} \times S_r$.

Mathematics Subject Classification numbers (2000). 20C20, 20C30

The representations of the symmetric group S_n over a field F have been much studied. Central are the Specht modules S^λ, a family of FS_n-modules indexed by the partitions λ of n that are defined in a way that does not depend essentially on the field characteristic. When F has prime characteristic p and S^λ is indecomposable it makes sense to speak of its vertex as an FS_n-module. In this article we prove the following two theorems concerning the vertices of Specht modules.

Theorem 1 *Let F be a field of prime characteristic p and let S^λ be the Specht module corresponding to the partition λ of n, defined over F. Suppose that S^λ is indecomposable. Then the vertex of S^λ is non-trivial cyclic if and only if λ has p-weight* 1.

Theorem 2 *Let F be a field of prime characteristic p and let $S_n(r)$ be the Specht module corresponding to the partition $(n-r, 1^r)$ of n, defined over F. Suppose that $S_n(r)$ is indecomposable and that p does not divide n. Then the vertex of $S_n(r)$ is a p-Sylow subgroup of $S_{n-r-1} \times S_r$.*

The only existing work in print in this area is a 1984 article by G. M. Murphy and M. H. Peel [5] where the vertices for some of the Specht module corresponding to hook partitions $(n-r, 1^r)$ are discussed. Unfortunately Theorem 4.6 in this paper (which deals with a case covered by our Theorem 2) is incorrect, and we point out at the end of §4 that it can be shown to lead to a contradiction with another theorem in [5].

The organisation of this article is as follows. In §1 we establish our notation and very briefly review some background material. In §2 it is proved that the Scopes functors (see [6]) preserve vertices of Specht modules; using this result we then prove Theorem 1 in §3. In §4 we state a fact about the Brauer correspondence for p-permutation modules and then use it to prove Theorem 2.

1 Notation

Let F be a field of prime characteristic p and let S^λ be the Specht module corresponding to the partition λ of n defined over F. (See [2] Ch. 4.) When we come to considering specific cases we will freely use James' notation of tableaux and polytabloids.

The dimension of S^λ is given by the hook-formula. If λ is a partition then a hook in λ consists of a node, together with all those nodes lying either directly below it or directly to its right. The hook-length, h_α of a node α is the number of such nodes. The hook-formula (see [2] Ch. 20) states that

$$\dim S^\lambda = \frac{n!}{\prod_{\alpha\in\lambda} h_\alpha}.$$

Hooks appear again when we consider the block of S^λ. This is given by the theorem known as Nakayama's conjecture (see [3] 6.1.21). It states that the block of S^λ is determined by the p-core of λ, which is the unique partition obtained by stripping off all rim-p-hooks from λ. (A rim-p-hook is a connected part of the rim of λ lying between the two rim nodes in an ordinary p-hook.) The p-weight of a partition λ is the number of rim-p-hooks we must strip off to obtain the p-core. The defect group of a weight w block is a p-Sylow subgroup in S_{wp} (see [3] 6.2.45).

Later in §3 we will need the following lemma:

Lemma 3 *If λ is a partition of n of p-weight w and p^a is the the highest power of p dividing $n!$ then the highest power of p dividing $\dim S^\lambda$ is at most p^{a-w}.*

This follows from the hook-formula together with the well known result (see [3] 2.7.40) that if λ is a partition of p-weight w then there are exactly w nodes in λ whose hook-length is divisible by p.

It is known that S^λ is indecomposable if either of the following hold:

1. λ is p-regular, i.e. there are no p parts in λ of the same size. (In this case S^λ has a unique top composition factor, D^λ.)

2. $p > 2$ (see [2] Corollary 13.18).

When $p = 2$ Specht modules may be decomposable, as $S^{(5,1,1)}$ shows (see [2] Example 23.10iii). Here we consider only the indecomposable case.

