COUNTING LATTICES POINTS AND WEAK ADMISSIBILITY OF A LATTICE AND ITS DUAL

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Abstract. We prove a counting theorem concerning the number of lattice points for the dual lattices of weakly admissible lattices in an inhomogeneously expanding box. The error term is expressed in terms of a certain function $\nu(\Gamma^{\perp},\cdot)$ of the dual lattice $\Gamma^{\perp}$, and we carefully analyse the relation of this quantity with $\nu(\Gamma,\cdot)$. In particular, we show that $\nu(\Gamma^{\perp},\cdot)=\nu(\Gamma,\cdot)$ for any unimodular lattice of rank 2, but that for higher ranks it is in general not possible to bound one function in terms of the other. This result relies on Beresnevich's recent breakthrough on Davenport's problem regarding badly approximable points on submanifolds of $\mathbb{R}^n$. Finally, we apply our counting theorem to establish asymptotics for the number of Diophantine approximations with bounded denominator as the denominator bound gets large.

Introduction

In the present article, we are mainly concerned with three objectives. Firstly, we prove a counting result for lattice points of unimodular weakly admissible lattices in inhomogeneously expanding, aligned boxes. A similar result for homogeneously expanding boxes was proven by Skriganov [22, Thm. 6.1] in 1998. Secondly, we carefully investigate the relation between $\nu(\Gamma,\cdot)$ (see (0.1) for the definition) and $\nu(\Gamma^{\perp},\cdot)$ of the dual lattice $\Gamma^{\perp}$ which captures the dependency on the lattice in these error terms. And thirdly, we apply our counting result to count Diophantine approximations.

To state our first result, we need to introduce some notation. By writing $f \ll g$ (or $f \gg g$) for functions f,g, we mean that there is a constant $c>0$ such that $f(x)\leq cg(x)$ (or $cf(x)\geq g(x)$) holds for all admissible values of x; if the implied constant depends on certain parameters, then this dependency will be indicated by an appropriate subscript. Let $\Gamma\subseteq\mathbb{R}^n$ be a unimodular lattice, and let $\Gamma^{\perp}:=\{w\in\mathbb{R}^n:\ \langle v,w\rangle\in\mathbb{Z}\quad\forall_{v\in\Gamma}\}$ be its dual lattice with respect to the standard inner product $\langle\cdot,\cdot\rangle$. Let γ_n denote the Hermite constant, and for $\rho>\gamma_n^{1/2}$ set

$$\nu(\Gamma,\rho):=\min\left\{|x_1\cdots x_n|:\ x:=(x_1,\ldots,x_n)^T\in\Gamma,\ 0<\|x\|_2<\rho\right\} \qquad (0.1)$$

where $\|\cdot\|_2$ denotes the Euclidean norm. We say Γ is weakly admissible if $\nu(\Gamma,\rho)>0$ for all $\rho>\gamma_n^{1/2}$. Note that this happens if and only if Γ has trivial intersection with every coordinate subspace. It is also worthwhile mentioning that the function $\nu(\Gamma,\rho)$ controls the rate of escape of the lattice Γ under the action of the diagonal subgroup of $\mathrm{SL}_n(\mathbb{R})$ (cf. (1.6)).

Furthermore, let $\mathcal{T}:=\mathrm{diag}(t_1,\ldots,t_n)$ for $t_i>0$ be the diagonal matrix with diagonal entries $t_1,\ldots,t_n$, and let $y\in\mathbb{R}^n$. We set

$$B:=\mathcal{T}\,[0,1]^n+y,$$

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and we call such a set an aligned box. Moreover, we define

$$T \coloneqq (\det \mathcal{T})^{1/n} \cdot \|\mathcal{T}^{-1}\|_2 = \frac{(t_1 \cdots t_n)^{1/n}}{\min\{t_1, \dots, t_n\}} \geq 1$$

where $\|\cdot\|_2$ denotes the operator norm induced by the Euclidean norm.

Theorem 1. *Let $n \geq 2$, let $\Gamma \subseteq \mathbb{R}^n$ be a unimodular lattice, and let $B \subseteq \mathbb{R}^n$ be as above. Suppose $\Gamma^\perp$ is weakly admissible, and $\rho > \gamma_n^{1/2}$. Then,*

$$|\#(\Gamma \cap B) - \mathrm{vol}(B)| \ll_n \frac{1}{\nu\big(\Gamma^\perp, T^\star\big)} \left(\frac{(\mathrm{vol}(B))^{1-1/n}}{\sqrt{\rho}} + \frac{R^{n-1}}{\nu(\Gamma^\perp, 2^R T)} \right) \tag{0.2}$$

where $x^\star := \max\{\gamma_n, x\}$, and $R \coloneqq n^2 + \log \frac{\rho^n}{\nu(\Gamma^\perp, \rho T)}$.

Note that $\rho^n/\nu(\Gamma^\perp, \rho) \geq n^{n/2}$ by the inequality between arithmetic and geometric mean. Since $T \geq 1$ and (cf. [15, Theorem 2.1.1])

$$\gamma_n \leq (4/3)^{(n-1)/2}, \tag{0.3}$$

we have $(2^R T)^\star = 2^R T$, and hence, the far right hand-side in (0.2) is well-defined.

The lattice Γ is called admissible if $\mathrm{Nm}\,(\Gamma) := \lim_{\rho \to \infty} \nu(\Gamma, \rho) > 0$. It is easy to show that if Γ is admissible then also $\Gamma^\perp$ is admissible (see [21, Lemma 3.1]). In this case we can choose $\rho = (\mathrm{vol} B)^{2-2/n}$, provided the latter is greater than $\gamma_n^{1/2}$, to recover the following impressive result of Skriganov ([21, Theorem 1.1 (1.11)])

$$|\#(\Gamma \cap B) - \mathrm{vol}(B)| \ll_{n, \mathrm{Nm}\,(\Gamma^\perp)} (\log(\mathrm{vol}(B))^{n-1}. \tag{0.4}$$

However, if Γ is only weakly admissible, then it can happen that $\Gamma^\perp$ is not weakly admissible; see Example 4. But this is a rather special situation and typically, e.g., if the entries of A are algebraically independent, see Lemma 4, then $\Gamma = A\mathbb{Z}^n$ and its dual are both weakly admissible. This raises the question whether, or under which conditions, one can control $\nu(\Gamma^\perp, \cdot)$ by $\nu(\Gamma, \cdot)$. We have the following result where we use the convention that for an integral domain I the group of all matrices in $I^{n\times n}$ with inverse in $I^{n \times n}$ is denoted by $\mathrm{GL}_n(I)$.

