Mathematisches Forschungsinstitut Oberwolfach

Report No. 9/2020

DOI: 10.4171/OWR/2020/9

Mini-Workshop: Kronecker, Plethysm, and Sylow Branching Coefficients and their Applications to Complexity Theory

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23 February – 29 February 2020

Modular plethysms for $\mathbf{SL_2}(F)$

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(joint work with Eoghan McDowell and Rowena Paget)

Let E be a two-dimensional complex vector space. The finite-dimensional irreducible polynomial representations of $\mathrm{SL}_2(\mathbf{C})$ are, up to isomorphism, the symmetric powers $\mathrm{Sym}^\ell E$ for $\ell \in \mathbf{N}_0$. Working in invariant theory, Hermite discovered the isomorphism

$$\mathrm{Sym}^r \mathrm{Sym}^\ell E \cong \mathrm{Sym}^\ell \mathrm{Sym}^r E. \tag{1}$$

This is one of many *plethystic isomorphisms* of $\mathrm{SL}_2(\mathbf{C})$-representations. Another important example is the Wronskian isomorphism $\mathrm{Sym}^r \mathrm{Sym}^\ell E \cong \bigwedge^r \mathrm{Sym}^{\ell+r-1} E$ (see for instance [1]). More generally, let ∇^λ denote the Schur functor canonically labelled by the partition λ. We ask: *when is there an* $\mathrm{SL}_2(\mathbf{C})$*-isomorphism* $\nabla^\lambda \mathrm{Sym}^\ell E \cong \nabla^\mu \mathrm{Sym}^m E$? In my talk I surveyed some of the answers to this question and then considered the modular analogue in which $\mathbf{C}$ is replaced with an infinite field of prime characteristic.

The first part is on joint work with Rowena Paget [6]; the second is on work in progress with my Ph.D. student Eoghan McDowell.

Part 1: Complex plethystic isomorphisms. Let s_λ denote the Schur function canonically labelled by the partition λ. By the bridge between representation theory and symmetric functions seen in my introductory talk, there is a plethystic isomorphism $\nabla^\lambda \operatorname{Sym}^\ell E \cong \nabla^\mu \operatorname{Sym}^m E$ if and only if $(s_\lambda \circ s_{(\ell)})(x^{-1}, x) = (s_\mu \circ s_\mu)(x^{-1}, x)$. (It is correct to specialize the variables x_1, x_2 so that they satisfy $x_1 x_2 = 1$ because this relation is satisfied by the eigenvalues of every matrix in $\mathrm{SL}_2(\mathbf{C})$.) Substituting $x = q^2$ one obtains (iii) in the theorem below; this is the combinatorial statement that the generating functions enumerating $\mathrm{SSYT}_{\{1,\ldots,\ell\}}(\lambda)$ and $\mathrm{SSYT}_{\{1,\ldots,m\}}(\mu)$ by the sum of the contents of each tableau are equal, up to a power of q.

Theorem 1. *The following are equivalent:*

(i) $\nabla^\lambda \operatorname{Sym}^\ell E \cong \nabla^\mu \operatorname{Sym}^m E$;
(ii) $(s_\lambda \circ s_{(\ell)})(x^{-1}, x) = (s_\mu \circ s_{(m)})(x^{-1}, x)$;
(iii) $s_\lambda(1, q, \ldots, q^\ell) = s_\mu(1, q, \ldots, q^m)$ *up to a (known) power of q;*
(iv) $C(\lambda) + \ell + 1 \big/ H(\lambda) = C(\mu) + m + 1 \big/ H(\mu)$.

In (iv), $C(\lambda) = \{j - i : (i,j) \in [\lambda]\}$ is the multiset of contents of λ, $H(\lambda) = \{h_{(i,j)} : (i,j) \in [\lambda]\}$ is the multiset of hook lengths, and $\big/$ denotes the difference of multisets, *allowing negative multiplicities.* (This is clarified in the example following Theorem 2 below.) The equivalence of (iii) and (iv) is proved using a unique factorization property of the quantum integers $[m]_q = (q^m - 1)/(q - 1) = 1 + \cdots + q^{m-1}$, and Stanley's *Hook Content Formula* [7, Theorem 7.12.2], namely that

$$s_\lambda(1, q, \ldots, q^\ell) = q^B \frac{\prod_{(i,j)\in[\lambda]} [j - i + \ell + 1]_q}{\prod_{(i,j)\in[\lambda]} [h_{(i,j)}]_q}$$

where q^B is a (known) power of q. For example, Hermite reciprocity (1) follows from (iv), since $\{1 + \ell, \ldots, r + \ell\} \big/ \{1, \ldots, r\} = \{1 + r, \ldots, \ell + r\} \big/ \{1, \ldots, \ell\}$. The Wronskian isomorphism may be established still more easily, because in this case the difference multisets on either side of (iv) are equal even before cancellation.

