Plethysms and decomposition matrices of symmetric groups

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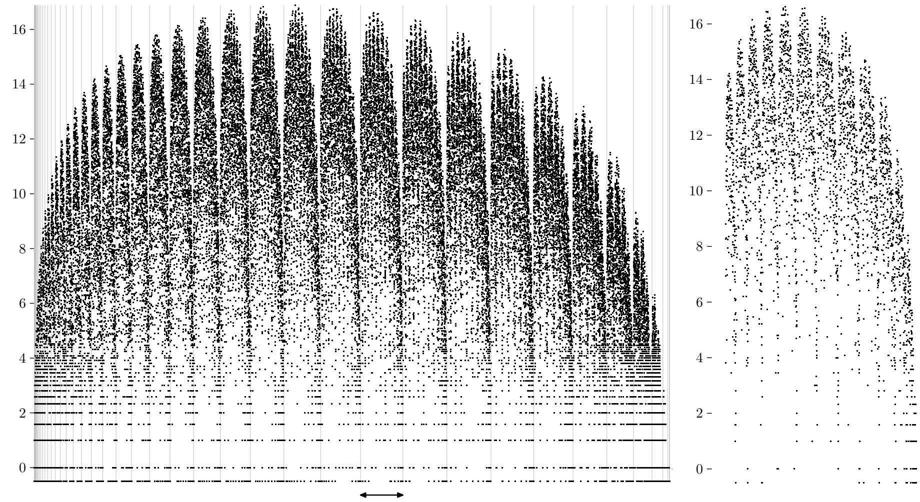
$[\mathrm{Sym}^7(\mathrm{Sym}^8 V) : \mathrm{Sym}^\lambda(V)]$ for $\lambda \vdash 56$, largest first



$V = \langle v_1, v_2, v_3, v_4 \rangle_{\mathbb{F}_2}$

$$(1,2,3,4) \mapsto \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \end{pmatrix} \end{array}$$

$V = \langle v_1, v_2, v_3, v_4 \rangle_{\mathbb{F}_2}$

$$(1,4,3) \quad \mapsto \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot \end{pmatrix} \end{array}$$

$V = \langle v_1, v_2, v_3, v_4 \rangle_{\mathbb{F}_2}$

$$(1,2)(3,4) \mapsto \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \end{pmatrix} \end{array}$$

$V = \langle v_1, v_2, v_3, v_4 \rangle_{\mathbb{F}_2}$

$$(1,2)(3,4) \mapsto \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \end{pmatrix} \end{array}$$

In the new basis

$$\begin{aligned} w_1 &= v_1 + v_2 + v_3 + v_4 \\ w_2 &= v_1 + v_2 \\ w_3 &= v_1 + v_3 \\ w_4 &= v_1 \end{aligned}$$

$V = \langle v_1, v_2, v_3, v_4 \rangle_{\mathbb{F}_2}$

$$(1,2)(3,4) \mapsto \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \end{pmatrix} \end{array} \qquad \mapsto \begin{array}{c} \begin{array}{cccc} w_1 & w_2 & w_3 & w_4 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 \\ \cdot & 1 & 0 & 1 \\ \cdot & 0 & 1 & 0 \\ \cdot & \cdot & \cdot & 1 \end{pmatrix} \end{array}$$

In the new basis

$$w_1 = v_1 + v_2 + v_3 + v_4$$
$$w_2 = v_1 + v_2$$
$$w_3 = v_1 + v_3$$
$$w_4 = v_1$$

$V = \langle v_1, v_2, v_3, v_4 \rangle_{\mathbb{F}_2}$

$$(1,2,3,4) \mapsto \begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 & v_4 \end{matrix} \\ \begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \end{pmatrix} \end{array} \qquad \mapsto \begin{array}{c} \begin{matrix} w_1 & w_2 & w_3 & w_4 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 1 & 0 \\ \cdot & 1 & 0 & 1 \\ \cdot & 1 & 1 & 0 \\ \cdot & \cdot & \cdot & 1 \end{pmatrix} \end{array}$$

In the new basis

$$\begin{aligned} w_1 &= v_1 + v_2 + v_3 + v_4 \\ w_2 &= v_1 + v_2 \\ w_3 &= v_1 + v_3 \\ w_4 &= v_1 \end{aligned}$$

Filtration:

$$0 \subseteq \langle w_1 \rangle \subseteq \langle w_1, w_2, w_3 \rangle \subseteq \langle w_1, w_2, w_3, w_4 \rangle$$