Proposition 1. *Let $\Gamma = A\mathbb{Z}^n$, and suppose there exist S, W both in $GL_n(\mathbb{Z})$ such that*

$$A^T S A = W,$$

and suppose S has exactly one non-zero entry in each column and in each row. Then, we have

$$\nu(\Gamma^\perp, \cdot) = \nu(\Gamma, \cdot). \tag{0.5}$$

A special case of Proposition 1 shows that $\nu(\Gamma^\perp, \cdot) = \nu(\Gamma, \cdot)$ whenever $\Gamma = A\mathbb{Z}^n$ with a symplectic matrix A, in particular, whenever[1] Γ is a unimodular lattice in $\mathbb{R}^2$. In these cases, one can directly compare Theorem 1 with a recent result [24, Theorem 1.1] of the second author, and we refer to [24] for more on that. On the other hand, our next result shows that in general $\nu(\Gamma, \cdot)$ can decay arbitrarily quickly even if we control $\nu(\Gamma^\perp, \cdot)$.

[1]Let us write $Sp_{2m}(\mathbb{R})$ for the symplectic subgroup of $GL_{2m}(\mathbb{R})$ and $SL_n(\mathbb{R})$ for the special linear subgroup of $GL_n(\mathbb{R})$. The fact $Sp_2(\mathbb{R}) = SL_2(\mathbb{R})$ can be checked directly.

Theorem 2. *Let $n \geq 3$, and let $\psi : (0,\infty) \to (0,1)$ be non-increasing. Then, there exists a unimodular, weakly admissible lattice $\Gamma \subseteq \mathbb{R}^n$, and a sequence $\{\rho_l\} \subseteq (\gamma_n^{1/2}, \infty)$ tending to ∞, as $l \to \infty$, such that*

$$\nu(\Gamma^\perp, \rho) \gg \rho^{-n^2},$$

and

$$\nu(\Gamma, \rho_l) \leq \psi(\rho_l)$$

for all $l \in \mathbb{N} = \{1,2,3,\dots\}$ and for all $\rho > \gamma_n^{1/2}$.

In the case where exactly one of the functions $\nu(\Gamma,\cdot)$, and $\nu(\Gamma^\perp,\cdot)$ is controllable while the other one decays very quickly either Theorem 1 or [24, Theorem 1.1] provides a reasonable error term, but certainly not both. This highlights the complementary aspects of Theorem 1, and [24, Theorem 1.1]. Theorem 2 is deeper than Proposition 1, and relies on Beresnevich's recent breakthrough on Davenport's longstanding question about the distribution of badly approximable points on certain submanifolds of $\mathbb{R}^n$. Going even beyond Davenport's original question, Beresnevich proved that the sets of these points have full Hausdorff-dimension, and it is the full power of this result that we require to prove Theorem 2.

Recently German [14] considered the so-called lattice exponent $\omega(\Gamma)$ which is a coarse measure for the rate of decay of the function $\nu(\Gamma,\rho)$; it can be expressed as

$$\omega(\Gamma) = \limsup_{\rho\to\infty} \frac{-\log \nu(\Gamma,\rho)}{n \log \rho}, \tag{0.6}$$

where for non-weakly admissible lattices this is interpreted as $\omega(\Gamma) = \infty$. German proposes the problem of studying the spectrum of the pairs $(\omega(\Gamma), \omega(\Gamma^\perp))$ as Γ runs over all unimodular lattices in $\mathbb{R}^n$. He constructs a non-weakly admissible lattice Γ with $\omega(\Gamma^\perp) = 1/(n(n-2))$ and hence, $(\omega(\Gamma), \omega(\Gamma^\perp)) = (\infty, 1/(n(n-2)))$. If we insist that Γ be also weakly admissible then we can use Theorem 2 but at the expense that we have only an estimate for $\omega(\Gamma^\perp)$. More precisely, there exists a weakly admissible lattice Γ such that $(\omega(\Gamma), \omega(\Gamma^\perp)) \in \{\infty\} \times [0,n]$.

Next, we apply Theorem 1 to deduce counting results for Diophantine approximations. We start with a bit of historical background on this, and related problems. Let $\alpha \in \mathbb{R}$, let $\iota : [1,\infty) \to (0,1]$ be a positive decreasing function, and let $N_\alpha^{loc}(\iota, t)$ be the number of integer pairs (p,q) satisfying $|p + q\alpha| < \iota(q)$, $1 \leq q \leq t$. In a series of papers, starting in 1959, Erdős [13], Schmidt [19, 20], Lang [9, 17, 18], Adams [1, 2, 3, 4, 5, 6, 7, 8], Sweet [23], and others, considered the problem of finding the asymptotics for $N_\alpha^{loc}(\iota, t)$ as t gets large.

Schmidt [19] has shown that for almost every[2] $\alpha \in \mathbb{R}$ the asymptotics are given by the volume of the corresponding subset of $\mathbb{R}^2$, provided the latter tends to infinity. This is false for quadratic α; there with $\iota(q) = 1/q$ the volume is $2\log(t) + O(1)$, and by Lang's result $N_\alpha^{loc}(1/q, t) \sim c_\alpha \log(t)$ but Adams [5] has shown that $c_\alpha \neq 2$.

Opposed to the above "non-uniform" setting, where the bound on $|p + q\alpha|$ is expressed as a function of q, we consider the "uniform" situation, where the bound is expressed as a function of t. Furthermore, we shall consider the more general asymmetric inhomogeneous setting. Let $\alpha \in (0,1)$ be irrational, $\varepsilon, t \in (0,\infty)$, and let $y \in \mathbb{R}$. We define the counting function

$$N_{\alpha,y}(\varepsilon, t) = \#\left\{ (p,q) \in \mathbb{Z} \times \mathbb{N} : \begin{array}{c} 0 \leq p + q\alpha - y \leq \varepsilon, \\ 0 \leq q \leq t \end{array} \right\}. \tag{0.7}$$

[2]Here "almost every" refers always to the Lebesgue measure.