The following theorem is a typical example of a plethystic isomorphism. It was first proved by King [5, §4.2]. A stronger version including a converse is proved using the equivalence of (i) and (iii) in Theorem 1.5 of [6].

Theorem 2. *Let λ be a partition contained in a box with $\ell+1$ rows and a columns. Let $\lambda^\bullet$ be its complement in this box. Then*

$$\nabla^\lambda \operatorname{Sym}^\ell E \cong \nabla^{\lambda^\bullet} \operatorname{Sym}^\ell E.$$

As a corollary of (iv) in Theorem 1 we obtain the following appealing result.

Corollary 3. *Let λ be a partition contained in a box with $\ell+1$ rows and a columns. Let $\lambda^\bullet$ be its complement in this box. There is an equality of multisets*

$$\big(C(\lambda) + \ell + 1\big) \cup H(\lambda^\bullet) = \big(C(\lambda^\bullet) + \ell + 1\big) \cup H(\lambda).$$

For example, if $\lambda = (4,3,3,1)$ and the box has 4 rows and 5 columns then $\lambda^\bullet = (4,2,2,1)$ and the equality in Corollary 3 may be checked using the bold numbers in the tableaux below.

$C(\lambda) + 4$

$\mathbf{4}_0$	$\mathbf{5}_1$	$\mathbf{6}_2$	$\mathbf{7}_3$	$\mathbf{1}_0$
$\mathbf{3}_0$	$\mathbf{4}_1$	$\mathbf{5}_2$	$\mathbf{1}_0$	$\mathbf{3}_1$
$\mathbf{2}_0$	$\mathbf{3}_1$	$\mathbf{4}_2$	$\mathbf{2}_0$	$\mathbf{4}_1$
$\mathbf{1}_0$	$\mathbf{1}_0$	$\mathbf{2}_1$	$\mathbf{5}_2$	$\mathbf{7}_3$

$H(\lambda^\bullet)$

$H(\lambda)$

$\mathbf{7}_3$	$\mathbf{5}_2$	$\mathbf{4}_1$	$\mathbf{1}_0$	$\mathbf{1}_0$
$\mathbf{5}_2$	$\mathbf{3}_1$	$\mathbf{2}_0$	$\mathbf{3}_1$	$\mathbf{2}_0$
$\mathbf{4}_2$	$\mathbf{2}_1$	$\mathbf{1}_0$	$\mathbf{4}_1$	$\mathbf{3}_0$
$\mathbf{1}_0$	$\mathbf{7}_3$	$\mathbf{6}_2$	$\mathbf{5}_1$	$\mathbf{4}_0$

$C(\lambda^\bullet) + 4$

The author is grateful for Christine Bessenrodt for observing that Corollary 3 holds in a stronger version also considering arm-lengths, as indicated above by subscripts. This was proved by Bessenrodt [3] by an ingenious application of [2, Theorem 3.2]. A longer inductive proof can be given by adapting the proof of Corollary 3 in [8]. Finding a representation theoretic interpretation of this stronger result was suggested at the workshop as an open problem.

In [6] many further plethystic isomorphisms, and obstructions to such isomorphisms, are proved. In particular, in [6, Theorem 1.4] we extend another result of King [5, §4] to give a complete classification of all isomorphisms between $\nabla^\lambda \operatorname{Sym}^\ell E$ and $\nabla^{\lambda'} \operatorname{Sym}^m E$, where λ' is the conjugate partition to λ. In [6, §10] we give a complete classification of all isomorphisms $\nabla^\lambda \operatorname{Sym}^\ell E \cong \nabla^\mu \operatorname{Sym}^m E$ in which λ and μ are (separately) either hook partitions, two-row partitions, or two-column partitions. One curious family we obtain is $\nabla^{(3\ell-3,2\ell-1)} \operatorname{Sym}^\ell E \cong \nabla^{(\ell+1,1^{\ell-2})} \operatorname{Sym}^{3\ell-4} E$ for all $\ell \geq 2$. The author suggests finding a geometric or invariant theory interpretation of this isomorphism as an open problem.