If G is a finite group and V is an indecomposable FG-module then a vertex of V is a minimal subgroup H of G such that there exists an indecomposable FH-module U with

$$V \mid U{\uparrow}_H^G.$$

If V has vertex H then we may always substitute $V{\downarrow}_H$ for U, i.e. $V \mid V{\downarrow}_H{\uparrow}^G$. If U is indecomposable then we say it is a source for V. If F has characteristic p then vertices are p-groups. All vertices of an indecomposable module are conjugate in G so we will sometimes abuse notation and speak of 'the' vertex of an indecomposable FG-module.

2 A theorem on the Scopes functor

In [6] Scopes defined Morita equivalences between certain blocks of symmetric groups. The results are stated in terms of James' abacus notation, a useful summary of which is given in §1 of [6]. In this section we prove that the Scopes functors preserve vertices of modules.

Let B and $\tilde{B}$ be blocks of S_n and S_{n-k} in the relationship described by Scopes and let G and G' be the Scopes functors

$$G : \textbf{mod}\text{–}B \to \textbf{mod}\text{–}\tilde{B},\ G' : \textbf{mod}\text{–}\tilde{B} \to \textbf{mod}\text{–}B.$$

By definition G is the composite of the following functors

$$\textbf{mod}\text{–}B \xrightarrow{G_1} \textbf{mod}\text{–}S_n \xrightarrow{G_2} \textbf{mod}\text{–}S_{n-k}\times S_k \xrightarrow{G_3} \textbf{mod}\text{–}S_{n-k} \xrightarrow{G_4} \textbf{mod}\text{–}\tilde{B}$$

where G_1 and G_4 are the usual functors for passing between modules for an algebra and modules for a block of that algebra, G_2 is the restriction functor $-\downarrow_{S_{n-k}\times S_k}$ and G_3 is the functor $-\otimes_{FS_k} F$ which has the effect of quotienting out the action of S_k on an $S_{n-k}\times S_k$-module by identifying all vectors in the same S_k-orbit.

By definition G' is the composite of the following functors

$$\textbf{mod}\text{–}B \xleftarrow{G'_1} \textbf{mod}\text{–}S_n \xleftarrow{G'_2} \textbf{mod}\text{–}S_{n-k}\times S_k \xleftarrow{G'_3} \textbf{mod}\text{–}S_{n-k} \xleftarrow{G'_4} \textbf{mod}\text{–}\tilde{B}$$

where G'_1 and G'_4 are again the usual functors for passing between modules for an algebra and modules for a block of that algebra and G'_2 is the coinduction functor $\mathrm{Hom}_{FS_{n-k}}(FS_n, -)$. We may think of G'_3 as the functor that takes each FS_{n-k}-module and regards it as a $FS_{n-k}\times FS_k$-module, on which S_k acts trivially.

All these functors are exact and G'_i is the right adjoint of G_i. It is important that the functor G gives a bijection between the Specht modules in the blocks B and $\tilde{B}$: if S^λ is in the block B then $G(S^\lambda)$ is the Specht module $S^{\tilde{\lambda}}$ in the block $\tilde{B}$ (see [6] Lemma 2.1 for details of the map $\lambda \mapsto \tilde{\lambda}$).

We will need the following lemma:

Lemma 4 *Let $J_0 \leq J$ and K be subgroups of a finite group G. Let U be a $F(J_0 \times K)$-module. Then $U{\uparrow}^{J\times K}_{J_0\times K} \otimes_{FK} F \cong (U \otimes_{FK} F){\uparrow}^{J}_{J_0}$ as an FJ-module.* □

Theorem 5 *Let V be an indecomposable B-module with vertex Q. Then $G(V)$ also has vertex Q.*