If the underlying set is not too stretched, then $N_{\alpha,y}(\varepsilon,t)$ is roughly the volume εt of the set in which we are counting lattice points. If we let $\varepsilon=\varepsilon(t)$ be a function of t with $t=o(t\varepsilon)$ we have, by simple standard estimates,

$$N_{\alpha,y}(\varepsilon,t)\sim\varepsilon t \tag{0.8}$$

for any pair $(\alpha,y)\in((0,1)\setminus\mathbb{Q})\times\mathbb{R}$ whatsoever. To get non-trivial estimates for our counting function, we need information on the Diophantine properties of α. Let $\phi:(0,\infty)\to(0,1)$ be a non-increasing function such that

$$q\big|p+q\alpha\big|\geq\phi\,(q) \tag{0.9}$$

holds for all $(p,q)\in\mathbb{Z}\times\mathbb{N}$. Then [24, Theorem 1.1] implies that

$$|N_{\alpha,y}(\varepsilon,t)-\varepsilon t|\ll_{\alpha}\sqrt{\frac{\varepsilon t}{\phi(t)}}. \tag{0.10}$$

Hence, unlike in the non-uniform setting, for badly approximable α the asymptotics are given by the volume as long as the volume tends to infinity.

Our next result significantly improves the error term in (0.10), provided α is "sufficiently" badly approximable, i.e., provided $\phi(t)$ decays slowly enough. We assume that

$$\varepsilon t>4\qquad\text{and}\qquad 0<\varepsilon<\sqrt{\alpha}. \tag{0.11}$$

Corollary 1. *Put $E:=\frac{\varepsilon t}{\phi(4t\sqrt{\varepsilon t})}$, and $E':=168\sqrt{\varepsilon t^3}E$. Then, we have*

$$|N_{\alpha,y}(\varepsilon,t)-\varepsilon t|\underset{\alpha}{\ll}\frac{\log E}{\phi^2(E')}. \tag{0.12}$$

In particular, if α is badly approximable then

$$|N_{\alpha,y}(\varepsilon,t)-\varepsilon t|\underset{\alpha}{\ll}\log(\varepsilon t). \tag{0.13}$$

The latter should be compared to a classical result of Ostrowski, cf. [16, p. 125, Thm. 3.4] which bounds the discrepancy of the sequence $(\langle q\alpha\rangle)_q$ of fractional parts[3] of $q\alpha$ from above in terms of the continued fraction expansion of $\alpha\in(0,1)\setminus\mathbb{Q}$. Thus for $y\in\mathbb{Z}$ Ostrowski's result implies that

$$|N_{\alpha,y}(\varepsilon,t)-\varepsilon t|\underset{\alpha}{\ll}\log t \tag{0.14}$$

for each badly approximable $\alpha\in(0,1)$ and all $\varepsilon\in(0,1)$. If, on the other hand, $y\notin\mathbb{Z}$ and $\varepsilon<\langle -y\rangle$, then it is easy to check that $\langle q\alpha-y\rangle\in[0,\varepsilon]$ if and only if $\langle q\alpha\rangle\in[1-\langle -y\rangle,1-\langle -y\rangle+\varepsilon]$. Thus, Ostrowski's result implies that (0.14) remains valid for any non-integral $y\in\mathbb{R}$, provided $\varepsilon\in(0,\langle -y\rangle)$.

1. Proof of Theorem 1 and Corollary 1

In the proof of Theorem 1, it is crucial for us to estimate the error in the lattice point counting problem for homogeneously expanding boxes such that the dependence on the Diophantine properties of the lattice is explicitly stated. To this end, we use an explicit version of Skriganov's result which is described in the next subsection.

[3]For $x\in\mathbb{R}$ we write $\langle x\rangle=x-\lfloor x\rfloor$ for the fractional part of x.

1.1. **An explicit version of Skriganov's counting theorem.** Let $\Gamma \subseteq \mathbb{R}^n$ be a lattice, and let $\lambda_i(\Gamma)$ denote the i-th successive minimum of Γ with respect to the Euclidean norm ($1 \leq i \leq n$). For $r > 0$ we introduce a special set of diagonal matrices

$$\Delta_r := \left\{\delta := \operatorname{diag}(2^{m_1}, \ldots, 2^{m_n}) : m = (m_1, \ldots, m_n)^T \in \mathbb{Z}^n,\ \|m\|_2 < r,\ \det \delta = 1\right\},$$

and we put

$$S(\Gamma, r) := \sum_{\delta \in \Delta_r} (\lambda_1(\delta\Gamma))^{-n}.$$

Now we can state Skriganov's result. In fact, his result is more general, and applies to any convex, compact polyhedron. On the other hand, the dependency on B and Γ in the error term is not explicitly stated in his counting result [22, Thm. 6.1]. By carefully following his reasoning we find the following explicit version of his result. Recall that γ_n denotes the Hermite constant.

Theorem 3 (Skriganov, 1998)**.** *Let $n \geq 2$ be an integer, let $\Gamma \subseteq \mathbb{R}^n$ be a unimodular lattice, and let $B \subseteq \mathbb{R}^n$ be an aligned box of volume 1. Suppose $\Gamma^\perp$ is weakly admissible, and $\rho > \gamma_n^{1/2}$. Then, for $t > 0$,*

$$|\#(\Gamma \cap tB) - t^n| \ll_n (|\partial B|\, \lambda_n(\Gamma))^n \cdot (t^{n-1}\rho^{-1/2} + S(\Gamma^\perp, r)) \tag{1.1}$$

where $r := n^2 + \log \frac{\rho^n}{\nu(\Gamma^\perp, \rho)}$, and $|\partial B|$ denotes the surface area of B.

