SOME THOUGHTS ON UNDERGRADUATE ASSIGNMENTS

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1. Motivation and Philosophy

I decided to write this note because I noticed that many students made similar mistakes in their solutions to problems sheets. Their maths was correct, but it was difficult to read and mark. Some academics in the department have made the same comments when talking about marking exams.

The issue is that writing maths is a skill in itself and understanding the maths is only half of the battle. Once you understand a definition or a theorem, you still need to explain to the reader why applying that theorem to this particular problem will give you the required result. The aim of a proof is that the reader is convinced by your argument and doesn't need to do much work (although they do need to read and understood it). The overall aim is to break the proof down into small enough pieces so that no single step requires too much thought from the reader. If you have convinced yourself of every step in the your proof, then this will not require much effort because you have already done the work. The final step is to present it to the reader in an easily digestible form.

For many PhD students, marking problem sheets is part of their funding arrangement so it's not something that they can avoid. Many complain about having to do it, but most of their complaints are about the perceived futility of the exercise: they see the same students handing work in, week after week, without learning from last week's feedback. While I don't claim that every PhD student will give every student excellent feedback, I suspect that this is in part because some markers have given up and all you will get is a tick or a cross at the bottom.

This note aims to help undergraduates get the most out of the feedback from the marking on their assignments. Obviously this process assumes that interested students will do more than quickly glance at the score out of 10 in the top corner of the front page. I absolutely encourage you to read all of the feedback and work through the problems that you got wrong, because a common strategy for setting exam questions is to reuse questions of the assignment sheets (with perhaps a little modification). This note describes some of the really frustrating things that are often seen on submitted solutions, so that the people marking the assignments remain enthusiastic about giving good feedback to the next generation of maths students and those students learn how to write good maths.

Personally, I tend to use roughly the same amount of effort when marking as I think that the student has taken when submitting it. If it looks like it's been thrown together in 2 minutes, then you can expect to get a single tick or cross at

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the bottom of the page; if it looks like you've invested some effort, then I want to give you advice on how best to use that effort next week. Also, if you've totally ignored the feedback that I gave you last week, that doesn't go down very well.

When I was starting my undergraduate degree, we were told (repeatedly) that 'Maths is not a spectator sport'. While you can learn every definition and the statement of every lemma in a course just by reading your notes, this is not maths. When you come to do exams, there is a massive correlation between students that attempt the assignments and students that do well in the exam. This is because you will learn the definitions and theorems and get used to identifying when you need to use them. You will also get used to writing maths and explaining your argument in a way that makes sense and is easy to follow. This is a skill by itself, it's something that takes time, effort and attempts at problems to develop. In an exam setting, if it is easier for the marker to read and understand your solutions, then it is easier for them to give you marks.

2. How to start

Knowing where to start is often the trickiest part of approaching a question. This section has some hints and tips about how to begin answering a question and suggests some things that the markers will appreciate.

2.1. **Work out what you know.** Writing down what you know is a good way of drawing all of the information out of the text of the question and can really help you get started. It also makes your answers more self-contained for the people marking your solutions.

It's not uncommon to see students set off in completely the wrong direction because they have misread or misinterpreted the question. It's really important to know what assumptions the question allows you to make, but also what those assumptions mean. I would often begin by writing down something like:

$$\begin{aligned}&\text{We are given that } \{a_n\}_{n=0}^{\infty} \to a\\&\text{Therefore } \forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } n > N \Rightarrow |a_n - a| < \epsilon\end{aligned}$$

Writing this means that if you have misunderstood the definition, or applied it to the wrong object to start with, then this can be picked out very quickly by the person marking.

2.2. **Establish what you need to do.** Read the question a couple of times and make sure you really understand what it is that the question is asking you to do. Perhaps make a note of any sufficient conditions that you recognise for the conclusion – this can give you an intermediate step to aim for.