Proof: Say V has source U. Then $G(V)$ is a direct summand of $G_3G_2(U{\uparrow}^{S_n}_Q)$. By Mackey's Lemma

$$G_3G_2\left(U{\uparrow}^{S_n}_Q\right) = \left(U{\uparrow}^{S_n}_Q{\downarrow}_{S_{n-k}\times S_k}\right) \otimes_{FS_k} F = \left(\bigoplus U^g{\uparrow}^{S_{n-k}\times S_k}_{S_{n-k}\times S_k \cap Q^g}\right) \otimes_{FS_k} F$$

where the sum is over a set of double coset representatives for $Q\backslash S_n/S_{n-k}\times S_k$. The direct product $S_{n-k}\times S_k$ comes endowed with projections into each of its factors: let

J_0 be the subgroup of S_{n-k} given by projecting $S_{n-k} \times S_k \cap P^g$ into S_{n-k}. Note that the order of J_0 does not exceed the order of P^g. Lemma 4 gives

$$U^g \uparrow_{S_{n-k}\times S_k \cap P^g}^{S_{n-k}\times S_k} \otimes_{FS_k} F = \left(U^g \uparrow_{S_{n-k}\times S_k \cap P^g}^{J_0 \times S_k} \otimes_{FS_k} F \right) \uparrow_{J_0}^{S_{n-k}}.$$

Thus $G(V)$ is a summand of a module, each of whose summands has vertex of order at most $|P|$.

Now we argue in the opposite direction — this is more straightforward. Let X be an indecomposable $\tilde{B}$-module with vertex R and source Y. Then we find

$$G'(X) \Bigm| G'_2 G'_3 \left(Y \uparrow_R^{S_{n-k}} \right) = (Y \otimes F_{S_k}) \uparrow_R^{S_n}$$

so $G'(V)$ has a vertex contained in R.

Thus if $G(V)$ had a vertex of smaller order than Q then $G'G(V) \cong V$ would be induced from a subgroup smaller than Q, a contradiction. □

Remarks:

1. Both the lemma and theorem have a simpler and more pleasant form if the module V is trivial source. In general, though, Specht modules are not trivial source.

2. Scopes has shown [6] that any weight w block of a symmetric group for the prime p is Morita equivalent, *via* a series of applications of the functor G, to a block of a symmetric group of degree at most

$$(w-1)^2 p^2 (p-1)^2/4 + wp.$$

3 Proof of Theorem 1

We are now ready to prove Theorem 1. We can make considerable progress just by dimension counting. Let S^λ be an indecomposable Specht module for FS_n, where F is a field of prime characteristic p. Let p^a be the highest power of p dividing $n!$; this is also the order of any p-Sylow subgroup of S_n. By a corollary to Green's indecomposability theorem, if S^λ has a vertex Q contained in a p-Sylow subgroup P of S_n then

$$|P| \,/\, |Q| \Bigm| \dim S^\lambda.$$

However if λ has p-weight w then we have seen that the highest power of p dividing $\dim S^\lambda$ is at most p^{a-w} (Lemma 3). So

$$p^a / |Q| \Bigm| p^{a-w} \quad \text{which implies} \quad p^w \mid |Q|\,.$$

Suppose now that Q is non-trivial cyclic of order p^c (where $c \geq 1$), so necessarily, $w \leq c$. The vertex Q is contained in a defect group of S_n of weight w, i.e. a p-Sylow subgroup of S_{wp}. There is an element of order p^c in S_{wp} if and only if $wp \geq p^c$, i.e. $w \geq p^{c-1}$. Thus

$$c \geq w \geq p^{c-1}.$$

If $c \geq 3$ this is a contradiction. So either $c = 1$ or $c = 2$.

If $c = 1$ then $1 \geq w \geq p^0 = 1$ so also $w = 1$. This case corresponds to the (known) fact that weight 1 Specht modules *do* have cyclic vertex, conjugate to $\langle(12\dots p)\rangle$.