1.2. **Proof of Theorem 1.** For proving Theorem 1, we want to exploit Theorem 3. To this end let $\bar{t} := (\det \mathcal{T})^{1/n}$, and let

$$U := \bar{t}\mathcal{T}^{-1}. \tag{1.2}$$

Thus,

$$\#(\Gamma \cap B) = \#(U\Gamma \cap U(\mathcal{T}\,[0,1]^n + y)) = \#(\Lambda \cap \bar{t}([0,1]^n + \mathcal{T}^{-1}(y)))$$

where $\Lambda := U\Gamma$. Moreover, we conclude by Theorem 3 that

$$|\#(\Gamma \cap B) - \operatorname{vol}(B)| \ll_n \lambda_n^n(\Lambda)\left(\frac{\bar{t}^{n-1}}{\sqrt{\rho}} + S(\Lambda^\perp, r)\right). \tag{1.3}$$

For controlling the quantities on the right hand side in terms of Γ, $\bar{t}$, ρ, and $\nu(\Gamma^\perp, \cdot)$, we need two lemmata. Their proofs use, in parts, arguments which are contained in the proof of [22, Lem. 4.1]; however, unlike in [22], we require to bound $\nu(\Lambda, \cdot)$ in terms of $\nu(\Gamma, \cdot)$. We will frequently use the fact that if $\Gamma = A\mathbb{Z}^n$ is unimodular then $\Gamma^\perp = (A^{-1})^T\mathbb{Z}^n$. As usual, we let $\mathrm{SL}_n(\mathbb{R})$ denote the group of all $\mathbb{R}^{n\times n}$ matrices with determinant 1.

Lemma 1. *Let $D := \operatorname{diag}(d_1, \ldots, d_n)$ be in $SL_n(\mathbb{R})$, and $\rho > \gamma_n^{1/2}$. Then,*

$$\nu((D\Gamma)^\perp, \rho) \geq \nu(\Gamma^\perp, \|D\|_2\, \rho), \tag{1.4}$$

and

$$\lambda_1^n(D\Gamma) \gg_n \nu(\Gamma, \left\|D^{-1}\right\|_2^\star). \tag{1.5}$$

Proof. For $v := (v_1, \ldots, v_n)^T \in \mathbb{R}^n$ define $\operatorname{Nm}(v) := |v_1 \cdots v_n|$. We remark that

$$\begin{aligned}\nu((D\Gamma)^\perp, \rho) &= \nu(D^{-1}\Gamma^\perp, \rho)\\ &= \min\left\{\operatorname{Nm}(D^{-1}v) : v \in \Gamma^\perp,\ 0 < \left\|D^{-1}v\right\|_2 < \rho\right\}\\ &= \min\left\{\operatorname{Nm}(v) : v \in \Gamma^\perp,\ 0 < \left\|D^{-1}v\right\|_2 < \rho\right\}.\end{aligned}$$

If $\|D^{-1}v\|_2 < \rho$, then $\|v\|_2 < \|D\|_2\rho$. Thus, (1.4) follows. Now let $Q > 0$, and $v \in \Gamma$ with $0 < \|v\|_2 \le Q$. By the inequality of arithmetic and geometric mean, we have

$$\|Dv\|_2^n \ge n^{n/2} \cdot \mathrm{Nm}(Dv) \underset{n}{\gg} \nu(\Gamma, Q^\star).$$

Now suppose $\|v\|_2 > Q$. Since $\|v\|_2 = \left\|D^{-1}Dv\right\|_2 \le \left\|D^{-1}\right\|_2 \|Dv\|_2$, we conclude that

$$\|Dv\|_2 > \left\|D^{-1}\right\|_2^{-1} Q.$$

Hence, we have

$$\|Dv\|_2 \underset{n}{\gg} \min\left\{(\nu(\Gamma, Q^\star))^{1/n},\ \left\|D^{-1}\right\|_2^{-1} Q\right\}.$$

Specialising $Q := \left\|D^{-1}\right\|_2$, and noticing that by the inequality of arithmetic and geometric mean, $\nu(\Gamma, \gamma_n) \underset{n}{\ll} 1$, we get (1.5). □

Note that $\left\|D^{-1}\right\|_2 \le \|D\|_2^{n-1}$, and hence by Lemma 1 that

$$\lambda_1^n(D\Gamma) \underset{n}{\gg} \nu(\Gamma, \|D\|_2^{n-1}), \tag{1.6}$$

at least if $\|D\|_2^{n-1} > \gamma_n^{1/2}$. Therefore, the function $\nu(\Gamma, \rho)$ controls the rate of escape of the lattice Γ under the action of the diagonal subgroup of $\mathrm{SL}_n(\mathbb{R})$.

Lemma 2. *Let U be as in (1.2), and let $s \ge 1$. Then, we have*

$$S(\Lambda^\perp, s) \underset{n}{\ll} \frac{s^{n-1}}{\nu(\Gamma^\perp, (2^s \|U\|_2)^\star)}.$$

Proof. Since $\Lambda^\perp = U^{-1}\Gamma^\perp$, we conclude by (1.5) that

$$S(\Lambda^\perp, s) = \sum_{\delta \in \Delta_s} \frac{1}{\lambda_1^n(\delta U^{-1}\Gamma^\perp)} \underset{n}{\ll} \sum_{\delta \in \Delta_s} \frac{1}{\nu(\Gamma^\perp, \|U\delta^{-1}\|_2^\star)}.$$

Since $\#\Delta_s \underset{n}{\ll} s^{n-1}$, and since $\nu(\Gamma^\perp, \cdot)$ is non-increasing, we get

$$S(\Lambda^\perp, s) \underset{n}{\ll} \frac{s^{n-1}}{\nu(\Gamma^\perp, (2^s \|U\|_2)^\star)}.$$ □

We can now easily complete the proof of Theorem 1. By (1.4), we conclude

$$r = n^2 + \log \frac{\rho^n}{\nu(\Lambda^\perp, \rho)} \le n^2 + \log \frac{\rho^n}{\nu(\Gamma^\perp, \|U\|_2 \rho)} = R$$

Since $\nu(\Lambda^\perp, \cdot)$ is non-increasing, and since $(2^R \|U\|_2)^\star = 2^R \|U\|_2$ Lemma 2 yields

$$S(\Lambda^\perp, r) \underset{n}{\ll} \frac{R^{n-1}}{\nu(\Gamma^\perp, 2^R \|U\|_2)}. \tag{1.7}$$

By using Mahler's relation $1 \le \lambda_i(\Gamma^\perp)\lambda_{n+1-i}(\Gamma) \le n!$, where $i = 1, \ldots, n$, and Lemma 1, we obtain

$$\lambda_n^n(\Lambda) \underset{n}{\ll} \frac{1}{\lambda_1^n(U^{-1}\Gamma^\perp)} \underset{n}{\ll} \frac{1}{\nu(\Gamma^\perp, \|U\|_2^\star)}. \tag{1.8}$$

Taking (1.7) and (1.8) in (1.3) into account, it follows that

$$|\#(\Gamma \cap B) - \mathrm{vol}(B)| \underset{n}{\ll} \frac{1}{\nu\big(\Gamma^\perp, \|U\|_2^\star\big)} \left(\frac{\bar{t}^{n-1}}{\sqrt{\rho}} + \frac{R^{n-1}}{\nu(\Gamma^\perp, 2^R \|U\|_2)}\right)$$

which is (0.2).