Something that often trips people up is proving theorems in the wrong direction; this is incorrect maths rather than poor style. If you are asked to prove that $A \Rightarrow B$, then proving $B \Rightarrow A$ is not an appropriate answer. The safest way to avoid this is to start from the assumptions (A) that you are given and work towards the conclusion (B). This isn't the only way and it doesn't mean that any solution that mentions the conclusions in the first line is wrong, but there is a risk you'll prove the wrong direction and it normally isn't the clearest way to explain your proof.

2.3. **Begin with an Introduction.** As a combination of the last two points, it's really nice to have a quick explanation of what is going on from the beginning. Just a couple of lines, like this:

> Consider $f : [0,1] \to [0,1]$ such that f is continuous.
> So we know that $\forall \epsilon > 0, \exists \delta > 0$ s.t. $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$
> We want to show that $\forall y \in [f(0), f(1)], \exists x \in [0,1]$s.t.$f(x) = y$.

3. The body of your answer

Once you've got started, there are a few things that you can do to make it easier for you and the person marking your solutions to follow your reasoning and identify any errors that you have made.

3.1. **Introducing Notation.** Whenever you introduce a new variable or function, you must let the reader know what it is that you are doing. A really nice way of doing this is to begin with a short introduction, as above.

Consider the following fragments of solutions, aiming to describe the number of real roots of a quadratic equation. This is just an example, so you don't need to give this much detail every time you use this result, but hopefully it makes the idea clear.

> Let $f(x) = x^2 + 6x + 2$
> Note that $D > 0$
> So f has two real roots

This is much less clear than the following example:

> Let $f(x) = x^2 + 6x + 2$
> Recall the discriminant: $D(ax^2 + bx + c) := b^2 - 4ac$
> So $D(f) = 6^2 - 4 \cdot 1 \cdot 2 = 28$
> Therefore $D(f) > 0$ and, by a theorem from the course,
> f has two real roots.

Here, I've used := to denote 'is defined to be equal to', which is a nice way to make it clear that you are introducing the function D.

3.2. **Write in Sentences.** We use lots of special symbols when we write maths, but they are just a convenient shorthand for proper words. When you write maths, what you are really doing is explaining why the result holds. In any context where you explain something, you do it in full sentences – otherwise it doesn't make sense and is really hard to read or follow.

Whenever you read maths and come across a symbol (like $\forall$, $\exists$, or $\Rightarrow$), you translate from the symbols to real words: $\forall$ because 'for all', $\exists$ becomes 'there exists' and $\Rightarrow$ becomes 'which implies'. When you use one of these symbols, it needs to be used in a way that makes sense when the symbol is replaced by the proper words.

3.3. **Explain how each line follows.** Another common issue is that students don't explain how each line follows from the previous. This makes it very difficult to follow the reasoning of the proof. It's very useful to know whether a line is a rearrangement of the previous line, or if you've applied a theorem from the lectures, or if you are making an assumption that you wish to later disprove by deriving a contradiction.

3.3.1. *A list of good words.* Having said that you should write in sentences and explain how each line follows from the previous, I thought a list of words that you might use to do this would be helpful.

Words to start a solution:

- Consider
- Let
- Take

Words to explain that the following line is a straightforward consequence of the previous:

- Therefore
- So
- Hence
- Thus

Words to give the reason (before the conclusion):

- Notice that
- Observe that
- If, then
- Because

Other useful words:

- Assume (for contradiction)
- Suppose that
- As Required (to show that you have proved the desired result).

3.4. **Spread Your Answers Out.** There aren't any bonus points available for fitting all of your assignments onto a single piece of paper. It is much easier to read and add comments if you leave plenty of space around your answers. It is also much easier if you use a new line for each equality, inequality or implication rather than working across the page and this gives you space to explain how each line follows on from the previous.

