Contributions to Metric Number Theory

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Abstract

The aim of this work is to investigate some arithmetical properties of real numbers, for example by considering sequences of the type $([b^n\alpha])$, $n = 1, 2, \ldots$ where $b \in \mathbb{N}, \alpha \in \mathbb{R}$, the terms of the sequences being in arithmetical progression, square-free, sums of two squares or primes. The results are most commonly proved for almost all $\alpha \in \mathbb{R}$ or $(\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$ (in the sense of Lebesgue measure).

In the first chapter normal numbers are studied. The concept of a normal number is generalised by defining normal points in higher dimensions, and through the link between normal numbers and uniform distribution, it is proved that almost all points on the curve $(\alpha, \alpha^2, \ldots, \alpha^m) \in \mathbb{R}^m$ are normal.

The second chapter includes a construction that yields normal numbers. This follows on from a result by Davenport and Erdős which shows that $0.f(1)f(2)f(3)\cdots$ is normal for any polynomial $f(x)$ which takes only positive integer values at $x = 1, 2, \ldots$. The result proved here replaces $f(x)$ by $[g(x)]$ where $g(x) = a_1x^{\alpha_1} + a_2x^{\alpha_2} + \cdots + a_kx^{\alpha_k}$ for the α_i, a_i any positive real numbers.

The third chapter considers square-free numbers and gives for almost all α, an asymptotic formula for the number of solutions in n to $[10^n\alpha^{a_1}]$, $[10^n\alpha^{a_2}], \ldots, [10^n\alpha^{a_k}]$ simultaneously square-free for $n \leq N$, where each $a_i \in \mathbb{N}$.

The fourth chapter considers sums of two squares and gives for almost all $(\alpha, \beta) \in \mathbb{R}^2$ an asymptotic formula for the number of solutions to $[10^n\alpha]$ and $[10^n\beta]$ simultaneously sums of two squares for $n \leq N$.

The final chapter investigates the set of $(\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$ such that

$[10^n\alpha_1], [10^n\alpha_2], \ldots, [10^n\alpha_m]$ are simultaneously prime infinitely often. This set is shown to have Hausdorff dimension m and to be dense in $\mathbb{R}^m$.

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Notation

$\lambda(\,)$	represents the Lebesgue measure of a subset of $\mathbb{R}$ or $\mathbb{R}^n$.
$A \ll B$	means $A = O(B)$.
$e(x)$	represents $e^{2\pi ix}$.
$\mu(S)$	where S is a set, represents any measure.
[x]	represents the integer part of x.
$A \sim B$	means $A = B(1 + o(1))$.
$\mu(d)$	where d is an integer, represents the Möbius function.
$\parallel x \parallel$	represents the distance from x to the nearest integer.
$\{x\}$	represents the fractional part of x.
(a, b)	represents the greatest common divisor of a and b.
$\phi(d)$	where d is an integer, represents the Euler function, except in chapter 3.
$\pi(x)$	represents the number of primes less than x.
$\tau(d)$	represents the number of divisors of d.
$\zeta(s)$	represents the Riemann zeta-function.

Chapter 1

Normal Numbers and Uniform Distribution

We begin by defining normal numbers. Let $b \geq 2$ be an integer. For any real number α, let $A(d,b,N)$ denote the number of occurrences of the digit d among the first N digits of the fractional part of the expansion of α to base b. We first define simply normal numbers, following Borel in [1]:

Definition A real number α is said to be simply normal to base b if

$$\lim_{N \to \infty} \frac{A(d,b,N)}{N} = \frac{1}{b}$$

for each d with $0 \leq d \leq b-1$.

Thus a number simply normal to base b has on average the same number of occurrences of each possible digit in its expansion to base b.

We also define entirely normal and absolutely normal numbers as in [2]:

Definition α is said to be entirely normal to base b if it is simply normal to all bases b^n for n a natural number.

α is said to be absolutely normal if it is simply normal to every base $b > 1$. If α is said to be normal it is usually meant that it is absolutely normal and this abbreviation is used here.

Thus the frequency with which any possible digit occurs in the decimal expansion of a normal number, to any base, is the same as that of any other possible digit.

