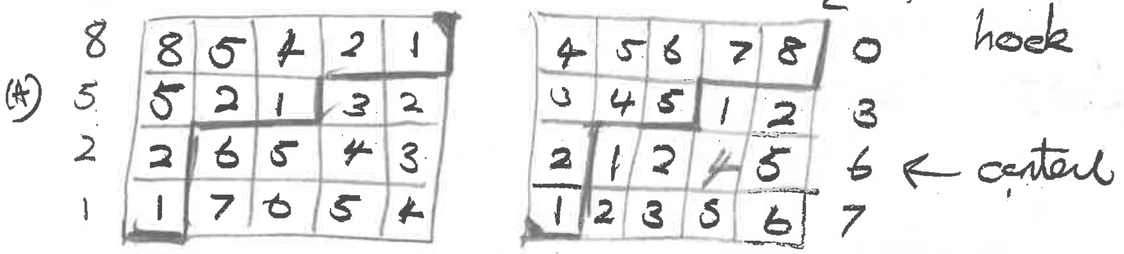
Plethysms and polynomial representations of $SL_2(\mathbb{C})$

w/ Rowena Paget THANKS

§1 Motivation

(A)



$\lambda_i + 4 - i$ $\searrow$ $\swarrow$ $\lambda_i^{\circ} + 4 - i$

$\{1^3, 2^4, 3^2, 4^3, 5^{④}, 6^2, 7, 8\}$

(B)
$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ \varepsilon^8 & \varepsilon^5 & \varepsilon^2 & \varepsilon \\ \varepsilon^{16} & \varepsilon^{10} & \varepsilon^4 & \varepsilon^2 \\ \varepsilon^{24} & \varepsilon^{15} & \varepsilon^6 & \varepsilon^3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ \varepsilon^7 & \varepsilon^6 & \varepsilon^3 & 1 \\ \varepsilon^{14} & \varepsilon^{12} & \varepsilon^6 & 1 \\ \varepsilon^{21} & \varepsilon^{18} & \varepsilon^9 & 1 \end{vmatrix}$$

(C) $\nabla^{(5,3,1,1)} \mathrm{Sym}^{③} \mathbb{C}^2 \cong_{SL_2(\mathbb{C})} \nabla^{(4,4,2)} \mathrm{Sym}^{③} \mathbb{C}^2$. NB

§2 Polynomial representations

Let $E = \langle e_1, \ldots, e_d \rangle_{\mathbb{C}}$.

E.g. if $d=2$, $SL_2(\mathbb{C}) \longrightarrow GL(\mathrm{Sym}^2 E)$

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \longrightarrow \begin{array}{c} \begin{array}{ccc} e_1^2 & e_1e_2 & e_2^2 \end{array} \\ \begin{pmatrix} \alpha^2 & \alpha\beta & \beta^2 \\ 2\alpha\gamma & \alpha\delta+\beta\gamma & 2\beta\delta \\ \gamma^2 & \gamma\delta & \delta^2 \end{pmatrix} \end{array}$$

degree 2 polys in $\alpha, \beta, \gamma, \delta$

~~so~~ not $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \to \begin{pmatrix} 1 & \log|\alpha\delta - \beta\gamma| \\ & 1 \end{pmatrix}$

Generally $E^{\otimes 2} = \mathrm{Sym}^2 E \oplus \Lambda^2 E$

$E^{\otimes 3} = \mathrm{Sym}^3 E \oplus \Lambda^3 E \oplus ?$ then $\nabla^{(2,1)} E^{\oplus 2}$

- Let $\nabla^{(2,1)} E = \left\langle F\left(\begin{array}{|c|c|}\hline a & b \\ \hline c & \\ \cline{1-1}\end{array}\right) : \begin{array}{c} a \leq b \\ \wedge \\ c \end{array} \right\rangle_{\mathbb{C}}$ where

$$F\left(\begin{array}{|c|c|}\hline a & b \\ \hline c & \\ \cline{1-1}\end{array}\right) = e_a e_b \otimes e_c - e_c e_b \otimes e_a \in \mathrm{Sym}^2 E \otimes E.$$

Thm $\{\nabla^\lambda E : \lambda \in \mathrm{Par}(r), \ell(\lambda) \leq d\}$ is a complete set of irreducible $GL_d(\mathbb{C})$ reps.

