ASYMPTOTIC DIOPHANTINE APPROXIMATION: THE MULTIPLICATIVE CASE

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ABSTRACT. Let α and β be irrational real numbers and $0 < \varepsilon < 1/30$. We prove a precise estimate for the number of positive integers $q \leq Q$ that satisfy $\|q\alpha\|\cdot\|q\beta\| < \varepsilon$. If we choose ε as a function of Q we get asymptotics as Q gets large, provided εQ grows quickly enough in terms of the (multiplicative) Diophantine type of (α, β), e.g., if (α, β) is a counterexample to Littlewood's conjecture then we only need that εQ tends to infinity. Our result yields a new upper bound on sums of reciprocals of products of fractional parts, and sheds some light on a recent question of Lê and Vaaler.

1. INTRODUCTION

Let α and β be irrational real numbers, and let $\|\cdot\|$ be the distance to the nearest integer. Littlewood's conjecture asserts that

$$\liminf_{q\to\infty} q \cdot \|q\alpha\|\cdot\|q\beta\| = 0.$$

We assume that $\phi : [1, \infty) \to (0, 1/4]$ is a a non-increasing[1] function (depending on (α, β)) such that

$$q \cdot \|q\alpha\|\cdot\|q\beta\| \geq \phi(q) \tag{1.1}$$

for all positive integers q. Note that ϕ can be chosen to be constant if and only if the pair (α, β) is a counterexample to Littlewood's conjecture. The condition (1.1) has been considered in various forms, e.g., by Badziahin [1]. He takes a function $f : \mathbf{N} \to (0, \infty)$ and considers the set

$$\mathbf{Mad}(f) = \left\{(\alpha, \beta) \in \mathbf{R}^2; \liminf_{q\to\infty} f(q) \cdot q \cdot \|q\alpha\|\cdot\|q\beta\| > 0\right\}. \tag{1.2}$$

Special cases of these sets already appeared in [2]. If we assume that $1/f$ is also non-increasing then (α, β) lies in $\mathbf{Mad}(f)$ if and only if (1.1) holds true with a ϕ satisfying $1/f(q) \ll_{\alpha,\beta} \phi(q) \ll_{\alpha,\beta} 1/f(q)$.

Throughout this article, let $Q \geq 1$, $\varepsilon > 0$, and $T > 0$ be real numbers, and assume

$$\varepsilon/T^2 \leq 1/e^2, \tag{1.3}$$

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[1]By non-increasing we mean that $x, y \in [1, \infty)$ and $x \leq y$ implies that $\phi(x) \geq \phi(y)$.

where e denotes the base of the (natural) logarithm. We consider the finite set

$$M_{\alpha,\beta}(\varepsilon,T,Q)=\Big\{(p_1,p_2,q)\in\mathbf{Z}^3; |p_1+q\alpha|\cdot|p_2+q\beta|<\varepsilon,\ \max\{|p_1+q\alpha|,|p_2+q\beta|\}\leq T,\ 0<q\leq Q\Big\}.$$

Theorem 1.1. *Suppose that (1.1) and (1.3) hold, and set $C_1=3^{28}$. Then we have*

$$\left||M_{\alpha,\beta}(\varepsilon,T,Q)|-4\varepsilon Q\left(\log\left(\frac{T^2}{\varepsilon}\right)+1\right)\right|\leq C_1(1+2T)^2\log\left(\frac{T^2}{\varepsilon}\right)\left(\frac{\varepsilon Q}{\phi(Q)}\right)^{2/3}.$$

We shall see that the main term is just the volume of the set Z defined in Section 2. The constant C_1 could easily be improved. Choosing $T=1/2$ we have

$$|M_{\alpha,\beta}(\varepsilon,1/2,Q)|=|\{q\in\mathbf{Z}; \|q\alpha\|\cdot\|q\beta\|<\varepsilon, 0<q\leq Q\}|$$

which is of particular interest, and hence we state this case of Theorem 1.1 as a corollary.