Borel, in 1909 [1], proved that almost all real numbers are absolutely normal. The aim here will be to generalise the concept of a normal number to higher dimensions and prove that $(\alpha, \alpha^2, \ldots \alpha^m)$ is a normal point (which will be defined) of $\mathbb{R}^m$ for almost all α. We first prove a simple consequence of Borel's normal number theorem:

Theorem 1.1. *Let $k \geq 1$. Then for almost all α, $\alpha, \alpha^2, \ldots, \alpha^k$ are all normal.*

Proof. Let $1 \leq i \leq k$. Since every positive real number can be written as α^i for some α, we have by Borel's theorem that almost all α^i are normal. In other words, the set of α^i which are not normal has measure zero. This implies that the set of α such that α^i is not normal has measure zero.

The above argument holds for all $1 \leq i \leq k$ and therefore the set of α such that for at least one $i \leq k$, α^i is not normal, has measure zero because it is the union of a finite number of sets each with measure zero. The complement of this set is the set of α such that $\alpha, \alpha^2, \ldots, \alpha^k$ are all normal and thus consists of almost all α as required.

For what follows we will need a different but equivalent formulation of an entirely normal number; it will be defined in terms of the number of occurrences of blocks of consecutive digits in the number.

Let B_k denote a block of k digits. We define $A(B_k, b, N)$ to be the number of occurrences of the block of digits B_k in the first N digits of the fractional part of α in base b. We will require the following theorem proved in [2] and proved earlier in [1] using a different, though again equivalent, definition of an entirely normal number (Borel defines α to be entirely normal to base b if $b^n\alpha$ is simply normal to base b^m for all n, m):

Theorem 1.2. *A real number α is entirely normal to the base b if and only if*

$$\lim_{N\to\infty} \frac{A(B_k, b, N)}{N} = \frac{1}{b^k}$$

for all $k \geq 1$ and all B_k.

In higher dimensions we define for an element $\alpha = (\alpha_1, \ldots \alpha_m)$ of $\mathbb{R}^m$, the expression $A(\mathcal{B}_k, \mathbf{b}, N)$ to be the number of occurrences of the m-dimensional element $\mathcal{B}_k \in \mathbb{Z}_{b_1^k} \times \cdots \times \mathbb{Z}_{b_m^k}$, formed of m blocks each of k digits, in the first N digits of the fractional part of $(\alpha_1, \ldots \alpha_m)$ to base $\mathbf{b} = (b_1, \ldots b_m)$.

We next define an m-dimensional analogue of a normal number.

Definition $(\alpha_1, \ldots \alpha_m)$ is a normal point of $\mathbb{R}^m$ to base $\mathbf{b} = (b_1, \ldots b_m)$ if

$$\lim_{N\to\infty} \frac{A(\mathcal{B}_k, \mathbf{b}, N)}{N} = \frac{1}{(b_1 \cdots b_m)^k}$$

for all $k \geq 1$ and all $\mathcal{B}_k \in \mathbb{Z}_{b_1^k} \times \cdots \times \mathbb{Z}_{b_m^k}$.

We define the point to be (absolutely) normal if it is normal to every base $\mathbf{b} = (b_1, \ldots b_m)$.

We now need to define uniform distribution of a sequence.

Definition Let (x_n) be a sequence of real numbers. The discrepancy of (x_n) modulo one, $D_N(x_n)$, is defined by

$$D_N(x_n) = \sup_{\mathcal{I}\subset[0,1)} \left| \sum_{\substack{n=1 \\ \{x_n\}\in\mathcal{I}}}^{N} 1 - N\lambda(\mathcal{I}) \right|$$

where $\mathcal{I}$ denotes an interval.

Thus the discrepancy measures the maximum deviation, over all intervals, of the distribution of the fractional parts of the sequence from a uniform distribution.

Definition A sequence (x_n) is said to be uniformly distributed modulo 1 if

$$\lim_{N\to\infty} \frac{D_N(x_n)}{N} = 0$$

The following well known theorem, proved in [2] (Theorem 5.2), shows how normality can be expressed in terms of uniform distribution of a sequence:

Theorem 1.3. *Let $b > 1$ be an integer. Then α is entirely normal to base b exactly when the sequence (αb^n) is uniformly distributed modulo 1.*

To produce a higher dimensional analogue of this theorem, first we define uniform distribution (and discrepancy) in $\mathbb{R}^m$. This is done in the expected way.

Definition The discrepancy of a sequence $(x_{n1},\ldots,x_{nm})$ in $\mathbb{R}^m$ modulo 1, is defined by

$$D_N(x_{n1},\ldots,x_{nm}) = \sup_{\mathcal{B}\subset[0,1)^m}\left|\sum_{\substack{n=1\\ \{x_{n1}\}\times\cdots\times\{x_{nm}\}\in\mathcal{B}}}^{N} 1 - N\lambda(\mathcal{B})\right|$$

where $\mathcal{B}$ is of the form $\mathcal{I}_1\times\cdots\times\mathcal{I}_m$ where the $\mathcal{I}_i$s are intervals.