Can apply ∇^λ to any rep V of $GL_d(\mathbb{C})$: replace $e_1 \cdots e_d$ with a basis of V.

Example $d=2$

$$\begin{tikzcd} \mathrm{Sym}^2(\mathrm{Sym}^2E) = \langle (e_1^2)(e_1^2), (e_1^2)(e_1e_2), (e_1^2)(e_2^2), (e_1e_2)(e_1e_2), (e_1e_2)(e_2^2), (e_2^2)(e_2^2) \rangle_{\mathbb{C}} \arrow[d] & = \mathcal{O}(\mathrm{Sym}^2E) = \mathbb{C}[A,B,C] \arrow[d, "\text{kernel ideal } AC-B^2"'] & \mathcal{C} = \{(x^2e_1^2+2xye_1e_2+y^2e_2^2)\} \\ \mathrm{Sym}^4 E & \mathcal{O}(E) = \mathbb{C}[X,Y] & E = \{x_1e_1+x_2e_2\} \arrow[u] \end{tikzcd}$$

kernel $\langle (e_1^2)(e_2^2)-(e_1e_2)^2\rangle \cong \nabla^{(2,2)}E$

$$\begin{tikzcd} A \arrow[d] & B \arrow[d] & C \arrow[d] \\ X^2 & XY & Y^2 \end{tikzcd}$$

$$\boxed{\cong \frac{\mathcal{O}(\mathrm{Sym}^2E)}{V(\mathcal{C})}}$$

$$\mathrm{Sym}^2(\mathrm{Sym}^2E) \cong \langle A^2\rangle^{\circlearrowleft GL_2(\mathbb{C})} \oplus \langle (AC-B^2)\rangle^{\circlearrowleft GL_2(\mathbb{C})}$$

$$\cong \nabla^{(4)}E \oplus \nabla^{(2^2)}E$$

$$\cong_{SL(E)} \mathrm{Sym}^4E \oplus \mathbb{C} \cong \det{}^2E$$

$$\mathrm{Sym}^4(\mathrm{Sym}^2E) \cong \langle A^4\rangle \oplus \langle A^2(AC-B^2)\rangle \oplus \langle (AC-B^2)^2\rangle$$

$$\cong \nabla^{(8)}E \oplus \nabla^{(6,2)}E \oplus \nabla^{(4,4)}E$$

$$\mathrm{Sym}^8E \oplus \mathrm{Sym}^4E \oplus \mathbb{C}$$

§3 Schur functions and plethysm

Let $s_\lambda(x_1,\dots,x_d) = \sum_{t\in SSYT_{\leq d}(\lambda)} x^t$. E.g.

$$s_{(2,1)}(x_1,x_2) = x^{\substack{11\\2}} + x^{\substack{12\\2}} = x_1^2x_2 + x_1x_2^2$$

$$s_{(\ell)}(x_1,x_2) = x^{11\cdots1} + x^{11\cdots2} + \cdots + x^{22\cdots2} = x_1^\ell + x_1^{\ell-1}x_2 + \cdots + x_2^\ell$$

Then $s_\lambda(x_1 \cdot x_d) = \mathrm{char}\ \nabla^\lambda$, ie

$$s_\lambda(\alpha_1 \cdot\cdot \alpha_d) = \mathrm{tr}_{\nabla^\lambda E}(\alpha_1\cdots\alpha_d)$$

Thm Two polynomial reps are iso $\Leftrightarrow$ same character πληθισμοσ (plethysm)

Def. $(s_\lambda \circ s_{(\ell)})(x_1, x_2) = s_\lambda$ (monomials in $s_{(\ell)}(x_1,x_2)$)
$$= s_\lambda(x_1^\ell, x_1^{\ell-1}x_2 \cdots, x_2^\ell)$$