Decomposition matrix of $\mathbb{F}_3 S_6$

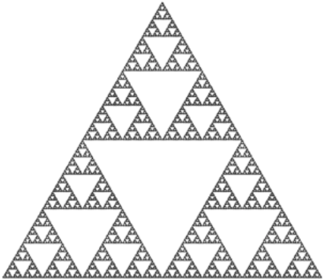
	(6)	(5,1)	(4,2)	(3,3)	(4,1,1)	(3,2,1)	(2,2,1,1)
(6)	1						
$(5,1)$	1	1					
$(4,2)$	·	·	1				
$(3,3)$	·	1	·	1			
$(4,1,1)$	·	1	·	·	1		
$(3,2,1)$	1	1	·	1	1	1	
$(2,2,1,1)$	·	·	·	·	·	·	1
$(2,2,2)$	1	·	·	·	·	1	·
$(3,1,1,1)$	·	·	·	·	1	1	·
$(2,1,1,1,1)$	·	·	·	1	·	1	·
$(1,1,1,1,1,1)$	·	·	·	1	·	·	·

Decomposition matrix of $\mathbb{F}_3 S_6$: two-row partitions

	(6)	**(5,1)**	**(4,2)**	**(3,3)**	(4,1,1)	(3,2,1)	(2,2,1,1)
$(\mathbf{6})$	1						
$(\mathbf{5}, \mathbf{1})$	1	1					
$(\mathbf{4}, \mathbf{2})$	·	·	1				
$(\mathbf{3}, \mathbf{3})$	·	1	·	1			
$(4, 1, 1)$	·	1	·	·	1		
$(3, 2, 1)$	1	1	·	1	1	1	
$(2, 2, 1, 1)$	·	·	·	·	·	·	1
$(2, 2, 2)$	1	·	·	·	·	1	·
$(3, 1, 1, 1)$	·	·	·	·	1	1	·
$(2, 1, 1, 1, 1)$	·	·	·	1	·	1	·
$(1, 1, 1, 1, 1, 1)$	·	·	·	1	·	·	·

General form of the two-row decomposition matrix

$$
\begin{array}{ccccccccccc}
1 & & & & & & & & & & \\
1 & 1 & & & & & & & & & \\
 & 1 & 1 & & & & & & & & \\
 & 1 & 1 & 1 & & & & & & & \\
1 & 1 & & 1 & 1 & & & & & & \\
1 & & & 1 & 1 & 1 & & & & & \\
 & & & 1 & & 1 & 1 & & & & \\
 & & & 1 & & 1 & 1 & 1 & & & \\
 & & & 1 & 1 & 1 & & 1 & 1 & & \\
 & & & 1 & 1 & & & 1 & 1 & 1 & \\
 & & & 1 & & & & 1 & & 1 & (1)
\end{array}
$$



Decomposition matrix of $\mathbb{F}_3 S_6$: separated into blocks

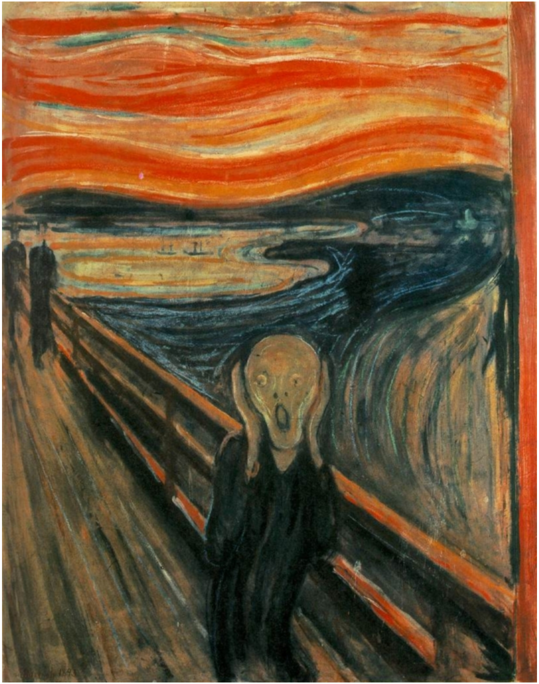
	(6)	(5,1)	(3,3)	(4,1,1)	(3,2,1)
(6)	1				
$(5,1)$	1	1			
$(3,3)$	·	1	1		
$(4,1,1)$	·	1	·	1	
$(3,2,1)$	1	1	1	1	1
$(2,2,2)$	1	·	·	·	1
$(3,1,1,1)$	·	·	·	1	1
$(2,1,1,1,1)$	·	·	1	·	1
$(1,1,1,1,1,1)$	·	·	1	·	·