Corollary 1.1. *Suppose that (1.1) holds, that $0<\varepsilon\leq 1/(2e)^2$, and set $C_2=4C_1=4\cdot 3^{28}$. Then we have*

$$\big||M_{\alpha,\beta}(\varepsilon,1/2,Q)|-4\varepsilon Q\left(1-\log(4\varepsilon)\right)\big|\leq -C_2\log(\varepsilon)\left(\frac{\varepsilon Q}{\phi(Q)}\right)^{2/3}.$$

If we choose a value $\varepsilon=\varepsilon(Q)\leq 1/(2e)^2$ for each value of Q, and we let Q tend to infinity then we get asymptotics for $|M_{\alpha,\beta}(\varepsilon,1/2,Q)|$ provided $1/\phi(Q)=o(\sqrt{\varepsilon(Q)Q})$. Let us write $\log^+ Q=\max\{1,\log Q\}$. A result of Gallagher [5] implies that for $f(q)=(\log^+ q)^\lambda$ the set $\mathbf{Mad}(f)$ has full Lebesgue measure if $\lambda>2$ and measure zero when $\lambda\leq 2$. Hence, if $\varepsilon(Q)\gg(\log^+ Q)^{2\lambda}/Q$ with $\lambda>2$ then the asymptotics are given by the main term in Corollary 1.1 for almost[2] every pair $(\alpha,\beta)\in\mathbf{R}^2$. Bugeaud and Moshchevitin [3] showed that when $\lambda=2$ the set $\mathbf{Mad}(f)$ still has full Hausdorff dimension. This was substantially improved by Badziahin [1] who showed that even with $f(q)=(\log^+ q)(\log^+(\log^+ q))$ the set $\mathbf{Mad}(f)$ has full Hausdorff dimension.

We now discuss an application of Corollary 1.1. In [6] Lê and Vaaler showed that

$$Q(\log^+ Q)^2\ll\sum_{q=1}^{\lfloor Q\rfloor}(\|q\alpha\|\cdot\|q\beta\|)^{-1}.$$

Motivated by this they raised the question whether there exist real irrational numbers α,β such that

$$\sum_{q=1}^{\lfloor Q\rfloor}(\|q\alpha\|\cdot\|q\beta\|)^{-1}\ll_{\alpha,\beta} Q(\log^+ Q)^2. \tag{1.4}$$

Lê and Vaaler showed that (1.4) holds for (α,β) provided the latter is a counterexample to Littlewood's conjecture, i.e., provided one can choose ϕ from (1.1) to be a constant function. We show that $\phi(Q)\gg_{\alpha,\beta}1/(\log^+ Q)$ suffices.

Corollary 1.2. *Suppose that (1.1) holds, and set $C_3=12$ and $C_4=3^{32}$. Then we have*

$$\sum_{q=1}^{\lfloor Q\rfloor}(\|q\alpha\|\cdot\|q\beta\|)^{-1}\leq C_3Q\left(\log\left(\frac{Q}{\phi(Q)}\right)\right)^2+C_4\frac{Q}{\phi(Q)}\log\left(\frac{Q}{\phi(Q)}\right).$$

[2] With respect to the Lebesgue measure.

Einsiedler, Katok and Lindenstrauss [4] showed that the set of counterexamples to Littlewood's conjecture has Hausdorff dimension zero, and it is widely believed that no such counterexample exists at all. On the other hand there is evidence for the existence of pairs (α, β) with $\phi(Q) \gg_{\alpha,\beta} 1/(\log^+ Q)$. In fact, Badziahin and Velani [2, (L2)] (see also [1, Conjecture 1]) conjectured that the set of these pairs has full Hausdorff dimension. Unfortunately, it is not known whether such a pair (α, β) really exists and so we cannot unconditionally answer Lê and Vaaler's question.

