DIOPHANTINE APPROXIMATION, FLOWS ON HOMOGENEOUS SPACES AND COUNTING

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1. Introduction

Let $\xi \in \mathbb{R}$, let $\iota : [1, \infty) \to (0, 1]$ be a positive decreasing function, and let $E_{\xi}^{loc}(\iota, Q)$ be the set of integer pairs $(p, q)$ satisfying $|p + q\xi| < \iota(q)$, $0 < q < Q$. In a series of papers starting in 1959 Erdős [14], Schmidt [22, 23], Lang [18, 19, 9], Adams [1, 2, 3, 4, 5, 7, 6, 8], Sweet [26], and others, considered the problem of finding the asymptotics for the cardinality $|E_{\xi}^{loc}(\iota, Q)|$ as $Q$ gets large.

Schmidt [22] has shown that for almost every $\xi \in \mathbb{R}$ the asymptotics are given by the volume of the corresponding subset of $\mathbb{R}^2$, provided the latter tends to infinity. Precursors of this important result are due to LeVeque [21] and Erdős [14]. Lang [18] proved $|E_{\xi}^{loc}(1/q, Q)| \sim c_\xi \log(Q)$ for quadratic $\xi$. In [19] he also proved a result for a general class of $\xi$ but with stronger constraints on the rate of decay of $\iota$. Adams [1, 2] established various counting results for more specific $\xi$. Finally, Schmidt [23], Adams [23, 3, 7, 6, 8] and Sweet [26] generalized some of these results to simultaneous approximations.

Opposed to the above “localised” setting, where the bound on $|p + q\xi|$ is expressed as a function of $q$, we consider the “non-localised” (sometimes called “uniform”) situation, where the bound is expressed as a function of $Q$.

Let $\xi$ be a vector in $\mathbb{R}^r$, and, throughout this article, let $\psi : [1, \infty) \to [1, \infty)$ be a function. We consider the set $E_{\xi}(\psi, Q) = \{(p, q) \in \mathbb{Z}^{r+1}; |p + q\xi|_\infty < \left(\frac{\psi(Q)}{Q}\right)^{1/r}, 0 < q < Q\}$, and we study its cardinality $|E_{\xi}(\psi, Q)|$ as $Q$ becomes large. Here, $| \cdot |_\infty$ denotes the maximum norm. If $\psi(Q) \to \infty$ and is monotone, then it is an easy consequence of Dirichlet’s approximation Theorem that $E_{\xi}(\psi, Q)$, as a function of $Q$, is unbounded (c.f. Section 12). However, unless stated explicitly we do not assume monotonicity. If $E_{\xi}(\psi, Q)$ is not too stretched then $E_{\xi}(\psi, Q)$ is roughly the volume of the corresponding subset in $\mathbb{R}^{r+1}$ which is $2^r \psi(Q)$. Specifically, for $Q = o(\psi(Q))$ we have, by simple standard estimates,

$$E_{\xi}(\psi, Q) \sim 2^r \psi(Q)$$

for any $\xi$ whatsoever. We are mainly interested in the case where $\psi(Q)/Q$ gets small, where, to the best of my knowledge, no general asymptotic counting result is known. In particular, we would like to know for what $\xi$ does (1.1) remain valid?

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\[1\text{In this article “almost every” refers always to the Lebesgue measure.}\]
If \((\psi(Q)/Q)^{1/r} < 1/2\) then a point \((p, q) \in E_\xi(\psi, Q)\) is uniquely determined by \(q\). Hence, one can bound the error term \(|E_\xi(\psi, Q) - 2^r\psi(Q)|\) by \(2^r\) times the star-discrepancy \(D_\xi^r(x_q)\) of the sequence \(x_q = q\xi\) mod 1. For instance, if \(r = 1\) and \(\xi = \xi\) is badly approximable, then, by the Erdős-Turán inequality, we can deduce \(D_\xi^r(x_q) \leq c_\xi \log(Q)^2\), and hence we get (1.1), provided \(\log(Q)^2 = o(\psi(Q))\). There are more refined discrepancy estimates that allow for improvements but estimating the error term by the star-discrepancy is rather wasteful. A famous result of Schmidt [24] shows that \(D_\xi^r(x_q) \geq 10^{-2}\log(Q)\) happens for infinitely many \(Q\). Consequently, even for badly approximable numbers \(\xi\) and \(\psi(Q) = \log(Q)\) we cannot deduce (1.1) by estimating the error term by the star-discrepancy.

We will apply a completely different strategy which invokes methods from dynamics on homogeneous spaces. This allows us to get a fairly complete answer to the above question, at least in the case \(r = 1\).

We write \(N = \{1, 2, 3, \ldots\}\) for the set of positive integers. Let \(\xi \in \mathbb{R}^r\) and \(\kappa \geq 1\), and let \(\varphi : [0, \infty) \to [0, 1]\) be a monotone decreasing (not necessarily strictly) function. We will assume that

\[(1.2) \quad \prod_{i=1}^r |p_i + q\xi_i| \geq q^{-\kappa}\varphi(q) \quad \text{for all } (p, q) \in \mathbb{Z}^r \times N.

\textbf{Theorem 1.1.} Suppose \(\xi \in \mathbb{R}^r\) is such that (1.2) holds. Then, we have

\[(1.3) \quad |E_\xi(\psi, Q)| \sim 2^r\psi(Q) \quad \text{as } Q \to \infty\]

for every function \(\psi : [1, \infty) \to [1, \infty)\) with \(\psi(Q) \leq Q\) and \(\varphi(Q)^{r/\kappa}\psi(Q)/Q^{r-\xi} \to \infty\) as \(Q \to \infty\).

For the moment let us suppose that \(r = 1\) so that \(\xi = \xi\) is a number. If \(\xi\) is badly approximable then we can take \(\kappa = 1\) and \(\varphi\) to be a positive constant. Hence, the asymptotics are given by the volume as long as the volume tends to infinity. Somewhat surprisingly, this is false in the localised situation, even with quadratic \(\xi\). Indeed, there with \(\varphi(q) = 1/q\) the volume is \(2\log(Q) + O(1)\), and by Lang’s result \(|E_\xi^{loc}(1/q, Q)| \sim c_\xi \log(Q)\) for quadratic \(\xi\), but Adams [5] has shown that \(c_\xi \neq 2\).

By taking multiples of good rational approximations it is easy to show that if there exists an infinite sequence \((p_n, q_n) \in \mathbb{Z} \times N\) and a monotone function \(\varphi(\cdot)\) tending “slowly” to zero such that

\[(1.4) \quad |p_n + q_n\xi| < q_n^{-1}\varphi(q_n)\]

for all \(n \in N\), then, with \(\psi(Q) = 4^{-5}\varphi(Q)^{-1}\), there exists an increasing sequence \(Q_n\) of positive integers with \(|E_\xi(\psi, Q_n)| \geq 4\psi(Q_n)\) for all \(n\) (c.f. Lemma 12.2). Hence, (1.3) fails in this case. An eligible choice is \(\varphi(x) = 1/\log(x)\) (for \(x \geq 3\)), and then the existence of such a sequence \((p_n, q_n)\) as in (1.4) is guaranteed by the “convergence part” of Khintchine’s Theorem [17] for almost every \(\xi \in \mathbb{R}\). Hence, for \(\psi(Q) = 4^{-5}\log(Q)\) the asymptotics are not given by the volume for almost every \(\xi \in \mathbb{R}\).

By virtue of Schmidt’s metrical result [22, Theorem 1] mentioned above, this is again in stark contrast to the localised situation. Schmidt’s metrical result raises the question for what \(\psi\) is (1.3) true for almost every \(\xi \in \mathbb{R}\). Using the “convergence part” of Khintchine Theorem we get that for almost every \(\xi \in \mathbb{R}\) there exists a positive constant \(c_\xi'\) such that with, e.g., \(\varphi(x) = c_\xi'/(\log(x)\log\log(x))\) and \(\kappa = 1\) the condition (1.2) holds true. Hence, if \(\log(Q)\log(\log(Q)) = o(\psi(Q))\) then the asymptotic relation (1.3) holds true for almost every \(\xi \in \mathbb{R}\).

Finally, we note that also for \(\kappa > 1\) the exponent \(r - r/\kappa = 1 - 1/\kappa\) is sharp (c.f. Lemma 12.4). So in the non-localised setting we have a simple and essentially sharp
criterion for $\psi$ in terms of the approximation properties (1.2) of $\xi$ to decide whether the asymptotics are given by the volume. In particular, for badly approximable numbers this is always the case.

For $r = 2$ the condition (1.2) is much less understood. The famous Littlewood conjecture asserts that with $\kappa = 1$ the condition (1.2) can only be satisfied if $\varphi$ tends to zero. On the other hand Bugeaud and Moshchevitin [11] showed that the set of $\xi = (\xi_1, \xi_2)$ for which (1.2) is satisfied with $\kappa = 1$ and $\varphi(x) = c_\xi / \log(x)^2$ for some $c_\xi > 0$ has full Hausdorff-dimension in $\mathbb{R}^2$. Badziahin [10] showed one can even take $\varphi(x) = c_\xi / (\log(x) \log(\log(x)))$.

Concerning general $r$, and $\kappa = 1$ Spencer [25] showed that if $\epsilon > 0$ then for almost every $\xi$ there exists $c_\xi > 0$ such that the condition (1.2) holds true with $\varphi(x) = c_\xi / \log(x)^{r+\epsilon}$. And it is known by a theorem of Gallagher [15] that for almost every $\xi$ the condition (1.2) fails for all $c_\xi > 0$ and $\varphi(x) = c_\xi / \log(x)^r$.

We also investigate the cardinality of the inhomogeneous asymmetric set

$$F_\xi(\psi, Q, y) = \{(p, q) \in \mathbb{Z}^{r+1}; 0 < p_i + q \xi_i - y_i < \left(\frac{\psi(Q)}{Q}\right)^{1/r} \quad (1 \leq i \leq r),$$

$$0 < q < Q\}$$

where $y = (y_1, \ldots, y_r) \in \mathbb{R}^r$. With

$$E_\xi(\psi, Q, y) = \{(p, q) \in \mathbb{Z}^{r+1}; |p + q \xi - y|_{\infty} < \left(\frac{\psi(Q)}{Q}\right)^{1/r}, 0 < q < Q\}$$

we have

$$|E_\xi(\psi, Q, y)| = \sum_{\sigma} |F_\xi(\psi, Q, \sigma y)|,$$

where the sum runs over all $2^r$ possible $\sigma$, and each such $\sigma$ acts on a vector in $\mathbb{R}^r$ by replacing some of its entries by their negative values. Thus, Theorem 1.1 follows immediately from the following, more general and more precise, result.