Part 2: Modular plethysms. Let F be an infinite field of prime characteristic p and let E be the natural representation of $\mathrm{SL}_2(F)$. It is now important to distinguish the two versions of the symmetric power. Given a polynomial representation V of $\mathrm{SL}_2(F)$, let $\operatorname{Sym}_r V = (V^{\otimes r})^{S_r}$ be the invariant submodule under the place permutation action of S_r on $V^{\otimes r}$ and let

$$\operatorname{Sym}^r V = V^{\otimes r} / \langle v^{(1)} \otimes \cdots \otimes v^{(r)} \cdot \sigma - v^{(1)} \otimes \cdots \otimes v^{(r)} \rangle$$

be the module of coinvariants. For example, the matrices giving the action of

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(F)$$

on $\operatorname{Sym}^2 E$ and $\operatorname{Sym}_2 E$ in a basis e_1, e_2 of E are

$$\begin{array}{ccc} e_1^2 & e_2^2 & e_1 e_2 \end{array} \qquad \begin{array}{ccc} e_1 \otimes e_1 & e_2 \otimes e_2 & e_1 \otimes e_2 + e_2 \otimes e_1 \end{array}$$

$$\begin{pmatrix} \alpha^2 & \beta^2 & \alpha\beta \\ \gamma^2 & \delta^2 & \gamma\delta \\ 2\alpha\gamma & 2\beta\delta & \alpha\delta + \beta\gamma \end{pmatrix} \qquad \begin{pmatrix} \alpha^2 & \beta^2 & 2\alpha\beta \\ \gamma^2 & \delta^2 & 2\gamma\delta \\ \alpha\gamma & \beta\delta & \alpha\delta + \beta\gamma \end{pmatrix}$$

respectively. (Here, as usual e_1^2 is the image of $e_1 \otimes e_1$ in the quotient module defined above.) Observe that if $p = 2$ then $\operatorname{Sym}^2 E$ has a 2-dimensional invariant submodule $\langle e_1^2, e_2^2 \rangle$, whereas $\operatorname{Sym}_2 E$ has this 2-dimensional module only as a quotient. More generally, it is known that $\mathrm{Sym}^r E \cong (\mathrm{Sym}_r E)^\circ$ where $\circ$ denotes contravariant duality, defined on a representation $\rho : \mathrm{SL}(E) \to \mathrm{GL}(V)$ by $\rho^\circ(g) = \rho(g^{\mathrm{t}})^{\mathrm{t}}$ (see [4, §2.7 and p44 Example 1]).

The distinction between the two versions of the symmetric power is critical in the following modular generalization of the Wronskian isomorphism.

Theorem 4. *For all r, $\ell \in \mathbf{N}$, there is an $\mathrm{SL}_2(F)$-isomorphism*

$$\mathrm{Sym}_r \mathrm{Sym}^\ell E \cong \bigwedge^r \mathrm{Sym}^{r+\ell-1} E.$$

We prove this isomorphism by an explicit construction: it is non-obvious and slightly subtle to prove $\mathrm{SL}_2(F)$-equivariance. We also generalize Theorem 2.

Theorem 5. *Let λ be a partition contained in a box with $\ell+1$ rows and a columns. Let $\lambda^\bullet$ be its complement in this box. Then*

$$\nabla^\lambda \mathrm{Sym}^\ell E \cong \nabla^{\lambda^\bullet} \mathrm{Sym}_\ell E.$$

One important idea in the proof is that if V is a polynomial representation of $\mathrm{SL}_2(F)$ of dimension d then $\bigwedge^r V \cong \bigwedge^{d-r} V^\star \cong \bigwedge^{d-r} V^\circ$.

It follows from the theorem of King on conjugation of partitions mentioned above that there is an $\mathrm{SL}_2(\mathbf{C})$-isomorphism $\nabla^{(a+1,1^b)} \mathrm{Sym}^\ell E \cong \nabla^{(b+1,1^a)} \mathrm{Sym}^{\ell+a-b} E$ for all $a, b \in \mathbf{N}$ and $\ell \geq b$. The final result below shows that this does not extend to the modular case.

Theorem 6. *There exist infinitely many pairs (a, b) such that, provided e is sufficiently large, the eight representations of $\mathrm{SL}_2(F)$ obtained from $\nabla^{(a+1,1^b)} \mathrm{Sym}^{p^e+b}$ by*

(i) *Replacing ∇ with its contravariant dual functor ∇°;*
(ii) *Replacing $(a+1, 1^b)$ with $(b+1, 1^a)$ and $p^e + b$ with $p^e + a$;*
(iii) *Replacing $\mathrm{Sym}^\ell E$ with $\mathrm{Sym}_\ell E$*

are all non-isomorphic.

Determining which of the other plethystic isomorphisms in [6] have modular generalizations appears to be a fruitful topic for further research.

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