1.3. **An Application — Proof of Corollary 1.** Throughout this subsection we fix the unimodular lattice $\Gamma = A\mathbb{Z}^2$ where

$$A := \frac{1}{\sqrt{\alpha}}\begin{pmatrix} 1 & \alpha \\ 1 & 2\alpha \end{pmatrix},$$

and we consider the aligned box

$$B := \frac{1}{\sqrt{\alpha}}\big([y,\, y+\varepsilon] \times [y,\, y+\alpha t]\big). \tag{1.9}$$

Then, the following relation holds

$$\#(B\cap\Gamma) = \#\left\{(p,q)\in\mathbb{Z}^2 : \begin{array}{c} 0 \le p+\alpha q - y \le \varepsilon, \\ 0 \le p + 2\alpha q - y \le \alpha t \end{array}\right\}.$$

Because of (0.11), we conclude that

$$|N_{\alpha,y}(\varepsilon,t) - \#(B\cap\Gamma)| \ll_{\alpha} 1. \tag{1.10}$$

In order to use Theorem 1, we need to control the characteristic quantity $\nu(\Gamma,\cdot)$ of the lattice Γ. This is where the Diophantine properties of α come into play.

Lemma 3. *Let ϕ be as in (0.9), and suppose $\rho > \gamma_2^{1/2}$. Then, we have*

$$\nu(\Gamma^{\perp},\rho) = \nu(\Gamma,\rho) \ge \frac{\phi(4\rho/\sqrt{\alpha})}{4}.$$

Proof. The claimed equality follows immediately from Proposition 1, and the remark thereafter. A vector $v\in\Gamma$ is of the shape

$$v = \frac{1}{\sqrt{\alpha}}\begin{pmatrix} z \\ z' \end{pmatrix}$$

where $z := p + q\alpha$, $z' := z + q\alpha$, and p,q denote integers. Assume that $\|v\|_2 \in (0,\rho)$. Observe that $q=0$ implies $\mathrm{Nm}(v) \ge 1 > 4^{-1}\phi(4\rho/\sqrt{\alpha})$. Therefore, we may assume $q\neq 0$. Since $z'-z = q\alpha$, one of the numbers $|z|,|z'|$ is at least $\frac{1}{2}\alpha|q|$, and both are bounded from below by $\frac{1}{2|q|}\phi(2|q|)$. Hence,

$$\mathrm{Nm}\,(v) \ge \frac{\alpha|q|}{2\sqrt{\alpha}}\cdot\frac{\phi(2|q|)}{2|q|\sqrt{\alpha}} \ge \frac{\phi(4\rho/\sqrt{\alpha})}{4}$$

where in the last step we used that $\frac{1}{2}\sqrt{\alpha}|q| \le \frac{1}{\sqrt{\alpha}}\min\{|z|,|z'|\} \le \|v\|_2 < \rho$. $\square$

Proof of Corollary 1. Let B be given by (1.9). Thus, B has sidelengths $t_1 = \alpha^{-1/2}\varepsilon$, and $t_2 = \sqrt{\alpha}t$. By (0.3) and (0.11), we are entitled to take $\rho := \varepsilon t > \gamma_2^{1/2}$ in Theorem 1. Moreover, (0.11) implies $t_1 < 1 < t_2$, and thus

$$T = \sqrt{\alpha\frac{t}{\varepsilon}} > \sqrt{\varepsilon t} > 2 > \gamma_2.$$

Hence, $T^{\star} = T$. By combining relation (1.10) and Theorem 1 with these specifications, it follows that

$$|N_{\alpha,y}(\varepsilon,t) - \varepsilon t| \ll_{\alpha} \frac{1}{\nu(\Gamma^{\perp},T)}\left(1 + \frac{R}{\nu(\Gamma^{\perp},2^{R}T)}\right). \tag{1.11}$$

By Lemma 3, the right hand side above is $\ll R(\phi(4T/\sqrt{\alpha})\phi(2^{R+2}T/\sqrt{\alpha}))^{-1}$. The first factor in the round brackets is larger than the second one, since ϕ is non-increasing. Hence, we conclude that the right hand-side of (1.11) is bounded by

$$\ll R(\phi(2^{R+2}T/\sqrt{\alpha}))^{-2}. \tag{1.12}$$

Furthermore, Lemma 3 yields

$$R \leq 4 + \log \frac{4(\varepsilon t)^2}{\phi(4t\sqrt{\varepsilon t})} \ll \log \frac{\varepsilon t}{\phi(4t\sqrt{\varepsilon t})}. \tag{1.13}$$

By using the first estimate from (1.13), we get

$$2^R \leq 2^4 \left(\frac{4(\varepsilon t)^2}{\phi(4t\sqrt{\varepsilon t})}\right)^{\log 2} < 2^{4+2\log 2} \frac{(\varepsilon t)^2}{\phi(4t\sqrt{\varepsilon t})}.$$

Hence, (1.12) is bounded from above by

$$\ll \frac{\log \frac{\varepsilon t}{\phi(4t\sqrt{\varepsilon t})}}{\phi^2\left(2^{6+2\log 2} \frac{(\varepsilon t)^2}{\phi(4t\sqrt{\varepsilon t})}\sqrt{\frac{t}{\varepsilon}}\right)} \leq \frac{\log E}{\phi^2(E')}.$$

This completes the proof of Corollary 1. □

2. Proof of Theorem 2 and Proposition 1

2.1. Comparing $\nu(\Gamma,\cdot)$ and $\nu(\Gamma^\perp,\cdot)$ and Proof of Proposition 1.

A natural question is whether one can state Theorem 1 in a way that is intrinsic in Γ, i.e. expressing $\nu(\Gamma^\perp,\cdot)$ in terms of $\nu(\Gamma,\cdot)$. However, for $n > 2$ there are weakly admissible lattices $\Gamma \subseteq \mathbb{R}^n$ such that $\Gamma^\perp$ is not weakly admissible as the following example shows.