**Theorem 1.2.** Suppose $\xi \in \mathbb{R}^r$ is such that (1.2) holds, and suppose $\psi(Q) \leq Q$. Then, for all $Q \geq 1$, we have

$$\left|\left|F_i(\psi, Q, y)\right| - \psi(Q)\right| \leq C_F \left(\frac{\psi(Q)Q^{r-1}}{\varphi(Q)}\right)^{r/(\kappa+r)}$$

where $C_F = (r + 1)^{9(r+1)}/2$.

Next we consider the subset of primitive points $(p, q)$, i.e., points with $\gcd(p, q) = \gcd(p_1, \ldots, p_r, q) = 1$. We define

$$F^*_\xi(\psi, Q, y) = \{(p, q) \in \mathbb{Z}^{r+1}; 0 < p_i + q \xi_i - y_i < \left(\frac{\psi(Q)}{Q}\right)^{1/r} \quad (1 \leq i \leq r)$$

$$0 < q < Q, \gcd(p, q) = 1\}.$$  

Obviously, $|F^*_\xi(\psi, Q, 0)|$ is the number of actual approximations $(-p_1/q, \ldots, -p_r/q)$ of $\xi$. We set

$$C_{F^*} = \frac{(r + 1)^{12(r+1)^2}}{\varphi(1)},$$

$$L_{F^*} = \begin{cases} 
\log(\max\{Q + |y|, \exp(1)\}) & \text{if } r = 1, \\
1 & \text{if } r > 1.
\end{cases}$$

Now we can state our next result.
Theorem 1.3. Suppose $\xi \in \mathbb{R}^r$ is such that (1.2) holds, and suppose $a > 2$ and $\psi(Q) \leq Q$. Then there exists a constant $Q_0 = Q_0(a,y,r)$ such that for all $Q \geq Q_0$

\[
\left| F_\xi^r(\psi;Q,y) - \frac{\psi(Q)}{\zeta(r+1)} \right| \leq C_{\psi} \left( \frac{\psi(Q)Q^{r-1}}{\varphi(Q)} \right)^{1/(r+\kappa)} + a^{2\log(Q)/\log(Q)} \left( \frac{\psi(Q)Q^{r-1}}{\varphi(Q)} \right)^{1/(r+\kappa)}.
\]

The localised situation

\[(1.6) \quad 0 < p + q\xi - y < \iota(q) \quad \text{and} \quad q > 0\]

for primitive points has been investigated already in 1959 by Chalk and Erdős [12]. Their result [12, Theorem] asserts the existence of an absolute constant $c$ such that with $\iota(x) = c\log(q)^2/(q\log(q))^2$ the number of coprime pairs $(p,q)$ satisfying (1.6) is infinite (a slightly weaker result has also been obtained by Haynes [16]). For badly approximable $\xi$ Theorem 1.3 yields

\[(1.7) \quad |F_\xi^r(\psi;Q,y)| \sim \frac{\psi(Q)}{\zeta(2)},\]

provided $a^{2\log(Q)/\log(Q)} = o(\psi(Q))$, and hence, with, e.g., $\iota(q) = q^{\log(q)/\log(q)}/q$ there are infinitely many coprime $(p,q)$ with (1.6). By the same argument as in the discussion after Theorem 1.1 we see that if, e.g., $\psi(Q) \geq 17^{\log(Q)/\log(Q)}$ then (1.7) holds for almost every $\xi$.

Erdős and Haynes results both rely on the continuous fraction expansion of $\xi$, and therefore, do not seem to generalize to simultaneous approximations. Indeed, for simultaneous approximations there seem to be no results available in the style of Chalk and Erdős'. Theorem 1.3 yields such a result if $\psi$ grows quickly enough and subject to the rather delicate condition (1.2).

Concerning metrical statements for simultaneous approximations with primitive points Laurent and Nogueira [20], and Dani, Laurent and Nogueira [13] have recently established various results. Theorem 1.2 and Theorem 1.3 are only very special instances of the two theorems presented in the next section.

## 2. A GENERALIZATION TO LINEAR FORMS

This section is devoted to the presentation of our main results. First we require to introduce some notation. Let $n, m$ be integers and suppose $n \geq m \geq 2$. Suppose the vectors

\[(2.1) \quad \alpha_1 = \begin{pmatrix} \alpha_{11} \\ \alpha_{12} \\ \vdots \\ \alpha_{1n} \end{pmatrix}, \ldots, \alpha_m = \begin{pmatrix} \alpha_{m1} \\ \alpha_{m2} \\ \vdots \\ \alpha_{mn} \end{pmatrix}\]

have their entries $\alpha_{i1}, \ldots, \alpha_{in}$ in $\mathbb{R}$. We consider the $n$ linear forms

\[(2.2) \quad L_i(\mathbf{z}) = \alpha_{i1}z_1 + \cdots + \alpha_{im}z_m \quad (1 \leq i \leq n).\]

For vectors $Q = (Q_{11}, Q_{12}, \ldots, Q_{m1}, Q_{m2}) \in \mathbb{R}^{2m}$ with $Q_{ii} < Q_{i2}$ ($1 \leq i \leq m$), and $y = (y_1, \ldots, y_m) \in \mathbb{R}^m$ we write $Z_{\alpha}(Q,y)$ for the set of vectors $\mathbf{z} \in \mathbb{R}^m \setminus \{0\}$ satisfying

\[(2.3) \quad Q_{ii} < L_i(\mathbf{z}) - y_i < Q_{i2} \quad (1 \leq i \leq m).\]

We set

\[(2.4) \quad Q_i = Q_{i2} - Q_{i1} \quad (1 \leq i \leq m).\]
We want to investigate \( N_\alpha(Q, y) = |Z_\alpha(Q, y) \cap \mathbb{Z}^m| \): the number of nonzero \( z \in \mathbb{Z}^m \) satisfying the condition (2.3). To ensure finiteness we assume the vectors \( \alpha_1, \ldots, \alpha_m \) are such that

\[
\text{rank}[\alpha_{ij}]_{1 \leq i, j \leq m} = m.
\]

Hence, the subspace

\[
S_\alpha = \alpha_1 \mathbb{R} + \cdots + \alpha_m \mathbb{R}
\]

has dimension \( m \), and

\[
\Lambda_\alpha = \alpha_1 \mathbb{Z} + \cdots + \alpha_m \mathbb{Z}
\]

is a lattice in \( \mathbb{R}^n \) of rank \( m \). Let \((\cdot, \cdot)\) be the standard scalar product on \( \mathbb{R}^n \). By the Cauchy-Binet formula we have

\[
\det \Lambda_\alpha = \sqrt{[\det[[\alpha_i, \alpha_j]]].}
\]

Let \( \beta = (\beta_1, \ldots, \beta_n) \in (0, \infty)^n \) and let \( \varphi : [0, \infty) \to (0, 1] \) be a monotone (not necessarily strict) decreasing function. We say that the vectors \( \alpha_1, \ldots, \alpha_m \) are \((\beta, \varphi)\)-badly approximable if

\[
\prod_{i=1}^n |L_i(z)|^{\beta_i} \geq \varphi(\max_{m \leq j \leq n} |L_j(z)|)
\]

for all \( z \in \mathbb{Z}^m \) with \( \max_{m \leq j \leq n} |L_j(z)| > 0 \). Note that a number \( \xi \) satisfies (1.2) (with \( r = 1 \)) precisely when the two vectors \( \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \left( \begin{array}{c} \xi \\ 1 \end{array} \right) \) are \((\beta, \varphi)\)-badly approximable with \( \beta = (1, \kappa) \). Another simple example comes from taking the Minkowski-embedding of a basis of a totally real number field.

Another condition we need to impose is the existence of a constant \( C = C(\alpha_1, \ldots, \alpha_m) > 0 \) such that

\[
\max_{1 \leq j \leq m-1} |L_j(z)| > C
\]

for all non-zero \( z \in \mathbb{Z}^m \) with \( \max_{m \leq j \leq n} |L_j(z)| = 0 \). In the first of the above examples one can take \( C = 1 \), and for the second example the condition is void \((L_j(z) = 0 \text{ implies } z = 0)\) and, hence one can take any value \( C > 0 \).

Let \( e_1, \ldots, e_n \) be the canonical standard basis of \( \mathbb{R}^n \). As a consequence of (2.5) we can find a basis \( u_1, \ldots, u_m \) of the vector space \( S_\alpha \) with

\[
(e_i, u_j) = \delta_{ij} \quad (1 \leq i, j \leq m).
\]

Let \( |\cdot| \) denote the Euclidean norm, and let \( M_u \) be such that

\[
M_u \geq \max_{1 \leq i \leq m} \{|u_i|\}.
\]

We set

\[
t = \beta_1 + \cdots + \beta_n,
\]

\[
s = \beta_m + \cdots + \beta_n,
\]

\[
R_0 = (Q_1^{\beta_1} \cdots Q_{m-1}^{\beta_{m-1}} Q_m^{\beta_m})^{1/t}.
\]

Theorem 2.1. Suppose that \( \alpha_1, \ldots, \alpha_m \) are \((\beta, \varphi)\)-badly approximable satisfying (2.5) and (2.10). Moreover, assume \( Q_1 \leq \cdots \leq Q_m \) and \( R_0/Q_m \geq 1 \). Then

\[
\left| N_\alpha(Q, y) - Q_1 \cdots Q_m \sqrt{[\det[[u_i, u_j]]]/[\det[[\alpha_i, \alpha_j]]]} \right| \leq C_1 \left( R_0^{m-1}/\min\{\varphi(Q_m)^{1/t}, C, R_0\} \right)^{m-1},
\]

where \( C_1 = (nm^{8m} M_u)^m \).
Our main application of Theorem 2.1 is to deduce Theorem 1.2. Hence, we content ourselves with a simple straightforward application here. Let $K$ be the totally real field generated by a root of the polynomial $x^3 + x^2 - 3x - 1$ and for $\alpha \in K$ let $\alpha'$ and $\alpha''$ denote its conjugates. As $Q \to \infty$ there are asymptotically
\[
\frac{1}{\sqrt{148}} \log(Q)
\]
algebraic integers $\alpha$ of $K$ that lie in the interval $(2014, 2015)$ and whose conjugates $\alpha'$ and $\alpha''$ lie in $(2015, 2015 + \exp(-Q))$ and $(2016, 2016 + \log(Q) \exp(Q))$ respectively.