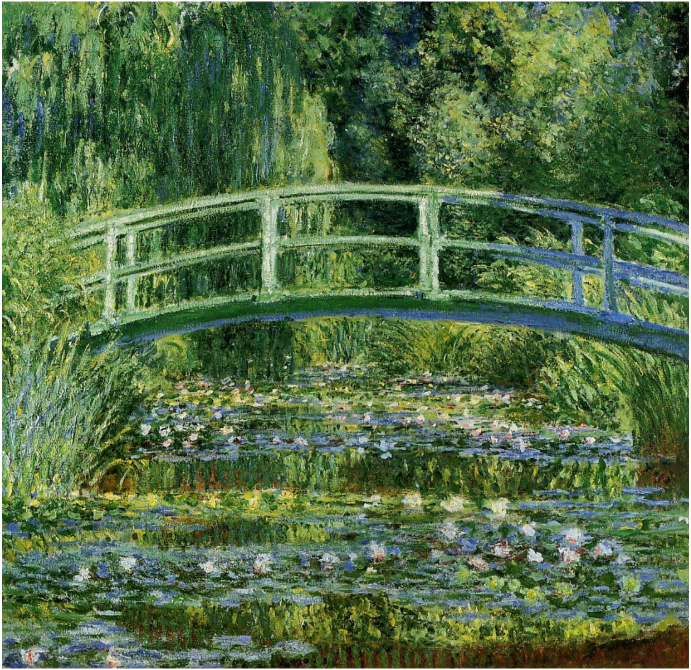
	(4,2)
$(4,2)$	1

	(2,2,1,1)
$(2,2,1,1)$	1

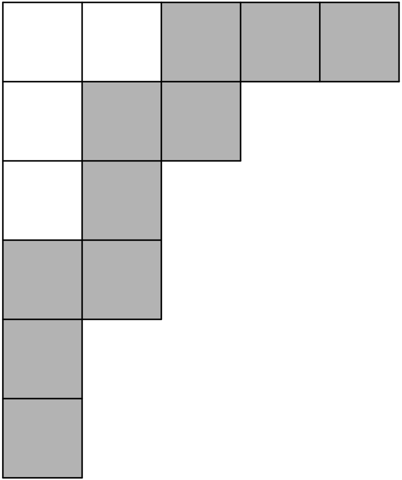
Decomposition matrix of $\mathbb{F}_2 S_{10}$: separated into blocks

	(10)	(9,1)	(8,2)	(7,3)	(6,4)	(6,3,1)	(5,3,2)	(7,2,1)	(5,4,1)	(4,3,2,1)
(10)	1									
$(9,1)$	1	1								
$(8,2)$	1	1	1							
$(7,3)$	1	·	1	1						
$(6,4)$	·	·	1	1	1					
$(6,3,1)$	1	·	2	1	1	1				
$(5,3,2)$	2	1	1	·	1	1	1			
$(5,5)$	·	·	1	·	1	·	·			
$(8,1,1)$	2	1	1	·	·	·	·			
$(6,2,2)$	1	·	1	·	·	1	·			
$(4,4,2)$	2	1	1	·	1	·	1			
$(4,3,3)$	2	1	·	·	·	·	1			
$(7,1,1,1)$	2	1	1	1	·	·	·			
$(6,2,1)$	2	1	3	1	1	1	·			
$(5,3,1,1)$	3	1	3	1	2	1	1			
$(4,4,1,1)$	2	1	1	1	1	·	1			
$(5,2,2,1)$	3	1	2	1	1	1	1			
$(6,1,1,1,1)$	2	1	2	1	1	·	·			
$(7,2,1)$								1	·	
$(5,4,1)$								1	1	
$(4,3,2,1)$										1