If $c = 2$ then the inequalities imply $2 \geq w \geq p$ so we must have $p = 2$ and $w = 2$. The defect group of a weight 2 block of S_n for $p = 2$ is a 2-Sylow subgroup of S_4 so Q is conjugate to $\langle(1234)\rangle \leq S_n$. We will deal with this case using the theory developed in the previous section. Any weight 2 block of a symmetric group for the prime $p = 2$ is Scopes equivalent to a block of some S_n for $n \leq 5$, so to prove Theorem 1 it is enough to prove that if S^λ is an indecomposable Specht module for S_4 or S_5 then S^λ does not have cyclic vertex. By the argument above, if S^λ has a vertex of order 4, then its dimension must be divisible by 2. This suffices to rule out the Specht modules for the following partitions: (4), $(3, 1)$, $(2, 1^2)$, (1^4) and (5), $(3, 2)$, $(2, 1^3)$, (1^5).

The remaining cases, $(2, 2)$ and $(3, 1, 1)$ are considered in the two lemmas below.

Lemma 6 *Over F, $S^{(2,2)}$ is indecomposable with vertex V, the normal order 4 subgroup of S_4.*

Proof: In fact $S^{(2,2)}$ is simple and is acted on trivially by the normal subgroup V. By dimension counting, $S^{(2,2)}$ is projective as a module for the quotient group S_4/V so $S^{(2,2)}$ has a vertex contained in V. □

Lemma 7 *Over F, $S^{(3,1,1)}$ is isomorphic to a direct summand of $M^{(3,2)}$.*

Proof: Let $M \cong M^{(3,2)}$ be the permutation module of S_5 acting on all 2-subsets of $[1..5]$, defined over F. M is induced from the Young subgroup $S_3 \times S_2$. Let V be the submodule of M generated by all sums of subsets of the form $\Delta_{ijk} = \{ij\} + \{jk\} + \{ki\}$. By identifying Δ_{ijk} with the $(3, 1, 1)$-polytabloid with first column i, j, k, we obtain an isomorphism $S^{(3,1,1)} \cong V$.

Let N be the submodule of M generated by elements of the form $v_i = \sum_{j \neq i} \{ij\}$. For $g \in S_5$, $v_i g = v_{ig}$ and $v_1 + v_2 + v_3 + v_4 + v_5 = 0$, so N is isomorphic to $S^{(4,1)}$, obtained as a 4-dimensional quotient module of $M^{(4,1)}$. We will prove that $M \cong V \oplus N$. By the hook-formula V is 6-dimensional, so it is enough to note that as

$$\{12\} = (\Delta_{123} + \Delta_{124} + \Delta_{125}) + (v_1 + v_2)$$

the subset $\{12\}$ is in the sum. □

Lemma 7 shows that if $S^{(3,1,1)}$ is indecomposable, then it has a vertex contained in $S_3 \times S_2$, so in particular its vertex cannot be cyclic of order 4. This completes the proof of Theorem 1.

Remark: Alternatively we may apply Theorem 2 to find the vertex of $S^{(3,1,1)}$ (assuming indecomposability). Later, however, we shall use Lemma 7 to show that $S^{(3,1,1)}$ is, in fact, indecomposable.

4 Proof of Theorem 2

In this section we prove Theorem 2. Recall that we write $S_n(r)$ for $S^{(n-r,1^r)}$. Our main tool is the Brauer correspondence, applied to p-permutation modules. A comprehensive account of the results we need can be found in [1] so below we just give the minimum for our use.

Definition 8 *Let G be a finite group and let U be an FG-module. Let P be a p-Sylow subgroup of G. Then U is p-permutation if there exists a P-invariant basis of U (i.e. a basis whose elements are permuted by the action of P).*

An indecomposable module is p-permutation if and only if it is a direct summand of a permutation module, i.e. if and only if it has trivial source. It turns out to be relatively easy to find the vertices of p-permutation modules; the theorem stated below is an immediate corollary of Theorem 3.2(1) in [1]. ($F\mathcal{B}^Q$ is just the Brauer correspondent of U with respect to Q.)