Let $Z^{m*} = \{z \in Z^m \setminus \{0\}; \gcd(z_1, \ldots, z_m) = 1\}$ be the subset of primitive lattice points, and set $N_{\alpha}(Q, y) = |Z_{\alpha}(Q, y) \cap Z^{m*}|$ for the number of primitive $z \in Z^m$ satisfying the condition (2.3). We set
\[
L_B = \max_{1 \leq i \leq m} \{|Q\alpha_1 + y_i|\},
\]
\[
U_B = \max_{1 \leq i \leq m} \{|Q\alpha_2 + y_i|\},
\]
\[
R_B = \max \{L_B, U_B\},
\]
\[
R_1 = nm^{2m+2}M_0R_0 \left( \frac{1 + L_B/Q_1}{\varphi(Q_m)^{1/t}} \right),
\]
\[
\mathcal{L} = \begin{cases} 
\log(\max\{R_B, \exp(1)\}) & \text{if } m = 2, \\
1 & \text{if } m > 2.
\end{cases}
\]

Let $T : [0, \infty) \to [1, \infty)$ be a monotone increasing function that is an upper bound for the divisor function, i.e., $T(k) \geq \tau(k) := \sum_{d \mid k} 1$ for all $k \in \mathbb{N}$.

**Theorem 2.2.** Suppose that $\alpha_1, \ldots, \alpha_m$ are $(\beta, \varphi)$-badly approximable satisfying (2.5), (2.10), and $\varphi(0)^{1/t} \leq \min\{C, R_0\}$. Moreover, assume $Q_1 \leq \cdots \leq Q_m$, and $R_0/Q_{m-1} \geq 1$. Then
\[
\left| N_{\alpha}(Q, y) - \frac{Q_1 \cdots Q_m}{\zeta(m)} \sqrt{\frac{|\det([u_i, u_j])|}{\det([\alpha_i, \alpha_j])}} \right| \leq C_2 \left( \frac{R_0}{\varphi(Q_m)^{1/t}} \right)^{m^{-1}} \mathcal{L} + \frac{T(R_1)R_0}{\varphi(Q_m)^{1/t}},
\]
where $C_2 = 2nm^2 \left( \frac{m^2M_0}{\varphi(1)^{1/t}} \right)^m C_1$.

### 3. Counting lattice points

Recall that $|\cdot|$ denotes the Euclidean norm. The closed Euclidean ball centered at $x$ with radius $R$ will be denoted by $B_x(R)$. Let $D \geq 2$ be an integer. Let $\Lambda$ be a lattice of rank $D$ in $\mathbb{R}^D$. We define the successive minima $\lambda_1(\Lambda), \ldots, \lambda_D(\Lambda)$ of $\Lambda$ as the successive minima in the sense of Minkowski with respect to the unit ball. That is
\[
\lambda_i = \inf \{\lambda; B_0(\lambda) \cap \Lambda \text{ contains } i \text{ linearly independent vectors} \}.
\]

**Definition 1.** Let $M$ be a positive integers and let $L$ be a non-negative real. We say that a set $S$ is in $\text{Lip}(D, M, L)$ if $S$ is a subset of $\mathbb{R}^D$, and if there are $M$ maps $\phi_1, \ldots, \phi_M : [0, 1]^{D-1} \to \mathbb{R}^D$ satisfying a Lipschitz condition
\[
|\phi_i(x) - \phi_i(y)| \leq L|x - y| \text{ for } x, y \in [0, 1]^{D-1}, i = 1, \ldots, M
\]
such that $S$ is covered by the images of the maps $\phi_i$.

For any set $A$ we write
\[
1^*(A) = \begin{cases} 
1 & \text{if } A \neq \emptyset, \\
0 & \text{if } A = \emptyset.
\end{cases}
\]

We will apply the following basic counting principle.
Lemma 3.1. Let $\Lambda$ be a lattice in $\mathbb{R}^D$ with successive minima $\lambda_1, \ldots, \lambda_D$. Let $S$ be a set in $\mathbb{R}^D$ such that the boundary $\partial S$ of $S$ is in $\text{Lip}(D, M, L)$, and suppose $S \subset B_P(L)$ for some point $P$. Then $S$ is measurable, and moreover,

\[
|S \cap \Lambda| - \frac{\text{Vol} S}{\det \Lambda} \leq c_p(D) M \left( \left( \frac{L}{\lambda_1} \right)^{D-1} + 1^*(S \cap \Lambda) \right),
\]

where $c_p(D) = D^{3D^2/2}$.

Proof. By [27, Theorem 5.4] the set $S$ is measurable, and moreover,

\[
|S \cap \Lambda| - \frac{\text{Vol} S}{\det \Lambda} \leq D^{3D^2/2} \max_{1 \leq j < D} \left\{ 1, \frac{L^j}{\lambda_1 \cdots \lambda_j} \right\}.
\]

First suppose $L \geq \lambda_1$. Then the lemma follows immediately from (3.1). Next we assume $L < \lambda_1$. We distinguish two subcases. First suppose $S \cap \Lambda \neq \emptyset$. Then

\[
\max_{1 \leq j < D} \left\{ 1, \frac{L^j}{\lambda_1 \cdots \lambda_j} \right\} = 1 = 1^*(S \cap \Lambda) \leq \left( \frac{L}{\lambda_1} \right)^{D-1} + 1^*(S \cap \Lambda).
\]

Now suppose $S \cap \Lambda = \emptyset$. As $L < \lambda_1$ we get, using Minkowski’s second Theorem,

\[
|S \cap \Lambda| - \frac{\text{Vol} S}{\det \Lambda} = \frac{\text{Vol} S}{\det \Lambda} \leq \left( \frac{(2L)^D}{\lambda_1 \cdots \lambda_D} \right) \leq 2^D \left( \frac{L}{\lambda_1} \right)^{D-1}.
\]

This proves the lemma. \hfill \Box

4. Volumes and Lipschitz parameterizations

Let $\phi$ be an automorphism of $\mathbb{R}^n$. From now on we shall identify the $m$-dimensional subspace $\phi S_{\alpha} \subset \mathbb{R}^n$ with $\mathbb{R}^m$ via any basis of $\phi S_{\alpha}$ that extends to an orthonormal basis of $\mathbb{R}^n$. The induced Lebesgue measure on $\phi S_{\alpha}$ will always be denoted by $\text{Vol}_m(\cdot)$, independent of the subspace $\phi S_{\alpha}$.

We set

\[
S_{\alpha}(Q, y) = \{ x \in S_{\alpha} \mid Q_{11} < x_i - y_i < Q_{12} \quad (1 \leq i \leq m) \} \backslash \{ 0 \}
\]

\[
= S_{\alpha} \cap (Q_{11} + y_1, Q_{12} + y_1) \times \cdots \times (Q_{m1} + y_m, Q_{m2} + y_m) \times \mathbb{R}^{n-m} \backslash \{ 0 \}.
\]

We can express $S_{\alpha}(Q, y)$ as a vector sum

\[
S_{\alpha}(Q, y) = (u_1(Q_{11} + y_1, Q_{12} + y_1) + \cdots + u_m(Q_{m1} + y_m, Q_{m2} + y_m)) \backslash \{ 0 \}
\]

\[
= (u_1(0, Q_{12} - Q_{11}) + \cdots + u_m(0, Q_{m2} - Q_{m1}) + v) \backslash \{ 0 \},
\]

where

\[
v = (Q_{11} + y_1)u_1 + \cdots + (Q_{m1} + y_m)u_m.
\]

The map $z \mapsto z_1\alpha_1 + \cdots + z_m\alpha_m$ gives a bijection between $Z_{\alpha}(Q, y)$ and $S_{\alpha}(Q, y)$. The above bijection induces also a bijection between $Z_{\alpha}(Q, y) \cap \mathbb{Z}^m$ and $S_{\alpha}(Q, y) \cap \Lambda_{\alpha}$.

Lemma 4.1. We have

\[
\text{Vol}_m(S_{\alpha}(Q, y)) = Q_{11} \cdots Q_{m} \sqrt{|\det([u_i, u_j])|}.
\]

Proof. Clearly, we have $\text{Vol}_m(S_{\alpha}(Q, y)) = \text{Vol}_m(u_1(0, Q_{12} - Q_{11}) + \cdots + u_m(0, Q_{m2} - Q_{m1})$. Hence,

\[
\text{Vol}_m(S_{\alpha}(Q, y)) = \sqrt{|\det([Q_{i1}u_i, Q_{j1}u_j])|}.
\]

Using the linearity of the determinant in the rows and in the columns proves the lemma. \hfill \Box
Lemma 4.2. Let $\phi$ be an automorphism of $\mathbb{R}^n$. Then we have
\[
\frac{\text{Vol}_m(\phi S_{\alpha}(Q,y))}{\det \phi \Lambda_{\alpha}} = \frac{\text{Vol}_m(S_{\alpha}(Q,y))}{\Lambda_{\alpha}}.
\]

Proof. This is well-known. \qed

Lemma 4.3. Let $\phi(z) = (\theta_1 z_1, \ldots, \theta_n z_n)$ be a diagonal automorphism of $\mathbb{R}^n$, and suppose $\theta_1 \geq \cdots \geq \theta_n > 0$. Then we have $\phi S_{\alpha}(Q,y) \subset B_{\phi}(R)$ with
\[
R = n \sum_{i=1}^{m} |\theta_i Q_i|.
\]

Proof. By (4.1) we have $\phi S_{\alpha}(Q,y) = (0,Q_{12} - Q_{11}) \phi u_1 + \cdots + (0,Q_{m2} - Q_{m1}) \phi u_m + \phi v$. Moreover, $|\phi u_i|^2 \leq (\theta_1^2 + \cdots + \theta_n^2) |u_i|^2 \leq (|\theta_1| + |\theta_m| + \cdots + |\theta_n|)^2 |u_i|^2$, and the lemma follows immediately. \qed

Lemma 4.4. Let $\phi$ and $R$ be as in Lemma 4.3. Then the boundary $\partial \phi S_{\alpha}(Q,y)$ lies in $\text{Lip}(m,2)$ with $L = 8 m^{5/2} R$.