2. Prerequisites

Lemma 2.1. *If $\varepsilon Q < \phi(Q)$ then the stated inequality in Theorem 1.1 holds true.*

Proof. Suppose $(p_1, p_2, q) \in M_{\alpha,\beta}(\varepsilon, T, Q)$. Hence, $1 \leq q \leq Q$ and $\|q\alpha\| \cdot \|q\beta\| \leq |p_1 + q\alpha| \cdot |p_2 + q\beta| < \varepsilon$. On the other hand by (1.1), and using the monotonicity of ϕ, we have $\|q\alpha\| \cdot \|q\beta\| \geq \phi(q)/q \geq \phi(Q)/Q$. Thus, if $\varepsilon Q < \phi(Q)$ then $|M_{\alpha,\beta}(\varepsilon, T, Q)| = 0$. It remains to show that the main term is covered by the error term. As $T^2/\varepsilon \geq e^2$ we have $\log(T^2/\varepsilon) + 1 < 2\log(T^2/\varepsilon)$. Using that $\varepsilon Q < \phi(Q) \leq 1/4$ we see that $4\varepsilon Q < (C_1/2)(\varepsilon Q/\phi(Q))^{2/3}$. This shows that the main term is bounded by the error term, and this proves the lemma. □

For the proof of Theorem 1.1 we thus can and will assume that

$$\varepsilon Q \geq \phi(Q). \tag{2.1}$$

For a vector $\mathbf{x}$ in $\mathbf{R}^n$ we write $|\mathbf{x}|$ for the Euclidean length of $\mathbf{x}$. Let Λ be a lattice of rank n in $\mathbf{R}^n$. We define the first successive minimum $\lambda_1(\Lambda)$ of Λ as the shortest Euclidean length of a non-zero lattice vector

$$\lambda_1 = \inf\{|\mathbf{x}|; \mathbf{x} \in \Lambda, \mathbf{x} \neq \mathbf{0}\}.$$

From now on suppose $n \geq 2$, $M \geq 1$ is also an integer, and let L be a non-negative real. We say that a set S is in $\mathrm{Lip}(n, M, L)$ if S is a subset of $\mathbf{R}^n$, and if there are M maps $\iota_1, \ldots, \iota_M : [0,1]^{n-1} \longrightarrow \mathbf{R}^n$ satisfying a Lipschitz condition

$$|\iota_i(\mathbf{x}) - \iota_i(\mathbf{y})| \leq L|\mathbf{x} - \mathbf{y}| \text{ for } \mathbf{x}, \mathbf{y} \in [0,1]^{n-1}, i = 1, \ldots, M$$

such that S is covered by the images of the maps ι_i.

We will apply the following counting result which is an immediate consequence of [7, Theorem 5.4].

Lemma 2.2. *Let Λ be a lattice of rank n in $\mathbf{R}^n$ with first successive minimum λ_1. Let S be a set in $\mathbf{R}^n$ such that the boundary ∂S of S is in $Lip(n, M, L)$. Then S is measurable, and moreover,*

$$\left| |\Lambda \cap S| - \frac{\mathrm{Vol} S}{\det \Lambda} \right| \leq D_n M \left(1 + \left(\frac{L}{\lambda_1}\right)^{n-1}\right),$$

where $D_n = n^{2n^2}$.

Next we introduce the sets

$$H = \{(x, y) \in \mathbf{R}^2; |xy| < \varepsilon, |x| \leq T, |y| \leq T\},$$
$$Z = H \times (0, Q],$$

and the lattice

$$\Lambda = (1,0,0)\mathbf{Z} + (0,1,0)\mathbf{Z} + (\alpha, \beta, 1)\mathbf{Z}.$$

Clearly,

$$|M_{\alpha,\beta}(\varepsilon, T, Q)| = |\Lambda \cap Z|. \tag{2.2}$$

Instead of working with Z it is more convenient to decompose Z into four identically shaped parts Z_j and two rectangles R_j. We set

$$H_1 = \{(x,y) \in \mathbf{R}^2; |xy| < \varepsilon, 0 < x \leq T, 0 < y \leq T\},$$
$$H_2 = \{(x,y) \in \mathbf{R}^2; |xy| < \varepsilon, -T \leq x < 0, 0 < y \leq T\},$$
$$H_3 = \{(x,y) \in \mathbf{R}^2; |xy| < \varepsilon, 0 < x \leq T, -T \leq y < 0\},$$
$$H_4 = \{(x,y) \in \mathbf{R}^2; |xy| < \varepsilon, -T \leq x < 0, -T \leq y < 0\}.$$