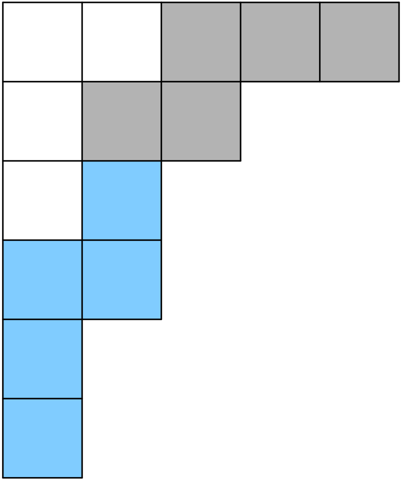


p-cores and p-hooks



The partition $(5, 3, 2, 2, 1, 1)$ is a 3-core. It has a unique 5-hook and two 7-hooks.

p-cores and p-hooks



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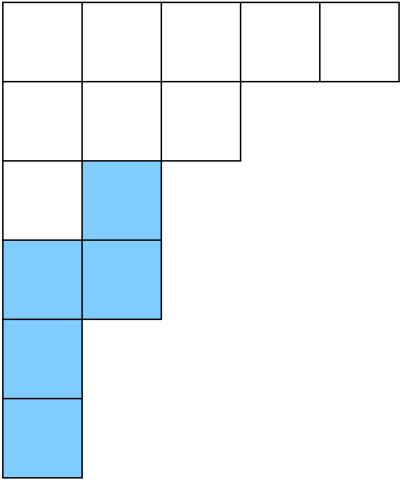
p-cores and p-hooks



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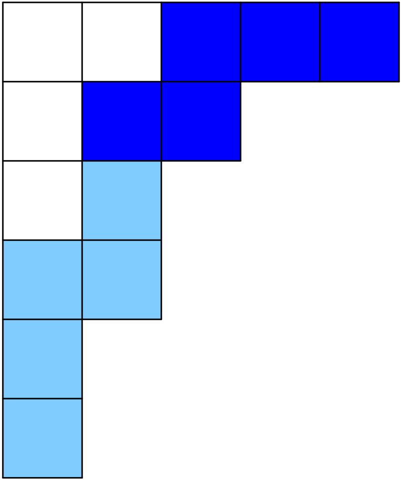
p-cores and p-hooks



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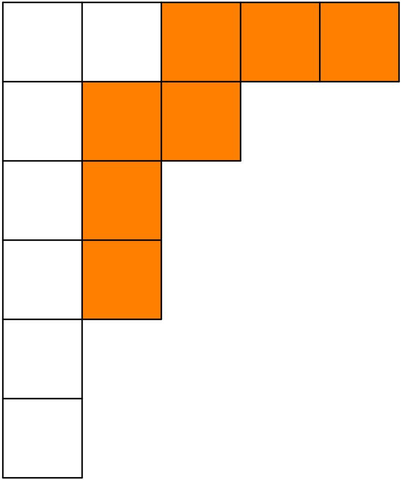
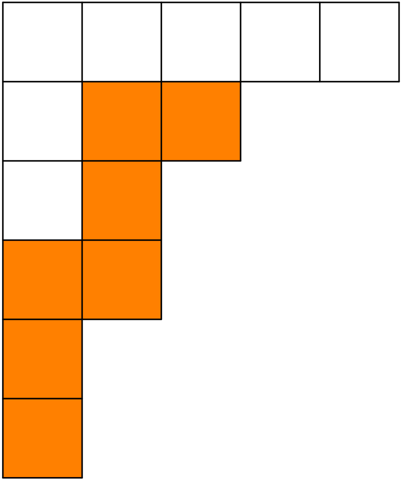
p-cores and p-hooks



The partition $(5, 3, 2, 2, 1, 1)$ is a 3-core. It has a unique 5-hook and two 7-hooks.

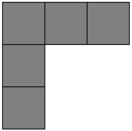
p-cores and p-hooks



The partition $(5, 3, 2, 2, 1, 1)$ is a 3-core. It has a unique 5-hook and two 7-hooks.

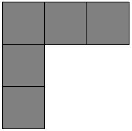
Decomposition matrices: 3-block of S_{12} with core $(3,1,1)$