Theorem 9 *Let G be a finite group and let U be an indecomposable p-permutation FG-module with p-permutation basis $\mathcal{B}$ with respect to the p-Sylow subgroup P. Let Q be a p-subgroup of G contained in P. Then U has a vertex containing Q if and only if $\mathcal{B}^Q \neq \emptyset$.* □

Next we show that if p does not divide n, the Specht module $S_n(r)$ is p-permutation. Let $\mathcal{B}$ be the standard basis of $S_n(r)$ consisting of all polytabloids based on tableau with a '1' in their top left corner — for $A \subset [2..n]$ with $|A| = r$ let e_A be the standard polytabloid corresponding to a tableau with first column $1 \cup A$. For $g \in S_{[2..n]}$,

$$e_A g = \pm e_{Ag}.$$

Now let P be a p-Sylow subgroup of S_n contained in $S_{[2..n]}$. If $p = 2$ the equation above shows that $\mathcal{B}$ is already a p-permutation basis for $S_n(r)$ (with respect to P). If $p > 2$ we must work a little harder.

Consider the orbits of P on the collection of all r-subsets of $[2..n]$. Let $A_1, \ldots, A_k$ be a set of representatives for each orbit, arbitrarily chosen. We claim that

$$\bigcup_{j=1}^{k} \{e_{A_j} p : p \in P\}$$

is a p-permutation basis for $S_n(r)$. To show this, it is sufficient to prove that if A is an r-subset of $[2..n]$ and $p, q \in P$ are such that $Ap = Aq$ then $e_A p = e_A q$, i.e. the signs match up. Suppose that $e_A p = -e_A q$. Then we have $e_A pq^{-1} = -e_A$ so in particular $Apq^{-1} = A$. But

$$\mathrm{Stab}_{S_{[2..n]}} A = S_{n-r-1} \times S_r$$

and pq^{-1} is a product of even cycles (as $p > 2$). Thus $pq^{-1} \in A_{n-r-1} \times A_r$ and $e_A pq^{-1} = e_A$. This shows that $S_n(r)$ is p-permutation with respect to P and our modified basis.

We may now prove Theorem 2. Assume that $S_n(r)$ is indecomposable. Theorem 9 tells us that any p-Sylow subgroup of the stabilizer in $S_{[2..n]}$ of an element of $\mathcal{B}$ is a vertex for $S_n(r)$. Thus the vertex of $S_n(r)$ is a p-Sylow subgroup of $S_{n-r-1} \times S_r$. □

Remark: In [5] the authors look mainly at the case when $p = 2$. If n is odd and r is even and $S_n(r)$ is indecomposable, they give the correct result for the vertex, but if both n and r are odd and $S_n(r)$ is indecomposable their Theorem 4.6 wrongly claims that the vertex of $S_n(r)$ is the defect group of its block, i.e. a 2-Sylow subgroup of S_{n-3}.

We give an independent example. Consider the Specht module $S_7(3)$. Applying the Scopes functor to move into the principal block of S_5 we see that $S_7(3)$ is sent to $S_5(2)$. We will use Lemma 7 to argue that this latter module is indecomposable. Any direct summand of $M^{(3,2)}$ is a direct sum of Young modules (see [4]), namely $Y^{(5)} = S^{(5)}$, $Y^{(4,1)} = S^{(4,1)}$ and $Y^{(3,2)}$. Decomposition numbers (see e.g. [3] p414) show that $S^{(3,1,1)}$ has two trivial composition factors and one non-trivial factor, $D^{(3,2)}$, so if $S^{(3,1,1)}$ decomposes then, as Young modules are self dual, the only possibility is that $V \cong D^{(3,2)} \oplus F \oplus F$. However M comes from a transitive permutation action so there is only one copy of the trivial module in soc$M$. Thus $S_7(3)$ is also indecomposable and by Theorem 5 it has the same vertex as $S_5(2)$, namely $C_2 \times C_2$. So Theorem 4.6 in [5] is inconsistent with their previous result.

Acknowledgment: I would like to thank the referee for his or her helpful comments on an earlier version of this article.

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The author is financially supported by EPSRC grant number 00801412.