Furthermore, we put for $1 \leq j \leq 4$

$$Z_j = H_j \times (0, Q],$$
$$R_1 = [-T, T] \times \{0\} \times (0, Q],$$
$$R_2 = \{0\} \times [-T, T] \times (0, Q],$$

so that we have the following partition

$$Z = Z_1 \cup Z_2 \cup Z_3 \cup Z_4 \cup R_1 \cup R_2.$$

Due to the irrationality of α and β we have

$$|\Lambda \cap R_1| = |\Lambda \cap R_2| = 2\lfloor T \rfloor + 1 < 2(T+1).$$

Hence,

$$\left| |\Lambda \cap Z| - \sum_{j=1}^{4} |\Lambda \cap Z_j| \right| < 4(T+1).$$

Using the automorphisms defined by $\tau_1(x,y,z) = (x,y,z)$, $\tau_2(x,y,z) = (-x,y,z)$, $\tau_3(x,y,z) = (x,-y,z)$, and $\tau_4(x,y,z) = (-x,-y,z)$ we have $\tau_j Z_j = Z_1$. Setting for $1 \leq j \leq 4$

$$\Lambda_j = \tau_j(\Lambda),$$

we find

$$\left| |M_{\alpha,\beta}(\varepsilon, T, Q)| - \sum_{j=1}^{4} |\Lambda_j \cap Z_1| \right| < 4(T+1). \tag{2.3}$$

Unfortunately, our set Z_1 is increasingly distorted when approaching the coordinate-axes. After the trivial decomposition of Z we shall now consider a less obvious decomposition of our new counting domain Z_1.

3. Partitioning the counting domain

First let us decompose H_1 into three disjoint pieces. Set

$$\Delta_x = \{(x,y); 0 < y < (\varepsilon/T^2)x, 0 < x \leq T\},$$
$$\Delta_y = \{(x,y); (T^2/\varepsilon)x \leq y \leq T, 0 < x < \varepsilon/T\},$$
$$S = \{(x,y); 0 < (\varepsilon/T^2)x \leq y < (T^2/\varepsilon)x, xy < \varepsilon\}.$$

Hence, we have

$$|\Lambda_j \cap Z_1| = |\Lambda_j \cap \Delta_x \times (0,Q]| + |\Lambda_j \cap \Delta_y \times (0,Q]| + |\Lambda_j \cap S \times (0,Q]|. \tag{3.1}$$

The sets Δ_x and Δ_y are long and thin triangles, distorted only in x-direction or y-direction respectively. The set S is more troublesome and requires a further decomposition into about $-\log(\varepsilon/T^2)$ pieces. Recall that by hypothesis $0 < \varepsilon/T^2 \leq 1/e^2$. Let $\nu \in [1/e^2, 1/e]$ be maximal such that $N = \log(\varepsilon/T^2)/\log \nu$ is an integer. Hence,

$$1 \leq N \leq -\log(\varepsilon/T^2). \tag{3.2}$$

Decompose S into the $2N$ pieces $S_{-N+1}, \ldots, S_N$, where

$$S_i = \{(x, y); 0 < \nu^i x \le y < \nu^{i-1} x, xy < \varepsilon\}.$$

Then we have the following partition

$$S = \bigcup_{-N+1 \le i \le N} S_i. \tag{3.3}$$

Note that

$$S_0 \subset [0, \sqrt{\varepsilon}] \times [0, \sqrt{\varepsilon/\nu}] \subset [0, 3\sqrt{\varepsilon}]^2. \tag{3.4}$$

A straightforward calculation yields

$$\mathrm{Vol}_2(S_0) = \frac{\sqrt{\varepsilon\nu}\sqrt{\varepsilon/\nu}}{2} + \int_{\sqrt{\nu\varepsilon}}^{\sqrt{\varepsilon}} \frac{\varepsilon}{x} dx - \frac{\varepsilon}{2} = -\frac{\varepsilon}{2}\log\nu.$$

Hence,

$$V = \mathrm{Vol}_3(S_0 \times (0, Q]) = -\frac{\varepsilon}{2} Q \log \nu.$$

Thus

$$\frac{\varepsilon Q}{2} \le V \le \varepsilon Q. \tag{3.5}$$

4. Applying flows

In this section we construct certain elements of the diagonal flow on $\mathbf{R}^3$ that transform our distorted sets into sets of small diameter.