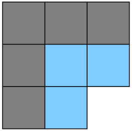
	$(12,1^2)$	$(9,4,1)$	$(9,3,2)$	$(8,4,2)$	$(6^2,2)$	$(6,4^4)$	$(6,4,2^2)$	$(6,3,2^2,1)$	$(5,4,2^2,1)$	$(4^2,2^2,1^2)$
$(12,1^2)=\langle 2\rangle$	1									
$(9,4,1)=\langle 2,2\rangle$	1	1								
$(9,3,2)=\langle 2,1\rangle$	2	1	1							
$(8,4,2)=\langle 1\rangle$	1	1	1	1						
$(6^2,2)=\langle 1,2\rangle$				1	1					
$(6,4^4)=\langle 1,2,2\rangle$			1	1	1	1				
$(6,4,2^2)=\langle 2,2,2\rangle$	1	1	1	1	1	1	1			
$(6,3,2^2,1)=\langle 1,1,2\rangle$	2	1	1				1	1		
$(5,4,2^2,1)=\langle 1,1\rangle$	1	1	1		1	1	1	1	1	
$(4^2,2^2,1^2)=\langle 3\rangle$	1		1		1	1		1	1	1
$(9,1^5)=\langle 2,3\rangle$			1							
$(6,4,1^4)=\langle 2,2,3\rangle$							1			
$(6,3,2,1^3)=\langle 1,2,3\rangle$			1			1	1	1		
$(6,2^3,1^2)=\langle 3,2\rangle$								1		
$(6,1^8)=\langle 2,3,3\rangle$						1				
$(5,4,2,1^3)=\langle 1,3\rangle$					2	1	1	1	1	
$(3^4,1^2)=\langle 3,1\rangle$	1		1			1				1
$(3^2,2^4)=\langle 1,1,3\rangle$	1									1
$(3^2,2^2,1^4)=\langle 1,1,1\rangle$					1	1			1	1
$(3^2,2,1^6)=\langle 1,3,3\rangle$					2	1			1	
$(3,2^3,1^5)=\langle 3,3\rangle$					1				1	
$(3,1^{11})=\langle 3,3,3\rangle$					1					



Decomposition matrices: 3-block of S_{12} with core $(3,1,1)$

	$(12,1^2)$	$(9,4,1)$	$(9,3,2)$	$(8,4,2)$	$(6^2,2)$	$(6,4^4)$	$(6,4,2^2)$	$(6,3,2^2,1)$	$(5,4,2^2,1)$	$(4^2,2^2,1^2)$
$(12,1^2)=\langle 2\rangle$	1									
$(9,4,1)=\langle 2,2\rangle$	1	1								
$(9,3,2)=\langle 2,1\rangle$	2	1	1							
$(8,4,2)=\langle 1\rangle$	1	1	1	1						
$(6^2,2)=\langle 1,2\rangle$				1	1					
$(6,4^4)=\langle 1,2,2\rangle$			1	1	1	1				
$(6,4,2^2)=\langle 2,2,2\rangle$	1	1	1	1	1	1	1			
$(6,3,2^2,1)=\langle 1,1,2\rangle$	2	1	1				1	1		
$(5,4,2^2,1)=\langle 1,1\rangle$	1	1	1		1	1	1	1	1	
$(4^2,2^2,1^2)=\langle 3\rangle$	1		1		1	1		1	1	1
$(9,1^5)=\langle 2,3\rangle$			1							
$(6,4,1^4)=\langle 2,2,3\rangle$							1			
$(6,3,2,1^3)=\langle 1,2,3\rangle$			1			1	1	1		
$(6,2^3,1^2)=\langle 3,2\rangle$								1		
$(6,1^8)=\langle 2,3,3\rangle$						1				
$(5,4,2,1^3)=\langle 1,3\rangle$					2	1	1	1	1	
$(3^4,1^2)=\langle 3,1\rangle$	1		1			1				1
$(3^2,2^4)=\langle 1,1,3\rangle$	1									1
$(3^2,2^2,1^4)=\langle 1,1,1\rangle$					1	1			1	1
$(3^2,2,1^6)=\langle 1,3,3\rangle$					2	1			1	
$(3,2^3,1^5)=\langle 3,3\rangle$					1				1	
$(3,1^{11})=\langle 3,3,3\rangle$					1					