Compare this:

$$(a+b)^5 + (a-b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 + a^5 - 5a^4b + 10a^3b^2$$

$$- 10a^2b^3 + 5ab^4 - b^5 = 2[a^5 + 10a^3b^2 + 5ab^4] = 2a^5 + 20a^3\left(\frac{a}{2}\right)^2 + 10a\left(\frac{a}{2}\right)^4$$

$$= (2 + 5 + \frac{10}{16})a^5$$

With this:

$$\begin{aligned}(a+b)^5+(a-b)^5 &= a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5 \\ &\quad +a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5 \quad \text{(expand brackets)} \\ &= 2[a^5+10a^3b^2+5ab^4] \quad \text{(collect like terms)} \\ &= 2a^5+20a^3\left(\frac{a}{2}\right)^2+10a\left(\frac{a}{2}\right)^4 \quad (\text{as } b=\frac{a}{2}) \\ &= (2+5+\frac{10}{16})a^5 \quad \text{As Required.}\end{aligned}$$

3.5. **= vs ⇒ vs ⇔ vs →.** These symbols seem to cause students no end of problems. Partly because students don't seem to understand that they mean different things.

The symbol '=' means that both sides are equal. I'm going to side-step the philosophical and set-theoretic definition of equality, but you can put almost anything on either side of an equals sign (as long as both sides are indeed equal).

One definition of $\Rightarrow$ is that $A \Rightarrow B$ if whenever A is True, then B is also True. Similarly, $\Leftrightarrow$ means that A is True precisely when B is True. This means that I need to be able to say that A is True, or that A is False, and that B is either True or False. For example, $z = 4$ is True whenever z is four and it is False if z is not four. So I can write $z = 4 \Rightarrow z^2 = 16$. However $z \Rightarrow 4$ is completely meaningless because z and 4 cannot be True or False and the sentence "z which implies 4" doesn't mean anything in English.

Also, you need to choose between $\Rightarrow$ and $\Leftrightarrow$ carefully. $\Leftrightarrow$ means both $\Leftarrow$ and $\Rightarrow$, so if you are going to write $\Leftrightarrow$ then you need to be sure that both directions of the implication hold. For example, writing $z = 4 \Rightarrow z^2 = 16$ is fine, because when $z = 4$ it is true that $z^2 = 16$. However, I can't write $z = 4 \Leftrightarrow z^2 = 16$, because I can find a z that satisfies $z^2 = 16$ that does not satisfy $z = 4$ (like -4).

We use $\rightarrow$ for limits. In this case I need to have a function of a variable on the left hand side and a constant on the right. I also need to illustrate somewhere which variable is changing and what the limit of that variable is. For example, if I define z as some function of a variable, maybe $z_n = 4 - \frac{1}{n}$, then I can write $z \rightarrow 4$, as long as I make it clear that this holds as $n \rightarrow \infty$. While I'm talking about limits, please don't ever write $\lim_{x\rightarrow\infty} e^x = e^\infty = \infty$. There is no definition of e^∞, because ∞ is a concept not a number.

3.6. **Getting Stuck.** In my mind, it is preferable for a student to acknowledge once they get stuck rather than to continue writing things in the hope of fooling a marker into giving them the benefit of the doubt. When you are doing this, summarising your progress so far and describing what you would like to show in order to reach the desired conclusion can be very helpful. There isn't anything wrong with saying 'I am unsure about this step' – it gives the marker a good idea where you think you might have made a mistake and means that it is much faster for us to help you. Also, most lecturers will have office hours, which are a really good place to ask for help.

Also, it saves you irritating the person marking your assignment by handing in loads of working that isn't going anywhere because you made a sign error 2 pages ago... That isn't to say that the same applies in exams – in that situation, there is a definite advantage to writing things rather than not, but the same short explanation can make it a lot easier to follow your logic and understand what you were thinking at the time.

3.7. **Go Right to the End.** Another common thing that many students do is to manage almost all of the working but not quite finish the question. If a question asks you to show that, for example, $f(z) = 0$ then the last line of your answer should be '... and so $f(z) = 0$.'. If you don't do this, you risk leaving the reader with quite a lot of work to do in the last step. A really nice way to finish these sort of questions is with the phrase 'As Required' once you have demonstrated the final answer.