We introduce the following automorphisms of $\mathbf{R}^2$

$$g_i(x, y) = (\nu^{i/2} x, \nu^{-i/2} y).$$

Then we have for $-N + 1 \le i \le N$

$$g_i S_i = S_0.$$

We extend g_i to an automorphism of $\mathbf{R}^3$

$$G_i(x, y, z) = (\nu^{i/2} x, \nu^{-i/2} y, z),$$

so that

$$G_i(S_i \times (0, Q]) = S_0 \times (0, Q].$$

Next we introduce a further automorphism of $\mathbf{R}^3$

$$G_\theta(x, y, z) = (\theta x, \theta y, \theta^{-2} z),$$

where

$$\theta = \frac{V^{1/3}}{\sqrt{\varepsilon}}.$$

Let us write

$$\varphi_i = G_\theta \circ G_i.$$

Then we have

$$\varphi_i(S_i \times (0, Q]) = \theta S_0 \times (0, \theta^{-2} Q].$$

Combining (3.4) and (3.5) we get

$$\varphi_i(S_i \times (0, Q]) = \theta S_0 \times (0, \theta^{-2} Q] \subset [0, 3\theta\sqrt{\varepsilon}]^2 \times (0, \theta^{-2} Q] \subset [0, 3V^{1/3}]^3. \tag{4.1}$$

Similarly, we find

$$\varphi_N(\Delta_x \times (0, Q]) \subset [0, 3V^{1/3}]^3, \tag{4.2}$$

$$\varphi_{-N+1}(\Delta_y \times (0, Q]) \subset [0, 3V^{1/3}]^3. \tag{4.3}$$

Lemma 4.1. *For $-N+1 \leq i \leq N$ the boundary of $\varphi_i(S_i \times (0,Q])$, $\varphi_N(\Delta_x \times (0,Q])$ and $\varphi_{-N+1}(\Delta_y \times (0,Q])$ lies in Lip$(3,M,L)$ where $M=5$, $L = C_L V^{1/3}$ and $C_L = 12$.*

Proof. The boundary of the set $\varphi_i(S_i \times (0,Q]) = \theta S_0 \times (0, \theta^{-2}Q]$ can be covered by 4 planes and the set

$$\{(x, \theta^2\varepsilon/x, z); \theta\sqrt{\nu\varepsilon} \leq x \leq \theta\sqrt{\varepsilon}, 0 \leq z \leq Q\theta^{-2}\}.$$

For the Jacobian J of the parameterising map

$$(t_1, t_2) \to (at_1 + b, \frac{\theta^2\varepsilon}{(at_1+b)}, ct_2)$$

with $a = \theta\sqrt{\varepsilon}(1-\sqrt{\nu})$, $b = \theta\sqrt{\nu\varepsilon}$, $c = Q/\theta^2$, and domain $[0,1]^2$ we get for its l_2-operator norm $\|J\|_2 \leq 4V^{1/3}$ which yields the required Lipschitz condition thanks to the Mean-Value Theorem. Hence, we are left with the linear pieces of the boundary. Clearly, a subset of a plane with diameter no larger than d can be parameterised by a single affine map with domain $[0,1]^2$ and Lipschitz constant $2d$. Thus, it suffices to show that the diameter of $\varphi_i(S_i \times (0,Q])$ is $\leq 6V^{1/3}$. But the latter holds due to (4.1).

Finally, the boundary of the set $\varphi_N(\Delta_x \times (0,Q])$ and of the set $\varphi_{-N+1}(\Delta_y \times (0,Q])$ can each be covered by 5 planes. Moreover, by (4.2) and (4.3) their diameter is also $\leq 6V^{1/3}$. This proves the lemma. □

5. Controlling the orbits

Our transformations of the previous section have brought our distorted sets into nice shapes. Unfortunately, they transform our lattices Λ_j in a less favourable manner. Indeed, the corresponding orbit of Λ_j escapes to infinity, i.e., the fist successive minimum gets arbitrarily small. However, the rate of escape is controllable and sufficiently slow.