Example 4. *Let $n \geq 3$, and let $A_0' \in GL_{n-1}(\mathbb{R})$ be such that the elements of each row of A_0' are $\mathbb{Q}$-linearly independent. Choose real $x_1, \ldots, x_{n-1}, y$ outside of the $\mathbb{Q}$-span of the entries of A_0', and suppose $y \neq x_{n-1}$. Let $x = (x_1, \ldots, x_{n-1})^T$ and let r_{n-1} be the last row of A_0'. Then, the matrix*

$$A_0 \coloneqq \begin{pmatrix} A_0' & x \\ r_{n-1} & y \end{pmatrix}$$

satisfies

- *(i) $A_0 \in GL_n(\mathbb{R})$, and*
- *(ii) the elements in each row of A_0 are $\mathbb{Q}$-linearly independent.*

The second assertion is clear and for the first suppose a linear combination of the rows vanishes. Using that the rows of A_0' are linearly independent over $\mathbb{R}$ and that $y \neq x_{n-1}$, the first claim follows at once. We now let A be the matrix we get from A_0 by swapping the first and the last row, and scaling each entry with $|\det A_0|^{-1/n}$. Clearly, (i) and (ii) remain valid for A, and the (n,n)-minor of A vanishes. We conclude that $\Gamma \coloneqq A\mathbb{Z}^n$ is a unimodular, and weakly admissible lattice; moreover, Cramer's rule implies that

$$(A^{-1})^T = \begin{pmatrix} \star & \star & \ldots & \star \\ \star & \ddots & \ddots & \vdots \\ \vdots & \ddots & \star & \star \\ \star & \ldots & \star & 0 \end{pmatrix}$$

where an asterisk denotes some arbitrary real number, possibly a different number each time. Hence, $\Gamma^\perp$ *contains a non-zero lattice point with a zero coordinate, and thus is not weakly admissible.*

Keeping Example 4 in mind, we now concern ourselves with finding large subclasses of lattices $\Gamma \subseteq \mathbb{R}^n$ such that

(1) Γ and $\Gamma^\perp$ are both weakly admissible,
(2) $\nu(\Gamma^\perp, \cdot) = \nu(\Gamma, \cdot)$.

It is easy to see that the first item holds for almost all lattices in the sense of the Haar-measure on the space $\mathcal{L}_n = \mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$ of unimodular lattices in $\mathbb{R}^n$. Moreover, we have the following criterion.

Lemma 4. *Suppose $A \in SL_n(\mathbb{R})$, and suppose that the entries of A are algebraically independent (over $\mathbb{Q}$). Then, $\Gamma \coloneqq A\mathbb{Z}^n$ and $\Gamma^\perp$ are both weakly admissible.*

Proof. First note that if K is a field and $X_1, \ldots, X_N$ are algebraically independent over K, then any non-empty collection of pairwise distinct monomials $X_1^{a_1} \cdots X_N^{a_N}$ is linearly independent over K. Next note that by Cramer's rule, each entry of $(A^{-1})^T$ is a sum of pairwise distinct monomials (up to sign) in the entries of A, and none of these monomials occurs in more than one entry of $(A^{-1})^T$. This shows that the entries of $(A^{-1})^T$ are linearly independent over $\mathbb{Q}$, in particular, the entries of any fixed row of $(A^{-1})^T$ are linearly independent over $\mathbb{Q}$. Thus, $\Gamma^\perp$ is weakly admissible. □

Next, we prove Proposition 1.

Proof (Proposition 1). Notice that S and S^{-1} are, up to signs of the entries, permutation matrices, and thus for every $w \in \mathbb{R}^n$

$$\mathrm{Nm}\,(w) = \mathrm{Nm}\,(Sw) = \mathrm{Nm}\,(S^{-1}w), \tag{2.1}$$

$$\|w\|_2 = \|Sw\|_2 = \left\|S^{-1}w\right\|_2. \tag{2.2}$$

Now let Aw be an arbitrary lattice point in $\Gamma = A\mathbb{Z}^n$. Then, since $W \in \mathbb{Z}^{n\times n}$, we get $(A^{-1})^T Ww \in \Gamma^\perp$. Since by hypothesis $A = S^{-1}((A^{-1})^T W)$, we conclude from (2.1) that $\mathrm{Nm}\,(Aw) = \mathrm{Nm}\,((A^{-1})^T Ww)$, and from (2.2) that $\|Aw\|_2 = \left\|(A^{-1})^T Ww\right\|_2$. This shows that $\nu(\Gamma^\perp, \cdot) \le \nu(\Gamma, \cdot)$.

Similarly, if $(A^{-1})^T w \in \Gamma^\perp$ then, since $W^{-1} \in \mathbb{Z}^{n\times n}$, we find that $AW^{-1}w \in \Gamma$, and using that $(A^{-1})^T = SAW^{-1}$ we conclude as above that $\nu(\Gamma, \cdot) \le \nu(\Gamma^\perp, \cdot)$. This proves Proposition 1. □

Remark 1. Let $I_m \coloneqq \mathrm{diag}(1, \ldots, 1)$ be the identity matrix, and 0_m the null matrix in $\mathbb{R}^{m\times m}$. Specialising

$$S = W = \begin{pmatrix} 0_m & I_m \\ -I_m & 0_m \end{pmatrix}$$

in Proposition 1, we conclude that if $\Gamma = A\mathbb{Z}^n$ with a symplectic matrix A, then

$$\nu(\Gamma^\perp, \cdot) = \nu(\Gamma, \cdot). \tag{2.3}$$

Moreover, it is easy to see that $\mathrm{Sp}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})$, and hence (2.3) holds for any unimodular lattice $\Gamma \subseteq \mathbb{R}^2$.