Proof. The set $\{0\}$ is clearly in $\text{Lip}(m,1,L)$. Adding the origin to the set $\phi S_{\alpha}(Q,y)$ if necessary yields a convex set contained in a ball of radius $R$. Hence, we can apply [28, Theorem 2.6] to conclude that the remaining part of the boundary lies also in $\text{Lip}(m,1,L)$. Hence, the entire boundary lies in $\text{Lip}(m,2,L)$.

5. Flows and their orbits

We remind the reader that $t = \beta_1 + \cdots + \beta_n$.

Lemma 5.1. Let $\phi(x) = (\theta_1 x_1, \ldots, \theta_n x_n)$ be a diagonal automorphism of $\mathbb{R}^n$, and let $\beta_1, \ldots, \beta_n > 0$. Suppose that $\theta_1^{\beta_1} \cdots \theta_n^{\beta_n} = 1$. Then,
\[
|\phi x| \geq \left( \prod_{i=1}^{n} |x_i|^{\beta_i} \right)^{\frac{1}{t}}.
\]

Proof. We have
\[
|\phi x|^2 = \sum_{i=1}^{n} |\theta_i x_i|^2 \geq \frac{1}{\max_i \beta_i} \sum_{i=1}^{n} \beta_i |x_i|^2.
\]

Applying the weighted arithmetic-geometric mean inequality we see that the latter is
\[
\geq \frac{t}{\max_i \beta_i} \left( \prod_{i=1}^{n} |x_i|^{2 \beta_i} \right)^{\frac{1}{2}} \geq \left( \prod_{i=1}^{n} \theta_i^{\beta_i} \right)^{\frac{1}{t}} \left( \prod_{i=1}^{n} |x_i|^{\beta_i} \right)^{\frac{1}{t}} = \left( \prod_{i=1}^{n} |x_i|^{\beta_i} \right)^{\frac{1}{t}},
\]
which proves the lemma. \qed

Lemma 5.2. Let $\phi(x) = (\theta_1 x_1, \ldots, \theta_n x_n)$ be a diagonal automorphism of $\mathbb{R}^n$ and let $\beta_1, \ldots, \beta_n > 0$. Suppose that $\theta_1 \geq \cdots \geq \theta_{m-1} \geq 1, \theta_m = \cdots = \theta_n$, and $\theta_1^{\beta_1} \cdots \theta_n^{\beta_n} = 1$. Moreover, let $A > 0$. Then,
\[
\lambda_1(\phi L_{\alpha}) \geq \min \{ (\varphi(A))^{1/t}, C, \theta_m A \}.
\]

Proof. Let $x = (L_1(z), \ldots, L_n(z)) \in L_{\alpha}$ be non-zero. Hence, $z$ is non-zero. First suppose that $\max_{m \leq i \leq n} |L_i(z)| = 0$. Using (2.10) and the assumption $\theta_1 \geq \cdots \geq \theta_{m-1} \geq 1$ we conclude
\[
|\phi(x)| \geq \max_{1 \leq i \leq m-1} |\theta_i L_i(z)| \geq \max_{1 \leq i \leq m-1} |L_i(z)| \geq C.
\]
Next suppose \( \max_{m \leq i \leq n} |L_i(z)| > 0 \). Then by Lemma 5.1 and (2.9)

\[
|\phi(x)| \geq \left( \prod_{i=1}^{n} |L_i(z)|^{|x^i|} \right)^{\frac{1}{t}} \geq \left( \varphi(\max_{m \leq i \leq n} |L_i(z)|) \right)^{\frac{1}{t}}.
\]

We now distinguish two subcases. Let us first assume that \( \max_{m \leq i \leq n} |L_i(z)| \leq A \). As \( \varphi \) is decreasing we get

\[
|\phi(x)| \geq (\varphi(A))^{\frac{1}{t}}.
\]

For the second subcase let us assume \( \max_{m \leq i \leq n} |L_i(z)| > A \). Then we have

\[
|\phi(x)| \geq \max_{m \leq i \leq n} |\theta_1 L_i(z)| = \theta_m \max_{m \leq i \leq n} |L_i(z)| > \theta_m A.
\]

\[\square\]

6. Proof of Theorem 2.1

We are now going to specify our diagonal automorphism \( \phi \) of \( \mathbb{R}^n \), that is we choose the numbers \( \theta_1, \ldots, \theta_n > 0 \). We set

\[
(6.1) \quad \theta_i = \left( Q_1^{\beta_1} \cdots Q_{m-1}^{\beta_{m-1}} Q_m^t \right)^{1/t} = R_0/Q_i, \quad (1 \leq i \leq m)
\]

\[
(6.2) \quad \theta_m = \theta_{m+1} = \cdots = \theta_n
\]

Now we have

\[
(6.3) \quad \theta_1^{\beta_1} \cdots \theta_n^{\beta_n} = 1,
\]

\[
(6.4) \quad \theta_1 Q_1 = \cdots = \theta_m Q_m = R_0.
\]

As \( Q_1 \leq \cdots \leq Q_m \) we get \( \theta_1 \geq \cdots \geq \theta_n > 0 \). Thus, using also the hypothesis of the theorem,

\[
(6.5) \quad \theta_1 \geq \cdots \geq \theta_{m-1} = R_0/Q_{m-1} \geq 1,
\]

\[
(6.6) \quad \|\phi\| = \theta_1 = R_0/Q_1.
\]

Proposition 6.1. We have

\[
\left| N_{\alpha}(Q, y) - \frac{Q_1 \cdots Q_m \sqrt{\det[(u_i, u_j)]}}{\det \Lambda_{\alpha}} \right| \leq c_0 \left( \left( \frac{R_0}{\lambda_1(\phi \Lambda_{\alpha})} \right)^{m-1} + \frac{1}{4^*} \left( \phi_{S_{\alpha}}(Q, y) \cap \phi_{\Lambda_{\alpha}} \right) \right),
\]

where \( c_0 = 2^{3m-3n-1} m^{7/2} 7^2 \frac{1}{4^*} c_{\alpha}(m) \), and \( c_{\alpha}(\cdot) \) is as in Lemma 3.1.

Proof. First note that

\[
N_{\alpha}(Q, y) = |Z_{\alpha}(Q, y) \cap \mathbb{Z}^m| = |S_{\alpha}(Q, y) \cap \Lambda_{\alpha}| = |\phi S_{\alpha}(Q, y) \cap \phi \Lambda_{\alpha}|.
\]

From (4.2) and (6.4) we get

\[
R = nmM_u R_0,
\]

and by Lemma 4.3 we conclude that

\[
(6.8) \quad \phi S_{\alpha}(Q, y) \subset B_{\phi v}(R).
\]

Moreover, by Lemma 4.4 we see that \( \partial \phi S_{\alpha}(Q, y) \) lies in \( \text{Lip}(\alpha, 2, L) \) with \( L = 8m^{5/2} R \). Hence, we can apply Lemma 3.1 to estimate \( |\phi S_{\alpha}(Q, y) \cap \phi \Lambda_{\alpha}| \). We get

\[
\left| N_{\alpha}(Q, y) - \frac{\text{Vol}_m(\phi S_{\alpha}(Q, y))}{\det \phi \Lambda_{\alpha}} \right| \leq c_0 \left( \left( \frac{R_0}{\lambda_1(\phi \Lambda_{\alpha})} \right)^{m-1} + \frac{1}{4^*} \left( \phi S_{\alpha}(Q, y) \cap \phi \Lambda_{\alpha} \right) \right),
\]

Finally, we apply Lemma 4.2 and Lemma 4.1 to substitute \( \text{Vol}_m(S_{\alpha}(Q, y)) / \det \phi \Lambda_{\alpha} \). This proves the claim. \[\square\]
We can now easily finish the proof of Theorem 2.1. We apply Lemma 5.2 with $A = Q_m$ to replace $\lambda_1(\phi \Lambda \alpha)$ in Proposition 6.1 by $\min\{\varphi(Q_m)1/t, C, \theta_m Q_m\}$, and we recall that by (6.4) we have $\theta_m Q_m = R_0$. After replacing $c_0$ by $2c_0 \leq (nm^{8m}M_n)^m = C_1$ we can omit $1^* (\phi S \alpha(Q, y) \cap \phi \Lambda \alpha)$. Finally, we use the Cauchy-Binet formula (2.8) to replace $\operatorname{det} \Lambda \alpha$. This completes the proof of Theorem 2.1.

7. Preparations for the Möbius inversion

Recall that $T : [0, \infty) \to [1, \infty)$ is a monotone increasing function that is an upper bound for the divisor function, i.e., $T(k) \geq \tau(k) := \sum_{d \mid k} 1$ for all $k \in \mathbb{N}$. In this section $D$ is a positive integer. For an endomorphism $\Psi$ of $\mathbb{R}^n$ we write $\|\Psi\|$ for the (Euclidean) operator norm.