Lemma 5.1. *For $1 \leq j \leq 4$, $-N+1 \leq i \leq N$, and $Q \geq 1$ we have*

$$\lambda_1(\varphi_i\Lambda_j) \geq \min\{1, 1/(2T)\}\phi(Q)^{1/3}.$$

Proof. Let $v \in \Lambda_j$ be an arbitrary non-zero lattice point. Then there exist ϵ_1 and ϵ_2 in $\{-1,1\}$ and $p_1, p_2, q \in \mathbf{Z}$, not all zero, such that $v = (\epsilon_1(p_1+q\alpha), \epsilon_2(p_2+q\beta), q)$. First suppose $q \neq 0$. Then by the inequality of arithmetic and geometric means we have

$$|\varphi_i v|^2 \geq 3(|p_1+q\alpha|\cdot|p_2+q\beta|\cdot|q|)^{2/3}.$$

Using our hypothesis (1.1) we get $|p_1+q\alpha|\cdot|p_2+q\beta|\cdot|q| \geq \phi(|q|)$. If $|q| \leq Q$ we conclude, by the monotonicity of ϕ, that $\phi(|q|) \geq \phi(Q)$, and hence

$$|\varphi_i v| \geq \phi(Q)^{1/3}.$$

If, on the other hand, $|q| > Q$ then, looking only at the last coordinate, and using (2.1), we find

$$|\varphi_i v| > \theta^{-2}Q \geq (\varepsilon Q)^{1/3} \geq \phi(Q)^{1/3}.$$

Suppose now that $q = 0$. Then p_1 and p_2 are not both zero, and hence

$$|\varphi_i v| \geq \max\{\theta\nu^{i/2}|p_1|, \theta\nu^{-i/2}|p_2|\} \geq \theta\nu^{|i|/2} \geq \theta\nu^{N/2}.$$

Recall that $\nu^{N/2} = \sqrt{\varepsilon}/T$, and $\theta = \mathrm{V}^{1/3}/\sqrt{\varepsilon} \geq 2^{-1/3}\left(Q/\sqrt{\varepsilon}\right)^{1/3}$. Hence,

$$\theta\nu^{N/2} \geq \frac{1}{2T}(\varepsilon Q)^{1/3} \geq \frac{1}{2T}\phi(Q)^{1/3}.$$

This proves the lemma. □

6. Proof of Theorem 1.1

Let $1 \leq j \leq 4$. Decomposing the set Z_1 using (3.1) and (3.3) and then applying the automorphisms φ_i yields

$$\begin{aligned}|\Lambda_j \cap Z_1| = & \ |\Lambda_j \cap \Delta_x \times (0, Q]| + |\Lambda_j \cap \Delta_y \times (0, Q]| + \sum_{i=-N+1}^{N} |\Lambda_j \cap S_i \times (0, Q]| \\ = & \ |\varphi_N \Lambda_j \cap \varphi_N(\Delta_x \times (0, Q])| \\ & + |\varphi_{-N+1}\Lambda_j \cap \varphi_{-N+1}(\Delta_y \times (0, Q])| \\ & + \sum_{i=-N+1}^{N} |\varphi_i \Lambda_j \cap \varphi_i(S_i \times (0, Q])|.\end{aligned}$$

Note that $\det \varphi_i \Lambda_j = 1$. Applying Lemma 2.2 to each summand, using Lemma 4.1, and collecting the main terms and the error terms terms yields

$$\big| |\Lambda_j \cap Z_1| - \mathrm{Vol}_3(Z_1) \big| \leq 2D_3 M C_L^2 \sum_{i=-N+1}^{N} \left(1 + \frac{\mathrm{V}^{2/3}}{\lambda_1(\varphi_i \Lambda_j)^2}\right). \tag{6.1}$$