4. Finishing Up

Once you've solved all of the problems on the sheet, there are a few finishing touches. None of these are arduous tasks, but the difference that it makes to the mood of the person marking your assignment is huge.

4.1. **Name.** Write your name on your assignment. It is rather optimistic to expect your work to get back to you if you don't write your name on it!

4.2. **Numbering.** Number your questions clearly, use the same numbering pattern as the question sheet does, and hand your solutions in the same order as the question sheet. This makes it much easier to find the relevant question, which means that the marker can spend more time providing useful feedback, rather than looking for the answers.

4.3. **Stapling.** Stapling all of your work together makes my life a lot easier; anything other than a staple is a pain. Any other sort of clip will come undone; screwing up/folding over the corner works well, until I turn the first page and it goes *everywhere*; and putting it in a plastic wallet is great, until I take it out so that I can mark it.

4.4. **Neat Version.** When I was studying for my undergraduate degree, everyone that I used to work with would work on rough paper first and then copy up a final neat version to hand in. This means that we did spend some time thinking about the neatest way to explain and lay out our work, but this is how you practise the skill of explaining your reasoning. Practising this on the assignment sheets means that this comes much more naturally in exams (and that you won't hand in a piece of paper on which everything had been scribbled out).

This doesn't mean that you delete all of your working, or that it needs to take very long to do this. The best advice is to give as much detail as you need to understand what is going on. Suppose you need to differentiate something using the product rule. If you can do it in one step, then that is fine (it's even better if you add 'using the product rule' on the end). If you want to show where you have considered $f := u \cdot v$, computed u' and v' and concluded that $f' = uv' + vu'$ then that is also fine.

4.5. **Presentation.** Most mathematicians have fairly awful handwriting, I know this and I'm not asking anyone to spend hours practicing their handwriting. However, I have a very vivid memory of my first week of my undergraduate being lectured for 90 minutes on the importance of making your notation clear. The 'highlight' was the lecturer writing a lot of vertical lines on the blackboard and telling us that what he really meant was:

$$|\{l \in I \mid \|l\|_{l_1} \,|\, |I_1|\}| = 1$$

That is, the size of the set of l in I so that the l_1-norm of l divides the size of I_1 is equal to 1. Even when it is typed up neatly, it is very difficult to read what this statement means. This is a bit of a silly example, but in the real world it can get incredibly difficult to tell the difference between u, v and U or 1 and 7, or m and M (particularly if you have written your solutions in a bit of a rush). Choosing more sensible notation is a good idea:

$$|\{c \in C : \|c\|_{l_1} \text{ divides } |C_1|\}| = 1$$

and it will be much easier to read if you write ls in script (curly/loopy), make sure that your curly brackets really are curly, and that your subscripts really are subscripts (not just slightly smaller letters!). Writing your 1s with a base bar and your 7s with a strike through (continental style) makes it much harder to confuse these.

5. An easy example

Show that, for x and y in $\mathbb{R}_{\geq 0}$, if $y > x$ then $y^2 > x^2$.
Take $x, y \in \mathbb{R}_{\geq 0}$ such that $y > x$.

Note that because $y > x$, $\exists \epsilon > 0$ s.t. $y = x + \epsilon$.

We want to show that $y^2 > x^2$.

So consider

$$\begin{aligned} y^2 - x^2 &= (x+\epsilon)^2 - x^2 \quad \text{(by definition of } \epsilon) \\ &= x^2 + 2x\epsilon + \epsilon^2 - x^2 \quad \text{(by expanding brackets)} \\ &= \epsilon(2x + \epsilon) \quad \text{(factorise)} \end{aligned}$$

Now, note that $\epsilon > 0$ (by definition) and $2x+\epsilon > 0$ (as both $2x$ and ϵ are), therefore $\epsilon(2x + \epsilon) > 0$.

Then, because $y^2 - x^2 = \epsilon(2x + \epsilon) > 0$, we conclude that $y^2 > x^2$, as required.