2.2. **Proof of Theorem 2.** Recall that $\alpha := (\alpha_1, \ldots, \alpha_n)^T \in \mathbb{R}^n$ is called badly approximable, if there is a constant $C = C(\alpha) > 0$ such that for any integer $q \geq 1$ the inequality

$$\max\{\|q\alpha_1\|, \ldots, \|q\alpha_n\|\} \geq \frac{C}{q^{1/n}} \tag{2.4}$$

holds where $\|\cdot\|$ denotes the distance to the nearest integer. By a well-known transference principle, cf. [12], assertion (2.4) is equivalent to saying that for all non-zero vectors $q := (q_1, \ldots, q_n)^T \in \mathbb{Z}^n$ the inequality

$$\|\langle \alpha, q\rangle\| \geq \frac{\tilde{C}}{\|q\|_2^n} \tag{2.5}$$

holds where $\tilde{C} = \tilde{C}(\alpha) > 0$ is a constant. Let $\mathbf{Bad}(n)$ denote the set of all badly approximable vectors in $\mathbb{R}^n$. The crucial step for constructing matrices generating the lattices announced in Theorem 2 is done by the following lemma.

Lemma 5. *Let $n \geq 3$ be an integer. Fix algebraically independent real numbers $c_{i,j}$ where $i, j = 1, \ldots, n$ and $i \neq j$. Then, there exist $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that the entries of each row of*

$$A := \begin{pmatrix} \lambda_1 & c_{1,2} & \cdots & c_{1,n} \\ c_{2,1} & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & c_{n-1,n} \\ c_{n,1} & \cdots & c_{n,n-1} & \lambda_n \end{pmatrix} \tag{2.6}$$

are algebraically independent, A is invertible, and each row-vector of $(A^{-1})^T$ is badly approximable.

For proving this lemma, we shall use the following special case of a recent theorem of Beresnevich concerning badly approximable vectors. We say that the map $F := (f_1, \ldots, f_n)^T : \mathcal{B} \to \mathbb{R}^n$, where $\mathcal{B} \subsetneq \mathbb{R}^m$ is a non-empty ball and $m, n \in \mathbb{N}$, is non-degenerate, if $1, f_1, \ldots, f_n$ are linearly independent functions (over $\mathbb{R}$).

Theorem 5 ([10, Thm. 1])**.** *Let n, m, k be positive integers. For each $j = 1, \ldots, k$ suppose that $F_j : \mathcal{B} \to \mathbb{R}^n$ is a non-degenerate, analytic map defined on a non-empty ball $\mathcal{B} \subsetneq \mathbb{R}^m$. Then,*

$$\dim_{\mathrm{Haus}} \bigcap_{j=1}^{k} F_j^{-1}(\mathbf{Bad}(n)) = m.$$

Proof of Lemma 5. We work in two steps. First, we set the scene to make use of Theorem 5.

(i) Let $M \in \mathbb{R}^{n\times n}$, and denote by $(M)_{i,j}$ the entry in the i-th row and j-th column of M. Moreover, we define a map $\tilde{F} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ by

$$\lambda := (\lambda_1, \ldots, \lambda_n)^T \mapsto \begin{pmatrix} \lambda_1 & c_{1,2} & \cdots & c_{1,n} \\ c_{2,1} & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & c_{n-1,n} \\ c_{1,n} & \cdots & c_{n,n-1} & \lambda_n \end{pmatrix}.$$

On a sufficiently small non-empty ball $\mathcal{B} \subsetneq \mathbb{R}^n$, centred at the origin, $\tilde{F}(\lambda)$ is invertible for every $\lambda \in \mathcal{B}$.[4] On this ball $\mathcal{B}$, we define F_j, for $j = 1, \ldots, n$, by mapping λ to the j-th row of $(\big(\tilde{F}(\lambda)\big)^{-1})^T$. We claim that F_j is a non-degenerate, and analytic map. By Cramer's rule, every entry of $((\tilde{F}(\lambda))^{-1})^T$ is the quotient of polynomials in $\lambda_1, \ldots, \lambda_n$ whereas the polynomial in the denominator does not vanish on $\mathcal{B}$. Hence, each F_j is an analytic function. Now we show that F_1 is non-degenerate, the argument for the other F_j being similar. The j-th component of F_1 is $(\big(\tilde{F}(\lambda)\big)^{-1})_{j,1}$ and, using Cramer's rule, is hence of the shape

$$(\det \tilde{F}(\lambda))^{-1}\left(\mathcal{R}_j + (-1)^{1+j} \prod_{k=2,\, k\neq j}^{n} \lambda_k\right)$$

where the polynomial $\mathcal{R}_j \in \mathbb{R}[\lambda_2, \ldots, \lambda_n]$ is of (total) degree $< n-1$, if $j = 1$, and of (total) degree $< n-2$, if $j = 2, \ldots, n$. Therefore, if a linear combination $k_0 + \sum_{j=1}^{n} k_j((\tilde{F}(\lambda))^{-1})_{j,1}$ with scalars $k_0, \ldots, k_n \in \mathbb{R}$ equals the zero-function $\mathbf{0} : \mathcal{B} \to \mathbb{R}$, then

$$\mathbf{0} = k_0 \cdot (\det \tilde{F}(\lambda)) + \sum_{j=1}^{n} k_j (-1)^{1+j} \prod_{k=2,\, k\neq j}^{n} \lambda_k + \sum_{j=1}^{n} k_j \mathcal{R}_j.$$

Comparing coefficients, we conclude that $k_0 = 0$ and thereafter $k_1 = k_2 = \cdots = k_n = 0$. Hence, F_1 is non-degenerate.