**Lemma 7.1.** Let $\Lambda$ be a lattice in $\mathbb{R}^D$ and let $\Psi$ be an automorphism of $\mathbb{R}^D$ with $\Psi \mathbb{Z}^D = \Lambda$. Then

$$|\{k \in \mathbb{N} : kv \in B_p(R) \text{ for some non-zero } v \in \Lambda\}| \leq T((R + |P|)\|\Psi^{-1}\|)(2R\|\Psi^{-1}\| + 1).$$

**Proof.** First assume $\Psi = \operatorname{id}$ so that $\Lambda = \mathbb{Z}^D$. Suppose $v = (a_1, \ldots, a_D) \in \mathbb{Z}^D$ is non-zero, $kv \in B_p(R)$ and $P = (x_1, \ldots, x_D)$. Then $ka_i$ lies in $[x_i - R, x_i + R]$ for $1 \leq i \leq D$. As $v \neq 0$ there exists an $i$ with $a_i \neq 0$. We conclude that $k$ is a divisor of some non-zero integer in $[x_i - R, x_i + R]$. There are at most $2R + 1$ integers in this interval, each of which of modulus at most $R + |P|$. Hence the number of possibilities for $k$ is $\leq T(|P| + R)(2R + 1)$. This proves the lemma for $\Psi = \operatorname{id}$. Next note that

$$\{k \in \mathbb{N} : kv \in B_p(R) \text{ for some non-zero } v \in \Lambda\} = \{k \in \mathbb{N} : kw \in \Psi^{-1}B_p(R) \text{ for some non-zero } w \in \mathbb{Z}^D\}.$$

Hence, the general case follows from the case $\Psi = \operatorname{id}$ upon noticing $\Psi^{-1}B_p(R) \subset B_{\Psi^{-1}(P)}(R\|\Psi^{-1}\|)$, and $\|\Psi^{-1}(P)\| \leq \|\Psi^{-1}\||P|$.

Next we estimate the operator norm $\|\Psi^{-1}\|$ for a suitable choice of $\Psi$.

**Lemma 7.2.** Let $\Lambda$ be a lattice in $\mathbb{R}^D$. There exists an automorphism $\Psi$ of $\mathbb{R}^D$ with $\Psi \mathbb{Z}^D = \Lambda$ and

$$\|\Psi^{-1}\| \leq \frac{c_{\operatorname{con}}(D)}{\lambda_1},$$

where $c_{\operatorname{con}}(D) = D^{2D+1}$.

**Proof.** Any lattice $\Lambda$ in $\mathbb{R}^D$ has a basis $v_1, \ldots, v_D$ with $\frac{|v_1| \cdots |v_D|}{|\operatorname{det}[v_1, \ldots, v_D]|} \leq D^{2D}$, see, e.g., [27, Lemma 4.4]. Let $\Psi$ be the map that sends the canonical basis $e_1, \ldots, e_n$ to $v_1, \ldots, v_n$. Now suppose $\Psi^{-1}$ sends $e_i$ to $(\varrho_1, \ldots, \varrho_n)$ then by Cramer’s rule

$$|\varrho_j| = \left| \frac{\operatorname{det}[v_1 \cdots e_i \cdots v_D]}{\operatorname{det}[v_1 \cdots v_j \cdots v_D]} \right| \leq \frac{|\operatorname{det}[v_1 \cdots e_i \cdots v_D]|}{|v_1| \cdots |v_j| \cdots |v_D|} D^{2D}.$$

Now we apply Hadamard’s inequality to obtain

$$\frac{|\operatorname{det}[v_1 \cdots e_i \cdots v_D]|}{|v_1| \cdots |v_j| \cdots |v_D|} \leq \frac{|v_1| \cdots |e_j| \cdots |v_D|}{|v_1| \cdots |v_i| \cdots |v_D|} = \frac{1}{|v_i|} \leq \frac{1}{\lambda_i}.$$

Next we use that for a $D \times D$ matrix $[a_{ij}]$ with real entries we have $\|a_{ij}\| \leq \sqrt{D} \max_{ij} |a_{ij}|$, and this proves the lemma.

We combine the previous two lemmas.

**Lemma 7.3.** Let $\Lambda$ be a lattice in $\mathbb{R}^D$, and let $\lambda_1 = \lambda_1(\Lambda)$. Then

$$\sum_{k=1}^{\infty} 1^*(B_p(R) \setminus \{0\} \cap k\Lambda) \leq T\left( c_{\operatorname{con}}(D) \left( \frac{R + |P|}{\lambda_1} \right) \right) \left( \frac{2c_{\operatorname{con}}(D) R}{\lambda_1} + 1 \right).$$
Proof. Note that \( \sum_{k=1}^{\infty} 1^{*}(k\Lambda \cap B_{p}(R)\setminus \{0\}) = |\{k \in \mathbb{N} : k\Lambda \cap B_{p}(R)\setminus \{0\} \neq \emptyset\}| = |\{k \in \mathbb{N} : kv \in B_{p}(R)\text{ for some nonzero } v \in \Lambda\}|. \) Hence, the lemma follows immediately from Lemma 7.1 and Lemma 7.2. \( \square \)

8. Proof of Theorem 2.2

We use the same notation as in Section 6. Let us start with the following lemma.

Lemma 8.1. We have

\[
\left| N_{\alpha}(Q, y) - \frac{Q_{1} \cdots Q_{m}}{\sqrt{\det\{u_{i}, u_{j}\}}} \right| \leq c_{0} \left( \frac{R_{0}}{\lambda_{1}(\phi\Lambda_{\alpha})} \right)^{m-1} + 1^{*}(B_{\phi\Lambda}(R)\setminus \{0\} \cap \phi\Lambda_{\alpha}) .
\]

Proof. Note that 0 is not in \( S_{\alpha}(Q, y) \), and thus, by (6.8) we get \( \phi S_{\alpha}(Q, y) \subset B_{\phi\Lambda}(R)\setminus \{0\} \). The lemma is an now an immediate consequence of Proposition 6.1. \( \square \)

Put

\[
F(d) = \{z \in \mathbb{Z}^{m}\setminus \{0\} : Q_{11} < L_{1}(z) - y_{1} < Q_{12} \quad (1 \leq i \leq m), \gcd(z) = d\} .
\]

Hence, \( F(d) \) is the set of non-zero lattice points \( z \in \mathbb{Z}^{m} \) in \( Z_{\alpha}(Q, y) \) with \( \gcd(z) = d \). In particular, \( N_{\alpha}(Q, y) = |F(1)| \). For positive integers \( k \) we set

\[
B_{k} = \{z \in k\mathbb{Z}^{m}\setminus \{0\} : Q_{11} < L_{1}(z) - y_{1} < Q_{12} \quad (1 \leq i \leq m)\} .
\]

We have the disjoint union

\[
(8.1) \quad \bigcup_{d \mid k} F(d) = B_{k} .
\]

Just as \( \alpha \) denotes the tuple \( (\alpha_{1}, \ldots, \alpha_{m}) \) we write \( k\alpha \) for the tuple \( (k\alpha_{1}, \ldots, k\alpha_{m}) \). Next, we note that the map \( z \rightarrow kz \) gives a bijection between \( Z_{k\alpha}(Q, y) \cap \mathbb{Z}^{m} \) and \( B_{k} \). Hence, \( N_{k\alpha}(Q, y) = |B_{k}| \). Clearly, \( \Lambda_{k\alpha} = k\Lambda_{\alpha} \), and \( S_{\alpha} = S_{k\alpha} \). Hence, our basis \( u_{1}, \ldots, u_{m} \) of \( S_{\alpha} \) can also be used for \( S_{k\alpha} \), and thus, we can choose the same \( M_{u} \) as for \( S_{\alpha} \). Consequently, Lemma 8.1 yields

\[
(8.2) \quad \left| B_{k} \right| - \frac{Q_{1} \cdots Q_{m}}{k^{m} \sqrt{\det\{u_{i}, u_{j}\}}} \leq c_{0} \left( \frac{R_{0}}{k\lambda_{1}(\phi\Lambda_{\alpha})} \right)^{m-1} + 1^{*}(B_{\phi\Lambda}(R)\setminus \{0\} \cap k\phi\Lambda_{\alpha}) .
\]

Lemma 8.2. We have \( \lambda_{1}(\Lambda_{\alpha}) \geq \varphi(1)^{1/t} \).

Proof. Applying Lemma 5.2 with \( \phi \) as the identity and \( A = 1 \) we conclude that \( \lambda_{1}(\Lambda_{\alpha}) \geq \min\{\varphi(1)^{1/t}, C, 1\} \). Using that \( \varphi : [0, \infty) \rightarrow (0, 1] \) is decreasing and the assumption of the theorem yields \( \varphi(1)^{1/t} \leq \varphi(0)^{1/t} \leq \min\{C, 1\} \). \( \square \)

We set

\[
R_{2} = mM_{u}R_{B}/\varphi(1)^{1/t} .
\]

Lemma 8.3. Suppose \( d > R_{2} \) then \( F(d) = \emptyset \). Moreover, \( R_{2} \geq Q_{m} \).

Proof. Suppose \( z \in F(d) \). Then \( x = (L_{1}(z), \ldots, L_{n}(z)) \in d\Lambda_{\alpha}\setminus \{0\} \) and \( |L_{1}(z)| \leq R_{B} \) for \( 1 \leq i \leq m \). Moreover, \( x = \sum_{i=1}^{m} L_{i}(z)u_{i} \), and so \( mM_{u}R_{B} \geq |x| \geq d\lambda_{1}(\Lambda_{\alpha}) \). Applying Lemma 8.2 yields \( d \leq mM_{u}R_{B}/\varphi(1)^{1/t} = R_{2} \).

For the second assertion we note that firstly \( Q_{m} = Q_{m2} - Q_{m1} = (Q_{m2} + y_{m}) - (Q_{m1} + y_{m}) \leq |Q_{m2} + y_{m}| + |Q_{m1} + y_{m}| \leq U_{B} + L_{B} \leq 2R_{B} \), and secondly \( m \geq 2, M_{u} \geq 1, \) and \( \varphi(1)^{1/t} \leq 1 \). \( \square \)
We use the Möbius function $\mu(\cdot)$ and the Möbius inversion formula, combined with (8.1) and Lemma 8.3, to get
\[
N^*_\alpha(Q, y) = |F(1)| = \sum_{k=1}^{\infty} \mu(k) \sum_{d \mid k} |F(d)| = \sum_{k=1}^{R_2} \mu(k) \sum_{d \mid k} |F(d)| = \sum_{k=1}^{R_2} \mu(k)|B_k|.
\]
Consequently,
\[
\left|N^*_\alpha(Q, y) - \frac{Q_1 \cdots Q_m \sqrt{|\det([u_i, u_j])|}}{\zeta(m) \det \Lambda_\alpha} \right| \leq \sum_{k=1}^{R_2} \mu(k)|B_k| - \sum_{k=1}^{\infty} \frac{Q_1 \cdots Q_m \sqrt{|\det([u_i, u_j])|}}{k^m \det \Lambda_\alpha}.
\]
Applying (8.2) provides
\[
(8.4) \quad \left|N^*_\alpha(Q, y) - \frac{Q_1 \cdots Q_m \sqrt{|\det([u_i, u_j])|}}{\zeta(m) \det \Lambda_\alpha} \right| 
\leq c_0 \sum_{k=1}^{R_2} 1^*(B_{\phi_v}(R) \setminus \{0\} \cap k\phi \Lambda_\alpha)
\quad + c_0 \sum_{k=1}^{R_2} \left( \frac{R_0}{k \lambda_1(\phi \Lambda_\alpha)} \right)^{m-1}
\quad + \sum_{k > R_2} \frac{Q_1 \cdots Q_m \sqrt{|\det([u_i, u_j])|}}{k^m \det \Lambda_\alpha}.
\]
Lemma 8.4. We have
\[
\sum_{k=1}^{R_2} 1^*(B_{\phi_v}(R) \setminus \{0\} \cap k\phi \Lambda_\alpha) \leq c_1 \frac{T(R_1)R_0}{\varphi(Q_m)^{1/t}},
\]
where $c_1 = 4c_{\text{on}}(m)nmM_\alpha$.