Decomposition matrices: 3-block of S_{12} with core $(3,1,1)$

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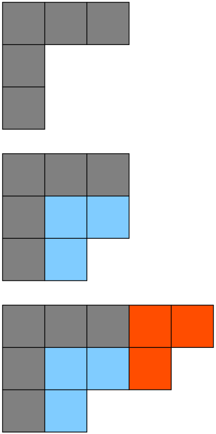
$(12,1^2)=\langle 2\rangle$	1									
$(9,4,1)=\langle 2,2\rangle$	1	1								
$(9,3,2)=\langle 2,1\rangle$	2	1	1							
$(8,4,2)=\langle 1\rangle$	1	1	1	1						
$(6^2,2)=\langle 1,2\rangle$				1	1					
$(6,4^4)=\langle 1,2,2\rangle$			1	1	1	1				
$(6,4,2^2)=\langle 2,2,2\rangle$	1	1	1	1	1	1	1			
$(6,3,2^2,1)=\langle 1,1,2\rangle$	2	1	1				1	1		
$(5,4,2^2,1)=\langle 1,1\rangle$	1	1	1		1	1	1	1	1	
$(4^2,2^2,1^2)=\langle 3\rangle$	1		1		1	1		1	1	1
$(9,1^5)=\langle 2,3\rangle$			1							
$(6,4,1^4)=\langle 2,2,3\rangle$							1			
$(6,3,2,1^3)=\langle 1,2,3\rangle$			1			1	1	1		
$(6,2^3,1^2)=\langle 3,2\rangle$								1		
$(6,1^8)=\langle 2,3,3\rangle$						1				
$(5,4,2,1^3)=\langle 1,3\rangle$					2	1	1	1	1	
$(3^4,1^2)=\langle 3,1\rangle$	1		1			1				1
$(3^2,2^4)=\langle 1,1,3\rangle$	1									1
$(3^2,2^2,1^4)=\langle 1,1,1\rangle$					1	1			1	1
$(3^2,2,1^6)=\langle 1,3,3\rangle$					2	1			1	
$(3,2^3,1^5)=\langle 3,3\rangle$					1				1	
$(3,1^{11})=\langle 3,3,3\rangle$					1					



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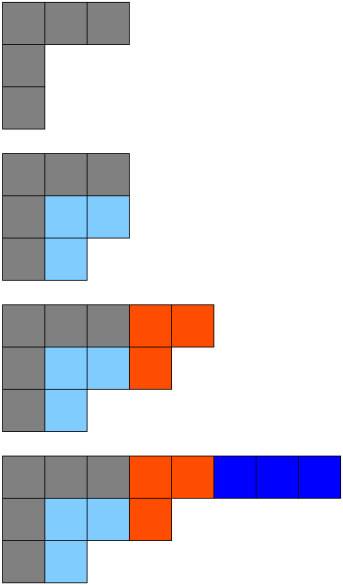
$(9,3,2)=\langle 2,1\rangle$	2	1	1							
$(8,4,2)=\langle 1\rangle$	1	1	1	1						
$(6^2,2)=\langle 1,2\rangle$				1	1					
$(6,4^4)=\langle 1,2,2\rangle$			1	1	1	1				
$(6,4,2^2)=\langle 2,2,2\rangle$	1	1	1	1	1	1	1			
$(6,3,2^2,1)=\langle 1,1,2\rangle$	2	1	1				1	1		
$(5,4,2^2,1)=\langle 1,1\rangle$	1	1	1		1	1	1	1	1	
$(4^2,2^2,1^2)=\langle 3\rangle$	1		1		1	1		1	1	1
$(9,1^5)=\langle 2,3\rangle$			1							
$(6,4,1^4)=\langle 2,2,3\rangle$							1			
$(6,3,2,1^3)=\langle 1,2,3\rangle$			1			1	1	1		
$(6,2^3,1^2)=\langle 3,2\rangle$								1		
$(6,1^8)=\langle 2,3,3\rangle$						1				
$(5,4,2,1^3)=\langle 1,3\rangle$					2	1	1	1	1	
$(3^4,1^2)=\langle 3,1\rangle$	1		1			1				1
$(3^2,2^4)=\langle 1,1,3\rangle$	1									1
$(3^2,2^2,1^4)=\langle 1,1,1\rangle$					1	1			1	1
$(3^2,2,1^6)=\langle 1,3,3\rangle$					2	1			1	
$(3,2^3,1^5)=\langle 3,3\rangle$					1				1	
$(3,1^{11})=\langle 3,3,3\rangle$					1					