(ii) By part (i), Theorem 5 implies that the set M of all $\lambda \in \mathcal{B}$ such that $F_1(\lambda), \ldots, F_n(\lambda)$ are all badly approximable, has full Hausdorff dimension. Moreover, we claim that there is a set $M^{(1)} \subseteq M$ of full Hausdorff dimension such that for every $\lambda \in M^{(1)}$ the entries of the first row of $\tilde{F}(\lambda)$ are algebraically independent. Let M_1 be the subset of M of all elements $\lambda := (\lambda_1, \ldots, \lambda_n)^T \in M$ satisfying that $\{\lambda_1, c_{1,j} : j = 2, \ldots, n\}$ is algebraically *dependent*; observe that the possible values for λ_1 are countable, since $\mathbb{Z}[c_{1,2}, \ldots, c_{1,n}, x]$ is countable and every complex, non-zero, univariate polynomial has only finitely many roots. Therefore, M_1 is contained in a countable union of hyperplanes. It is well-known that if a sequence of sets $\{E_i\} \subseteq \mathbb{R}^n$ is given, then $\dim_{\mathrm{Haus}} \bigcup_{i\geq 1} E_i = \sup_{i\geq 1}\{\dim_{\mathrm{Haus}} E_i\}$, cf. [11, p. 65]. Consequently,

$$n = \dim_{\mathrm{Haus}} M = \max\{\dim_{\mathrm{Haus}}(M \setminus M_1),\, \dim_{\mathrm{Haus}} M_1\} = \dim_{\mathrm{Haus}}(M \setminus M_1),$$

and we define $M^{(1)} := M \setminus M_1$. Using the same argument, we conclude that there is a set $M^{(2)} \subseteq M^{(1)}$ of full Hausdorff dimension such that each of the first two rows of $\tilde{F}(\lambda)$ has algebraically independent entries for every $\lambda \in M^{(2)}$. Iterating this construction, we infer that there is a subset $M^{(n)} \subseteq M^{(n-1)} \subseteq \ldots \subseteq M$ of full Hausdorff dimension such that for every $\lambda \in M^{(n)}$ each row of the matrix $A := \tilde{F}(\lambda)$ has algebraically independent entries, and $(A^{-1})^T$ has badly approximable row vectors. Moreover, $\lambda \in M^{(n)} \subseteq \mathcal{B}$ implies that A is invertible.

□

We also need the following easy fact whose proof is left as an exercise.

[4]To see this, it suffices to show $\det \tilde{F}((0, \ldots, 0)^T) \neq 0$. However, by the Leibniz formula,

$$\det \tilde{F}(0, \ldots, 0) = \sum_{\sigma} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} c_{i,\sigma(i)}$$

where the sum runs through all fixpoint-free permutations of $\{1, \ldots, n\}$. Since $\{c_{i,j} : i, j = 1, \ldots, n,\, i \neq j\}$ is algebraically independent, the evaluation of the polynomial on the right hand side above cannot vanish, cf. proof of Lemma 4.

Lemma 6. *Let $m \in \mathbb{N}$, and let $\alpha \in \mathbb{R}$ be transcendental. Then, there are real numbers $\beta_1, \ldots, \beta_m$ such that $\beta_1, \alpha\beta_1, \beta_2, \ldots, \beta_m$ are algebraically independent.*

We are now in position to prove Theorem 2. First, we set $\tilde{\psi}(x) = \psi(x^2)$ such that for every $c > 0$ and $x \geq c$ we have $\tilde{\psi}(x) \leq \psi(cx)$. We may assume that $\tilde{\psi}(q) \ll \exp(-q)$. By writing down a suitable decimal expansion, we conclude that there exists a number $\alpha \in (0,1)$ such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{\tilde{\psi}(q)}{q^{n+1}} \tag{2.7}$$

has infinitely many coprime integer solutions p, $q \in \mathbb{Z}$; observe that such an α is necessarily transcendental. We apply Lemma 6 with $m = n^2 - n - 1$ and we set $c_{1,2} := \beta_1, c_{1,3} := \alpha\beta_1$, and we choose exactly one value β_k $(k \geq 2)$ for each of the remaining $c_{i,j}$ $(i \neq j)$. Thus, the real numbers $c_{i,j}$ are algebraically independent. We use Lemma 5 with these specifications to find A as in (2.6). For $l \in \mathbb{N}$ let p_l, q_l denote distinct solutions to (2.7), and put $v_l := (0, -p_l, q_l, 0, \ldots, 0)^T \in \mathbb{Z}^n$. Set $\tilde{A} := |\det A|^{-1/n} A$, and let us consider the unimodular, weakly admissible lattice $\Gamma := \tilde{A}\mathbb{Z}^n$. Then, the first coordinate of $\tilde{A}v_l$ equals

$$|\det A|^{-1/n} \left|-p_l c_{1,2} + q_l c_{1,3}\right| = |\det A|^{-1/n} \left|c_{1,2}\right| \left|q_l \alpha - p_l\right| \underset{A}{\ll} \frac{\tilde{\psi}(q_l)}{q_l^n}.$$

Since $\alpha \in (0,1)$, we may assume, by choosing l large enough, that $p_l \leq q_l$. Hence, the j-th coordinate for $j = 2, \ldots, n$ of $\tilde{A}v_l$ is $\underset{A}{\ll} q_l$. Thus, for l sufficiently large,

$$\mathrm{Nm}(\tilde{A}v_l) \underset{A}{\ll} \frac{\tilde{\psi}(q_l)}{q_l^n} \cdot q_l^{n-1} = \frac{\tilde{\psi}(q_l)}{q_l} \leq \frac{\psi(2\|\tilde{A}\|_2 q_l)}{q_l} \leq \frac{\psi(\|\tilde{A}v_l\|_2)}{q_l}.$$

Choosing $\rho_l = \|\tilde{A}v_l\|_2$, we conclude that $\nu(\Gamma, \rho_l) \leq \psi(\rho_l)$ for all l sufficiently large.

Because the rows of $(A^{-1})^T$ are badly approximable vectors by construction, $\Gamma^\perp$ is weakly admissible. Moreover, by (2.5), we conclude that $\mathrm{Nm}((A^{-1})^T v) \underset{A}{\gg} \|v\|_2^{-n^2}$ for every non-zero $v \in \mathbb{Z}^n$. Also note that $\left\|(A^{-1})^T v\right\|_2 < \rho$ implies $\|v\|_2 < \|A^T\|_2 \rho$. This implies that $\nu(\Gamma^\perp, \rho) \underset{A}{\gg} \rho^{-n^2}$. Hence, Γ has the desired properties.

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