Proof. Applying Lemma 7.3 to the lattice $\phi(\Lambda_\alpha)$ which sits in some $\mathbb{R}_m$, yields
\[
\sum_{k=1}^{R_2} 1^*(B_{\phi_v}(R) \setminus \{0\} \cap k\phi \Lambda_\alpha) \leq T \left( c_{\text{on}}(m) \left( \frac{R + |\phi(v)|}{\lambda_1(\phi \Lambda_\alpha)} \right) \left( \frac{2c_{\text{on}}(m)R}{\lambda_1(\phi \Lambda_\alpha)} + 1 \right) \right).
\]
Recalling that $v = (Q_{11} + y_1)u_1 + \cdots + (Q_{m1} + y_m)u_m$ and $R = nmM_\alpha R_0$, and applying (6.6) we conclude $R + |\phi(v)| \leq nmM_\alpha R_0(1 + L_B/Q_1)$. By Lemma 5.2 we have $\lambda_1(\phi \Lambda_\alpha) \geq \varphi(Q_m)^{1/t}$, and thus $c_{\text{on}}(m)(R + |\phi(v)|)/\lambda_1(\phi \Lambda_\alpha) \leq R_1$. As $T$ is monotone increasing we get $T(c_{\text{on}}(m)(R + |\phi(v)|)/\lambda_1(\phi \Lambda_\alpha)) \leq T(R_1)$.

Recall that $\varphi(Q_m)^{1/t} \leq \varphi(0)^{1/t} \leq R_0$. Therefore, $2c_{\text{on}}(m)R/\lambda_1(\phi(\Lambda_\alpha)) + 1 \leq 4c_{\text{on}}(m)nmM_\alpha R_0/\varphi(Q_m)^{1/t} = c_1 R_0/\varphi(Q_m)^{1/t}$. This completes the proof. \[\square\]

Lemma 8.5.
\[
\sum_{k=1}^{R_2} \left( \frac{R_0}{k \lambda_1(\phi \Lambda_\alpha)} \right)^{m-1} \leq c_2 \left( \frac{R_0}{\varphi(Q_m)^{1/t}} \right)^{m-1} L,
\]
where $c_2 = 4^{mM_\alpha/\varphi(1)^{1/t}}$.

Proof. For $m = 2$ this is obvious. For $m = 2$ we use $\sum_{k=1}^{R_2} 1/k \leq 1 + \log(\max\{R_2, \exp(1)\}) \leq 2 \log(\max\{R_2, \exp(1)\})$, and $\log(\max\{R_2, \exp(1)\}) = \log(\max\{(mM_\alpha/\varphi(1)^{1/t})R_B, \exp(1)\}) \leq 2(mM_\alpha/\varphi(1)^{1/t}) \log(\max\{R_B, \exp(1)\})$. \[\square\]
Lemma 8.6. 
\[
\sum_{k > R_2} Q_1 \cdots Q_m \frac{\sqrt{\det[(u_i, u_j)]}}{k^m \det \Lambda_\alpha} \leq c_3 R_0,
\]
where \(c_3 = 2 \left( \frac{m^{3/2} M_u}{\varphi(1)^{1/t}} \right)^m\).

Proof. From Lemma 8.3 we know that \(R_2 \geq Q_m\). Hence,
\[
\sum_{k > R_2} Q_1 \cdots Q_m \frac{\sqrt{\det[(u_i, u_j)]}}{k^m \det \Lambda_\alpha} \leq \sum_{k > Q_m} Q_1 \cdots Q_m \frac{\sqrt{\det[(u_i, u_j)]}}{k^m \det \Lambda_\alpha}.
\]
Using Cauchy-Schwarz we obtain \(|\det[(u_i, u_j)]| \leq m! M_u^2 \leq m^m M_u^2\). By Minkowski’s second Theorem we get
\[
\det \Lambda_\alpha \geq (\sqrt{\pi/2})^m \Gamma\left(m/2 + 1\right) \lambda_1(\Lambda_\alpha) \cdots \lambda_m(\Lambda_\alpha) \geq m^{-m} \lambda_1(\Lambda_\alpha)^m.
\]
Applying Lemma 8.2 shows that \(\det \Lambda_\alpha \geq m^{-m} \varphi(1)^{m/t}\).

Next we note that
\[
\sum_{k > Q_m} Q_1 \cdots Q_m \frac{1}{k^m} \leq \frac{2Q_1 \cdots Q_m}{Q_m^{m-1}} \leq 2Q_1.
\]
Finally, we observe that
\[
Q_1 = \left( Q_1^{\sum_{i=1}^n \beta_i} \right)^{1/t} \leq \left( Q_1^{\beta_1} \cdots Q_{m-1}^{\beta_{m-1}} Q_m^{\beta_m + \cdots + \beta_n} \right)^{1/t} = R_0.
\]
Putting all pieces together proves the lemma. \(\square\)

We can now finish the proof of Theorem 2.2. As \(c_0 \geq 1\) and \(C_1 \geq 2c_0\) we get
\[
c_0 c_1 + c_3 = c_0 4nm^{2m+2} M_u + 2 \left( m^{3/2} M_u \right)^m \leq c_0 4nm^2 \left( \frac{m^2 M_u}{\varphi(1)^{1/t}} \right) \leq C_2.
\]
Moreover, \(\frac{T(R_1)}{\varphi(Q_m)^{1/t}} \geq 1\), and thus
\[
c_0 c_1 \frac{T(R_1) R_0}{\varphi(Q_m)^{1/t}} + c_3 R_0 \leq C_2 \frac{T(R_1) R_0}{\varphi(Q_m)^{1/t}}.
\]
Obviously, \(C_2 \geq c_0 4 \frac{m^2 M_u}{\varphi(1)^{1/t}} = c_0 c_2\). Hence, we get that the right hand-side in the inequality (8.4) is bounded from above by
\[
C_2 \left( \frac{T(R_1) R_0}{\varphi(Q_m)^{1/t}} + \frac{R_0}{\varphi(Q_m)^{1/t}} \right)^{m-1} \mathcal{L}
\leq C_2 \left( \frac{T(R_1) R_0}{\varphi(Q_m)^{1/t}} + \left( \frac{R_0}{\varphi(Q_m)^{1/t}} \right)^{m-1} \mathcal{L} \right).
\]
It remains to apply the Cauchy-Binet formula (2.8) to replace \(\det \Lambda_\alpha\) in the left hand-side of the inequality (8.4). This completes the proof of Theorem 2.2.
9. Proof of Theorem 1.2

For \( z \in \mathbb{Z}^{n+1} \) we shall use the notation \( z = (p, q) \). Let \( e_1, \ldots, e_n \) be the canonical basis of \( \mathbb{R}^n \). We will apply Theorem 2.1 with \( m = n = r + 1 \) and

\[
(9.1) \quad \alpha_1 = e_1, \ldots, \alpha_r = e_r, \quad \alpha_{r+1} = \xi_1 e_1 + \cdots + \xi_r e_r + e_{r+1} = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_r \\ 1 \end{pmatrix}
\]

in \( \mathbb{R}^n \). Clearly, we have

\[
(9.2) \quad \text{rank}[\alpha_{ij}]_{1 \leq i, j \leq r+1} = r + 1.
\]

Furthermore, we get our \( \alpha_{r+1} \) as corresponding linear forms in \( r + 1 \) variables

\[
L_i(z) = z_i + \xi_i z_{r+1} \quad (1 \leq i \leq r),
\]

\[
L_{r+1}(z) = z_{r+1}.
\]

Let \( \varphi(\cdot) \) be the function from the hypothesis (1.2) of Theorem 2.1. Let \( \beta_1 = \cdots = \beta_r = 1 \) and \( \beta_{r+1} = \kappa \). Hence,

\[
\begin{align*}
t &= r + \kappa, \\
s &= \kappa.
\end{align*}
\]

Lemma 9.1. The vectors \( \alpha_1, \ldots, \alpha_{r+1} \) are \((\beta, \varphi)\)-badly approximable.