Then, applying Lemma 5.1, we see that the right hand-side of (6.1) is bounded by

$$\leq 4D_3 M C_L^2 \max\{1, 2T\}^2 N \left(1 + \frac{\mathrm{V}^{2/3}}{\phi(Q)^{2/3}}\right).$$

Using that by (3.5) $\mathrm{V} \leq \varepsilon Q$, and then again that $\varepsilon Q \geq \phi(Q)$ we conclude that the latter is bounded by

$$\leq 8D_3 M C_L^2 (1 + 2T)^2 N \left(\frac{\varepsilon Q}{\phi(Q)}\right)^{2/3}.$$

Putting $C_5 = 8D_3 M C_L^2 = 8 \cdot 3^{18} \cdot 5 \cdot 12^2$, and recalling that $N \leq \log(T^2/\varepsilon)$ we conclude that

$$\big| |\Lambda_j \cap Z_1| - \mathrm{Vol}_3(Z_1) \big| \leq C_5 (1 + 2T)^2 \log\left(\frac{T^2}{\varepsilon}\right) \left(\frac{\varepsilon Q}{\phi(Q)}\right)^{2/3}.$$

By virtue of inequality (2.3), we get

$$\big| |M_{\alpha,\beta}(\varepsilon, T, Q)| - 4\mathrm{Vol}_3(Z_1) \big| \leq 5C_5 (1 + 2T)^2 \log\left(\frac{T^2}{\varepsilon}\right) \left(\frac{\varepsilon Q}{\phi(Q)}\right)^{2/3}.$$

Finally, we note that $5C_5 < 3^{28} = C_1$ and

$$\mathrm{Vol}_3(Z_1) = \varepsilon Q \left(\log\left(\frac{T^2}{\varepsilon}\right) + 1\right),$$

and this completes the proof of Theorem 1.1.

7. Proof of Corollary 1.2

We have

$$\begin{aligned}\sum_{q=1}^{\lfloor Q \rfloor} (\|q\alpha\| \cdot \|q\beta\|)^{-1} &\leq \sum_{k=1}^{\infty} 2^{k+1} |\{q; 1 \leq q \leq Q, 2^{-k-1} \leq \|q\alpha\| \cdot \|q\beta\| < 2^{-k}\}| \\ &\leq \sum_{k=1}^{\infty} 2^{k+1} |\{q; 1 \leq q \leq Q, \|q\alpha\| \cdot \|q\beta\| < 2^{-k}\}| \\ &= \sum_{k=1}^{\infty} 2^{k+1} |M_{\alpha,\beta}(2^{-k}, 1/2, Q)|.\end{aligned}$$

Moreover, in the proof of Lemma 2.1 we have seen that $M_{\alpha,\beta}(\varepsilon, T, Q) = \emptyset$ when $\varepsilon < \phi(Q)/Q$. We apply this with $\varepsilon = 2^{-k}$ and $T = 1/2$. Hence,

$$\begin{aligned}\sum_{k=1}^{\infty} 2^{k+1}|M_{\alpha,\beta}(2^{-k}, 1/2, Q)| &= \sum_{k=1}^{\lfloor \log_2(Q/\phi(Q))\rfloor} 2^{k+1}|M_{\alpha,\beta}(2^{-k}, 1/2, Q)| \\ &< 4\cdot 2^5 Q + \sum_{k=5}^{\lfloor \log_2(Q/\phi(Q))\rfloor} 2^{k+1}|M_{\alpha,\beta}(2^{-k}, 1/2, Q)|. \end{aligned} \tag{7.1}$$

From Corollary 1.1 we get for integers $k \geq 5$

$$|M_{\alpha,\beta}(2^{-k}, 1/2, Q)| \leq 4(\log 2)Qk2^{-k} + C_2(\log 2)k\left(\frac{2^{-k}Q}{\phi(Q)}\right)^{2/3}. \tag{7.2}$$

Combining (7.1) and (7.2) Corollary 1.2 follows from a straightforward calculation using the trivial estimates $\sum_{k=1}^{K} k \leq K^2$ and $\sum_{k=1}^{K} kx^k \leq Kx^{K+1}/(x-1)$ (where $x > 1$) and that $\phi(Q) \leq 1/4$.

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