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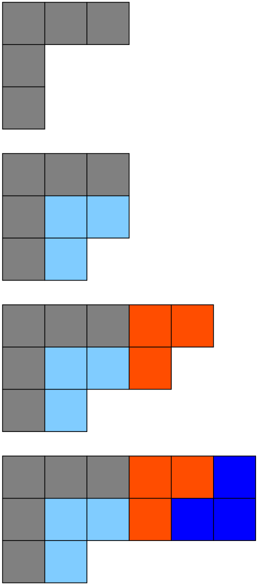
$(9,4,1)=\langle 2,2\rangle$	1	1								
$(9,3,2)=\langle 2,1\rangle$	2	1	1							
$(8,4,2)=\langle 1\rangle$	1	1	1	1						
$(6^2,2)=\langle 1,2\rangle$				1	1					
$(6,4^4)=\langle 1,2,2\rangle$			1	1	1	1				
$(6,4,2^2)=\langle 2,2,2\rangle$	1	1	1	1	1	1	1			
$(6,3,2^2,1)=\langle 1,1,2\rangle$	2	1	1				1	1		
$(5,4,2^2,1)=\langle 1,1\rangle$	1	1	1		1	1	1	1	1	
$(4^2,2^2,1^2)=\langle 3\rangle$	1		1		1	1		1	1	1
$(9,1^5)=\langle 2,3\rangle$			1							
$(6,4,1^4)=\langle 2,2,3\rangle$							1			
$(6,3,2,1^3)=\langle 1,2,3\rangle$			1			1	1	1		
$(6,2^3,1^2)=\langle 3,2\rangle$								1		
$(6,1^8)=\langle 2,3,3\rangle$						1				
$(5,4,2,1^3)=\langle 1,3\rangle$					2	1	1	1	1	
$(3^4,1^2)=\langle 3,1\rangle$	1		1			1				1
$(3^2,2^4)=\langle 1,1,3\rangle$	1									1
$(3^2,2^2,1^4)=\langle 1,1,1\rangle$					1	1			1	1
$(3^2,2,1^6)=\langle 1,3,3\rangle$					2	1			1	
$(3,2^3,1^5)=\langle 3,3\rangle$					1				1	
$(3,1^{11})=\langle 3,3,3\rangle$					1					



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Let $M^{(2^n)} = \langle \Omega^{(2^n)} \rangle_{\mathbb{F}_p} \cong \mathbb{F}_p \big\uparrow_{S_2 \wr S_n}^{S_{2n}}$ where $\Omega^{(2^n)}$ is the set of set partitions of $\{1, \ldots, 2n\}$ into n sets each of size 2.

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$$\begin{aligned}\langle {\langle \Omega^{(2^n)}}^{Q} \rangle_{\mathbb{F}_p} &= \langle \{\mathcal{P} \in \Omega^{(2^n)} : \sigma(\mathcal{P}) = \mathcal{P} \text{ for all } \sigma \in Q \rangle_{\mathbb{F}_p} \\ &\cong V \boxtimes M^{(2^{n-rp})}\end{aligned}$$

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- Hence U is projective with character inside $\sum_{2\nu \in \mathcal{E}(\gamma)} \chi^{2\nu}$.
- By Brauer reciprocity the column for the greatest element of $\mathcal{E}(\gamma)$ is as claimed.

Example of a more general theorem

Start with the empty 3-core $\varnothing$ and try to reach a partition with 2 odd parts. This can't be done by adding one 3-hook. But it can be done by adding two 3-hooks, giving

$$\mathcal{O} = \{(5,1),(4,1,1),(3,3),(3,2,1)\}.$$

The column of the decomposition matrix labelled by $(5,1)$ is given by a generalization of the theorem.

	(6)	**(5,1)**	(4,2)	(3,3)	(4,1,1)	(3,2,1)	(2,2,1,1)
(6)	1						
$(5,1)$	1	**1**					
$(4,2)$	·	·	1				
$(3,3)$	·	**1**	·	1			
$(4,1,1)$	·	**1**	·	·	1		
$(3,2,1)$	1	**1**	·	1	1	1	
$(2,2,1,1)$	·	·	·	·	·	·	1
$(2,2,2)$	1	·	·	·	·	1	·
$(3,1,1,1)$	·	·	·	·	1	1	·
$(2,1,1,1,1)$	·	·	·	1	·	1	·
$(1,1,1,1,1,1)$	·	·	·	1	·	·	·

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	(6)	**(5,1)**	(4,2)	(3,3)	(4,1,1)	(3,2,1)	(2,2,1,1)
(6)	1						
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$(3,2,1)$	1	**1**	·	1	1	1	
$(2,2,1,1)$	·	·	·	·	·	·	1
$(2,2,2)$	1	·	·	·	·	1	·
$(3,1,1,1)$	·	·	·	·	1	1	·
$(2,1,1,1,1)$	·	·	·	1	·	1	·
$(1,1,1,1,1,1)$	·	·	·	1	·	·	·