Proof. Let \( z = (p, q) \in \mathbb{Z}^r \times \mathbb{Z} \). Then,

\[
|L_1(z)|^{\beta_1} \cdots |L_n(z)|^{\beta_n} = |L_1(z) \cdots L_r(z) L_{r+1}(z)|^\kappa = |q|^\kappa \prod_{i=1}^r |p_i + q \xi_i|.
\]

Now \( \max_{r+1 \leq j \leq n} |L_j(z)| = |q| \), and thus if \( \max_{r+1 \leq j \leq n} |L_j(z)| > 0 \) then by the hypothesis (1.2)

\[
|L_1(z)|^{\beta_1} \cdots |L_n(z)|^{\beta_n} = |q|^\kappa \prod_{i=1}^r |p_i + q \xi_i| \geq \varphi(|q|) = \varphi(\max_{r+1 \leq j \leq n} |L_j(z)|).
\]

This proves the claim. \( \square \)

For non-zero \( p \in \mathbb{Z}^r \) we have \( \max_{1 \leq i \leq r} |L_i((p, 0))| = \max_{1 \leq i \leq r} |p_i| \geq 1 \). Hence, we can choose

\[
(9.3) \quad C = 1.
\]

As \( \varphi : [0, \infty) \to (0, 1] \) is decreasing we have

\[
(9.4) \quad \varphi(Q_m)^{1/t} \leq \varphi(0)^{1/t} \leq C.
\]

As \( S_\alpha = \mathbb{R}^n \) we can choose \( u_1, \ldots, u_{r+1} \) to be the canonical basis of \( \mathbb{R}^n \)

\[
(9.5) \quad u_1 = e_1, \ldots, u_{r+1} = e_{r+1}.
\]

We have \( \max_i |u_i| = 1 \), and so we can choose

\[
(9.6) \quad M_n = 1.
\]

Lemma 9.2. We have

\[
(9.7) \quad \text{det}[(u_i, u_j)] = \text{det}[(\alpha_i, \alpha_j)] = 1.
\]

Proof. The \((r+1)\times(r+1)\)-matrix \([u_i, u_j]\) is the identity matrix. To calculate the determinant of \([(\alpha_i, \alpha_j)] \) we subtract \( \xi_i \) times the \( i \)-th row for \( i = 1, \ldots, r \) from the last row. This shows that \( \text{det}[(\alpha_i, \alpha_j)] = 1 \), which proves the claim. \( \square \)
For $1 \leq i \leq r$ we choose
\[ Q_{i1} = 0, \]
\[ Q_{i2} = (\psi(Q)/Q)^{1/r}, \]
and for $i = r + 1$ we choose
\[ Q_{r+11} = 0, \]
\[ Q_{r+12} = Q. \]
Recalling that $1 \leq \psi(Q) \leq Q$ we get
\begin{equation}
Q_1 \leq \cdots \leq Q_m. \tag{9.8}
\end{equation}
\begin{equation}
R_0/Q_{m-1} = (Q_1^{\beta_1} \cdots Q_{m-2}^{\beta_{m-2}} Q_{m-1}^{\beta_{m-1} - 1})^{1/t} = (Q/(Q/\psi(Q)))^{1/r} \geq 1, \tag{9.9}
\end{equation}
\[ R_0 = (Q_1^{\beta_1} \cdots Q_{m-2}^{\beta_{m-2}} Q_{m-1}^{\beta_{m-1}})^{1/t} = \psi(Q)^{1/(r+\kappa)} Q^{(\kappa-1)/(r+\kappa)} \geq 1. \]
In conjunction with (9.4) we get
\begin{equation}
\varphi(Q_m)^{1/t} \leq \varphi(0)^{1/t} \leq \min\{C, R_0\}. \tag{9.10}
\end{equation}
Moreover, $Q_1 \cdots Q_{r+1} = \psi(Q)$, and by virtue of Lemma 9.2 we find
\begin{equation}
\sqrt{\left| \det([u_i, u_j]) \right|} = 1. \tag{9.11}
\end{equation}
Finally, we take
\[ y' = (y_1, 0) = (y_1, \ldots, y_r, 0). \]
Thus,
\begin{equation}
|F_{\xi}(\psi, Q, y)| = \left| \{ (p, q) \in \mathbb{Z}^{r+1}; 0 < p_i + q\xi_i - y_i < (\psi(Q)/Q)^{1/r} (1 \leq i \leq r), \ 0 < q < Q \} \right| = N\alpha(Q, y'). \tag{9.12}
\end{equation}
We are now in position to apply Theorem 2.1. We get
\begin{equation}
|N\alpha(Q, y') - \psi(Q)| \leq C_1 \left( \frac{(Q/Q)^{(\kappa-1)}}{\varphi(Q)} \right)^{r/(r+\kappa)}. \tag{9.13}
\end{equation}
Here
\begin{equation}
C_1 = (nm^{8m} M_q)^m \leq (r + 1)^{9(r+1)^2} = C_F. \tag{9.14}
\end{equation}
This completes the proof of Theorem 1.2.

10. SIMPLE BOUNDS FOR THE DIVISOR FUNCTION

Let $\tau(n) = \sum_{d|n} 1$ be the divisor function, and for $a > 2$ and $b \geq 3$ set
\[ g_{a,b}(x) = \begin{cases} a \log x \log \log x & \text{if } x \geq b, \\ a \log \log x & \text{if } 0 \leq x < b. \end{cases} \]
Let $b_0 = \exp(\exp(1))$.

**Lemma 10.1.** For any $a > 2$ there exists a minimal $b \geq b_0$ such that we can take $T(x) = g_a(x) = g_{a,b}(x)$, i.e., $g_a(x)$ is monotone increasing and we have $\tau(n) \leq g_a(n)$ for all positive integers $n$. Moreover, for any $t > 0$ and $x \geq \exp(a \log t/\log 2)$
\[ g_a(tx) \leq 2g_a(x). \]
Proof. The function $a^t$ is monotone increasing for $x \geq b_0$. Hence, $g_{a,b}(x)$ is a monotone increasing function of $x$ for all $b \geq b_0$. Then for any $t > 0$, any $b \geq b_0$ and for any $x \geq \exp(a^{t/\log 2})$ we have

$$g_{a,b}(tx) \leq 2g_{a,b}(x).$$

The above claim is clear for $t \leq 1$ as $g_{a,b}(x)$ is increasing. If $t > 1$ and $tx \leq b$ we have $g_{a,b}(tx) = g_{a,b}(x)$. For $t > 1$ and $x \geq b$ we note that $\log tx/\log tx \leq \log t/\log x + \log x/\log x$, which proves the claim in this case. Finally, if $t > 1$ and $x < b$ we use the monotonicity to get $g_{a,b}(tx) \leq g_{a,b}(tb) \leq 2g_{a,b}(b) = 2g_{a,b}(x)$.

For any $\varepsilon > 0$ there exists $C_\varepsilon$ such that $\tau(k) \leq C_\varepsilon e^{(\log 2+\varepsilon) \log k} \log 2$ for all $k \geq 3$. Taking $\varepsilon = (\log a - \log 2)/2$ we conclude for $k$ with $\varepsilon \log k/\log k \geq \log C_\varepsilon$ that $g_{a,b_0}(k) \geq \tau(k)$. In particular, there are only finitely many $k \in \mathbb{N}$ with $\tau(k) > g_{a,b_0}(k)$. If $\tau(k) \leq g_{a,b_0}(k)$ for all $k \in \mathbb{N}$ then we take $b = b_0$, and otherwise, we take $b > b_0$ such that $a^{\log k/\log \log k} = \max\{\tau(k); \tau(k) > g_{a,b_0}(k), k \in \mathbb{N}\}$. 

11. PROOF OF THEOREM 1.3

Let $\alpha, Q, y, y'$ be as in Section 9. From (9.12) we conclude that

$$|F_\alpha^t(\psi, Q, y)| = N^\alpha(Q, y').$$

Because of (9.2), Lemma 9.1, (9.3), (9.8), (9.9), and (9.10) the assumptions of Theorem 2.2 are satisfied. Note that

$$L_B \leq |y| = |y|,
U_B \leq Q + |y|,
R_B \leq Q + |y|,
R_0 = \left(\psi(Q)Q^{(\kappa-1)} \varphi(Q)\right)^{1/(r+\kappa)}.$$

Recalling the definition of $\mathcal{L}$ and $R_1$, just before the statement of Theorem 2.2, a simple calculation shows that

$$\mathcal{L} \leq \mathcal{L}_{F^*} = \begin{cases} 
\log(\max\{Q + |y|, \exp(1)\}) & \text{if } r = 1, \\
1 & \text{if } r > 1,
\end{cases}$$

and

$$R_1 \leq (r + 1)^{2r+5}(1 + |y|) \left(\frac{Q^{(r+1)/r}}{\varphi(Q)}\right)^{1/(r+\kappa)} \leq (r + 1)^{2r+5}(1 + |y|) \left(\frac{Q}{\varphi(Q)}\right)^2.$$

Let $a > 2$. We apply Lemma 10.1 to get

$$T(R_1) \leq g_a \left((r + 1)^{2r+5}(1 + |y|) \left(\frac{Q}{\varphi(Q)}\right)^2\right) \leq 2g_a \left(\left(\frac{Q}{\varphi(Q)}\right)^2\right),$$

provided $(Q/\varphi(Q))^2 \geq \exp(a^{\log((r+1)^{2r+5}(1+|y|))/\log 2})$. The latter is satisfied if $Q \geq Q_1 = \exp(a^{\log((r+1)^{2r+5}(1+|y|))/\log 2}) > 1$. Next, there exists $Q_2 = Q_2(a)$ such that $g_a((Q/\varphi(Q))^2) = a^{\log((Q/\varphi(Q))^2)/\log \log((Q/\varphi(Q))^2)}$ for all $Q \geq Q_2$. Hence, for $Q \geq Q_0 = \max\{Q_1, Q_2\}$ we have

$$T(R_1) \leq 2g_a \left(\left(\frac{Q}{\varphi(Q)}\right)^2\right) = 2a^\frac{\log((Q/\varphi(Q))^2)}{\log \log((Q/\varphi(Q))^2)} \leq 2a^\frac{2\log((Q/\varphi(Q))^2)}{\log \log((Q/\varphi(Q))^2)}.$$
Hence, if $Q \geq Q_0$ then Theorem 2.2 yields
\[
\left| \frac{N\alpha(Q,y') - \psi(Q)}{\zeta(r + 1)} \right| 
\leq C_2 \left( \frac{(\psi(Q)Q^{\kappa - 1})}{\varphi(Q)} \right)^{r/(r + \kappa)} \mathcal{L}_{F^*} + 2a^{2 \log Q/\varphi(Q)} \left( \frac{\psi(Q)Q^{\kappa - 1}}{\varphi(Q)} \right)^{1/(r + \kappa)},
\]
where $2C_2 = 4nm^2 \left( \frac{m^2M_n}{\varphi(1)} \right)^m$. Now, by (9.13) we have $C_1 \leq C_F$, and thus,
\[
2C_2 \leq 4nm^2 \left( \frac{m^2M_n}{\varphi(1)} \right)^m C_F = 4 \left( r + 1 \right)^{9(r+1)^2+2r+5} \varphi(1)/(r+1)/(r + \kappa) \leq \frac{(r + 1)^{12(r+1)^2}}{\varphi(1)} = C_{F^*}.
\]
This completes the proof of Theorem 1.3.