Eg $(s_{(2)} \circ s_{(2)})(x_1, x_2) = s_{(2)}(x_1^2, x_1x_2, x_2^2)$
$$= (x_1^2)^2 + (x_1^2)(x_1x_2) + (x_1^2)(x_2^2) + (x_1x_2)^2 + (x_1x_2)x_2^2 + (x_2^2)^2$$
$$= s_{(4)}(x_1,x_2) + (x_1x_2)^2$$
$$= s_{(4)}(x_1,x_2) + s_{(2,2)}(x_1,x_2) \quad \boxed{\begin{matrix}1&1\\2&2\end{matrix}}$$

Generally $(s_\lambda \circ s_{(\ell)})(x_1,x_2)$ = char $\nabla^\lambda \mathrm{Sym}^\ell \mathbb{C}^2$

Theorem TFAE
(i) $\nabla^\lambda \mathrm{Sym}^\ell \mathbb{C}^2 \cong_{SL_2(\mathbb{C})} \nabla^\mu \mathrm{Sym}^m \mathbb{C}^2$
(ii) $(s_\lambda \circ s_\ell)(q^{-1}, q) = (s_\mu \circ s_m)(q^{-1}, q)$
(iii) $s_\lambda(1, q \cdots q^\ell) = s_\mu(1, q \cdots, q^\ell)$

Rmk (A) uses Stanley's Hook Content Formula for (iii) (like hook formula / weyl char formula), (B) uses det form of (ii)

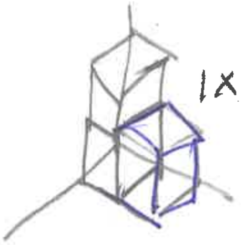
§4 Rectangles

Thm [King '85, Manivel '08] $\nabla^{(a^b)} \mathrm{Sym}^{b+c-1}\mathbb{C}^2$ is invariant up to $SL_2(\mathbb{C})$-iso under permutations of a, b, c.

E.g. take $b=1$, swap $a \leftrightarrow c$: $\mathrm{Sym}^a \mathrm{Sym}^c \mathbb{C}^2 \cong \mathrm{Sym}^c \mathrm{Sym}^a \mathbb{C}^2$
Hermite reciprocity

Thm [MacMahon 96] $q^{-a\binom{b}{2}} s_{(a^b)}(1, q \cdots q^{b+c-1})$

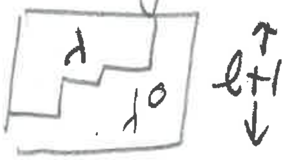
joke C19.
$$= \prod_{\substack{1\le i\le a\\1\le j\le b\\1\le k\le c}} \frac{q^{i+j+k-1}-1}{q^{i+j+k-2}-1} = \sum_{R \in PP(a\times b\times c)} q^{|R|}$$


1x1x2 plane p

§5 Some new results

Thm [P, W] If $\ell \leq 4$ then $\nabla^{\lambda} \mathrm{Sym}^{\ell} \mathbb{C}^2 \underset{SL_2(\mathbb{C})}{\cong} \nabla^{\mu} \mathrm{Sym}^{m} \mathbb{C}^2$

$\iff \lambda = \mu$ or $\lambda = \mu^{\circ}$



. If $\ell \geq 5$ infinitely many 'exotic' isos.

Thm $\nabla^{(a,b)} \mathrm{Sym}^{\ell} \mathbb{C}^2 \underset{SL_2(\mathbb{C})}{\cong} \nabla^{(a',b')} \mathrm{Sym}^{\ell'} \mathbb{C}^2 \iff \ldots$

$\ldots$ inc $\nabla^{(2\ell, \ell+2)} \mathrm{Sym}^{\ell} E \cong \nabla^{(2\ell-2, \ell-2)} \mathrm{Sym}^{\ell+2} E$

§6 Prime characteristic

Thm [Eggleton McDowell] $\mathrm{Sym}^2 \mathrm{Sym}^{\ell} E \underset{SL_2(\mathbb{F}_{p^r})}{\cong} \left(\Lambda^2 \mathrm{Sym}^{\ell+1} E\right)^{*}$

$\iff p > 2$ or $p = 2$ and all $r \gg 1$ $\ell = 2^e - 1$, $e \in \mathbb{N}$