12. LOWER BOUNDS FOR $|E_\xi(\psi, Q)|$

**Lemma 12.1.** Let $A > 0$ and suppose that $(p, q) \in \mathbb{Z}^r \times \mathbb{N}$ such that
\[
|p + q\xi|_\infty < q^{-\kappa} A.
\]
Moreover, suppose $N$ is an integer such that
\[
N^{1+1/r} < \psi(Nq)^{1/r} q^{\kappa - 1/r} A^{-1}.
\]
Then, with $Q = Nq$ we have $|E_\xi(\psi, Q)| \geq N$.

**Proof.** Consider the $N$ pairs $(mp, mq)$ for $1 \leq m \leq N$. It is straightforward to check that they all lie in $E_\xi(\psi, Nq)$, provided $N^2 < \psi(Nq)q^{\kappa - 1} A$, and this shows the claim.

Suppose $\psi$ is monotone\(^2\) increasing and unbounded, and $\xi$ is not rational. Applying Dirichlet’s approximation Theorem we get a sequence $(p_n, q_n)$ such that with $\kappa = 1+1/r$ and $A = 1$ the condition (12.1) holds for all $(p_n, q_n)$. Then (12.2) is fulfilled when we choose $N_n = \left[ \psi(q_n)^{1/(r+1)} \right] - 1$. Hence, with $Q_n = N_nq_n$ we get $|E_\xi(\psi, Q_n)| \geq N_n \to \infty$.

For the rest of the section whenever we specify $\psi$, we will do this only for large arguments and assume $\psi$ is adjusted to satisfy $\psi : [1, \infty) \to [1, \infty)$.

**Lemma 12.2.** Suppose $\varphi : [1, \infty) \to (0, 1]$ is a monotone decreasing function tending to zero such that for all $b > 0$ we have $\lim_{x \to \infty} \varphi(x)/\varphi(bx/\varphi(x)) = 1$. Moreover, suppose there exists an infinite sequence $(p_n, q_n) \in \mathbb{Z} \times \mathbb{N}$ such that
\[
|p_n + q_n\xi| < q_n^{-1}\varphi(q_n)
\]
for all $n \in \mathbb{N}$. Then, with $\psi(Q) = 4^{-5}\varphi(Q)^{-1}$, there exists an increasing sequence $Q_n$ of positive integers with $|E_\xi(\psi, Q_n)| \geq 4\psi(Q_n)$ for all $n$.

**Proof.** After passing to a subsequence we can assume that the sequence $q_n$ is increasing. Fix $n$ and let $Q_n = N_nq_n$ for some parameter $N_n$ to be specified later. Let us assume
\[
N_n^2 < \psi(N_nq_n)\varphi(q_n)^{-1}.
\]
Then by Lemma 12.1 $|E_\xi(\psi, Q_n)| \geq N_n$. We also want $N_n \geq 4\psi(Q_n)$, that is
\[
N_n \geq 4\psi(N_nq_n).
\]
We choose $\psi(Q) = a\varphi(Q)^{-1}$ and $N_n = \lfloor b\varphi(q_n)^{-1} \rfloor$ for positive $a, b$ satisfying $b^2 < a < b/4$ (e.g., $a = 4^{-5}$, $b = 4^{-3}$) and use $\lim_{n \to \infty} \varphi(x)/\varphi(bx/\varphi(x)) = 1$. This shows that for all

---

\(^2\)The assertion of monotonicity is used only here.
n large enough we can find $N_n$ satisfying (12.3) and (12.4). Now $q_n$ and $N_n$ are both increasing, hence $Q_n$ is increasing and this provides the desired sequence $Q_n = N_nq_n$. □

Note that if $\Phi$ is a monotone, decreasing function tending to zero then there exists a monotone, decreasing function $\varphi \geq \Phi$ tending to zero with $\lim_{x \to \infty} \varphi(x)/\varphi(bx/\varphi(x)) = 1$ for all $b > 0$. Therefore, the assumption $\lim_{x \to \infty} \varphi(x)/\varphi(bx/\varphi(x)) = 1$ is merely a technical one.

Lemma 12.3. Suppose $\xi$ is a real irrational number and let $\beta > 1$ be such that there exists an infinite sequence $(p_n, q_n) \in \mathbb{Z} \times \mathbb{N}$ with

$$|p_n + q_n\xi| \leq q_n^{-\beta}.$$  

Let $\psi(Q) = 2^{-4}Q^{1-1/\beta}$. Then there exists a monotone increasing unbounded sequence $Q_n$ such that

$$|E(\psi, Q_n)| \geq 4\psi(Q_n)$$

for all $n \in \mathbb{N}$.

Proof. After passing to a subsequence we can assume that the sequence $q_n$ is increasing. Fix $n$ and let $Q_n = N_nq_n$ for some parameter $N_n$ to be specified later. Suppose that

$$N_n^2 < \psi(N_nq_n)q_n^{\beta-1}. \tag{12.6}$$

By Lemma 12.1 we have $|E(\psi, Q_n)| \geq N_n$. Hence, we want $N_n \geq 4\psi(Q_n)$, that is

$$N_n \geq 4\psi(N_nq_n). \tag{12.7}$$

Let us momentarily write $\gamma = 1 - 1/\beta$. Using that $\psi(Q) = 2^{-4}Q^\gamma$ and rearranging terms we see that (12.6) means

$$N_n < 2^{4/(\gamma-2)}q_n^{(\beta-1+\gamma)/(2-\gamma)} = 2^{4/(\gamma-2)}q_n^{\gamma/(1-\gamma)}. \tag{12.8}$$

Similarly, we see that (12.7) is equivalent to

$$N_n \geq 2^{2/(\gamma-1)}q_n^{\gamma/(1-\gamma)}. \tag{12.9}$$

As $\beta > 1$ we have $\gamma \in (0, 1)$ and so $\gamma/(1-\gamma) > 0$ and $2^{4/(\gamma-2)} > 2^{2/(\gamma-1)}$. Hence, for all $n$ large enough there exists a natural number $N_n$ satisfying (12.8) and (12.9). Passing again to a subsequence we can assume that $q_{n+1}/q_n > 4^{\gamma/(2-\gamma)}$, and this implies $Q_{n+1} > Q_n$. Hence, $Q_n$ is a monotone increasing sequence of natural numbers satisfying the desired property $|E(\psi, Q_n)| \geq 4\psi(Q_n)$. □

Lemma 12.4. Suppose $\varepsilon > 0$ and $\xi$ has finite irrationality measure $\mu > 2$. Then there exists an increasing sequence $Q_n$ of positive integers, $\kappa > 1$ such that (1.2) holds with the constant function $\varphi(x) = 1$, and a monotone increasing function $\psi : [1, \infty) \to [1, \infty)$ with $\psi(Q)/Q^{1-\varepsilon/2+\varepsilon} \to \infty$ such that

$$|E(\psi, Q_n)| \geq 4\psi(Q_n)$$

for all $n \in \mathbb{N}$.

Proof. It suffices to prove the lemma for small $\varepsilon$. Hence, we can assume $0 < \varepsilon < 1$ and $\mu - 2 > \varepsilon/4$. By choosing $\kappa = \mu - 1 + \varepsilon/4$ and $\beta = \kappa - \varepsilon/2 = \mu - 1 - \varepsilon/4 > 1$ the hypothesis (12.5) is satisfied. Choosing $\psi(Q) = 2^{-4}Q^{1-1/\beta}$ and applying Lemma 12.3 we get the claim, and it remains to check that $\psi(Q)/Q^{1-1/\kappa+\varepsilon} \to \infty$. Now $1 - 1/\beta > 1 - 1/\kappa - \varepsilon$ means $\kappa(\kappa - \varepsilon/2) > 1/2$ which is true as $\kappa > 1$ and $\varepsilon < 1$. □
13. Using Discrepancy

Let \( x_q \) be a subsequence of the torus \((\mathbb{R}/\mathbb{Z})^r\), let \( B = (0, b)^r \), and let \( \chi_B \) be its characteristic function. Then we set

\[
D^*_Q(x_q) = \sup_{b \in (0, 1)} \left| \sum_{q=1}^{Q} \chi_B(x_q) - Q\text{Vol}(B) \right|
\]

In the definition of \( D^*_Q(x_q) \) the supremum is taken over the larger set of all aligned boxes \( \prod_{i=1}^{r}(0, b_i) \) in \((0, 1)^r\). Hence, we have \( D^*_Q(x_q) \leq D^*_Q(x_q) \). Now let \( x_q = q\xi \mod 1 \) (coordinatewise reduced to get a subsequence of the torus \((\mathbb{R}/\mathbb{Z})^r\)), and suppose \((\psi(Q)/Q)^{1/r} < 1/2\). Then, with \( B = (0, (\psi(Q)/Q)^{1/r})^r \), we have

\[
|E^*_\xi(\psi, Q)| = \sum_{\sigma} \sum_{q=1}^{Q} \chi_B(\sigma x_q),
\]

where the outer sum runs over all \( 2^r \) possible \( \sigma \), and each such \( \sigma \) acts on \( x_q \), by replacing some of its entries by their negative values. As \( D^*_Q(x_q) = D^*_Q(\sigma x_q) \) we get

\[
\left| |E^*_\xi(\psi, Q)| - 2^r \psi(Q) \right| \leq 2^r D^*_Q(x_q),
\]

and thus, we obtain the asymptotic relation (1.1) whenever \( D^*_Q(x_q) = o(\psi(Q)) \).

Instead of fixing a corner as in the star discrepancy, one could also take the supremum over boxes of fixed size and shape but varying positions. Moreover, one could choose the size of the boxes as a function of \( Q \), e.g., boxes of equal edge length \((\psi(Q)/Q)^{1/r} < 1/2\). Note that the error term in Theorem 1.2 is independent of \( y \). Hence, Theorem 1.2 provides an upper bound for this notion of discrepancy

\[
\sup_{y} \left| \sum_{q=1}^{Q} \chi_B(x_q) - \psi(Q) \right| \leq C_F \left( \psi(Q)^{Q_{k-1}} \varphi(Q)^{r/(k+r)} \right),
\]

where \( B_y \) is the box \( \prod_{i=1}^{r}(y_i, y_i + (\psi(Q)/Q)^{1/r}) \).

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