The sequence is said to be uniformly distributed modulo 1 in $[0,1)^m$ if

$$\lim_{N\to\infty}\frac{D_N(x_{n1},\ldots,x_{nm})}{N} = 0.$$

We can now prove the analogue of theorem 1.3 for higher dimensions. The proof is a generalisation of the proof in [2] of Theorem 1.2.

Theorem 1.4. *$(b_1^n\alpha_1,\ldots,b_m^n\alpha_m)$ is uniformly distributed modulo 1 in $[0,1)^m$ if and only if $(\alpha_1,\ldots,\alpha_m)$ is a normal point of $\mathbb{R}^m$ to base $\mathbf{b}=(b_1,\ldots,b_m)$.*

Proof. Suppose $\alpha=(\alpha_1,\ldots,\alpha_m)$ is a normal point. Let $\varepsilon>0$ and let k be an integer such that $4mb_i^{-k}<\varepsilon$ for all i. Since $(\alpha_1,\ldots,\alpha_m)$ is a normal point,

$$\left|\frac{A(\mathcal{B}_k,\mathbf{b},N)}{N}-\frac{1}{(b_1\cdots b_m)^k}\right|<\frac{\varepsilon}{2(b_1\cdots b_m)^k} \tag{1.1}$$

for every block $\mathcal{B}_k$ and every $N>N(\varepsilon,\mathcal{B}_k)$ for some $N(\varepsilon,\mathcal{B}_k)$.

Given any $\mathcal{B}\subset[0,1)^m$ we can find integers $c_1,\ldots c_m, d_1,\ldots d_m$ such that

$$\mathcal{B}\subset\left[\frac{c_1-1}{b_1^k},\frac{d_1+1}{b_1^k}\right]\times\cdots\times\left[\frac{c_m-1}{b_m^k},\frac{d_m+1}{b_m^k}\right]$$

and

$$\left[\frac{c_1}{b_1^k},\frac{d_1}{b_1^k}\right]\times\cdots\times\left[\frac{c_m}{b_m^k},\frac{d_m}{b_m^k}\right]\subset\mathcal{B}$$

and

$$\left(\frac{d_1-c_1}{b_1^k}\right)\cdots\left(\frac{d_m-c_m}{b_m^k}\right)\le\lambda(\mathcal{B})\le\left(\frac{d_1-c_1+2}{b_1^k}\right)\cdots\left(\frac{d_m-c_m+2}{b_m^k}\right).$$

If $g_1,\ldots,g_m$ are integers with $0\le g_i\le b_i^k-1$ for all i, write $\mathcal{B}_k(\mathbf{g})$ to denote the expansion of $\mathbf{g}=(g_1,\ldots,g_m)$ to base $\mathbf{b}$. Also define $\{\mathbf{b}^n\alpha\}=(\{b_1^n\alpha_1\},\ldots,\{b_m^n\alpha_m\})$. Then by the restrictions on $\mathcal{B}$ we have that:

$$\sum_{g_1=c_1}^{d_1-1}\cdots\sum_{g_m=c_m}^{d_m-1}A(\mathcal{B}_k(\mathbf{g}),\mathbf{b},N)\le\sum_{\substack{n=1\\ \{\mathbf{b}^n\alpha\}\in\mathcal{B}}}^{N}1$$

$$\le\sum_{g_1=c_1-1}^{d_1}\cdots\sum_{g_m=c_m-1}^{d_m}A(\mathcal{B}_k(\mathbf{g}),\mathbf{b},N).$$

Now using the second of these inequalities with the lower bound for $\lambda(\mathcal{B})$, we obtain

$$\frac{\sum' 1-N\lambda(\mathcal{B})}{N}\le$$

$$\frac{\sum_{g_1=c_1-1}^{d_1}\cdots\sum_{g_m=c_m-1}^{d_m}A(\mathcal{B}_k(\mathbf{g}),\mathbf{b},N)}{N}-$$

$$\left(\frac{d_1-c_1}{b_1^k}\right)\cdots\left(\frac{d_m-c_m}{b_m^k}\right)$$

$$<\left(\frac{1}{(b_1\cdots b_m)^k}+\frac{\varepsilon}{2(b_1\cdots b_m)^k}\right)\prod_{i=1}^{m}(d_i-c_i+2)-\frac{\prod_{i=1}^{m}(d_i-c_i)}{(b_1\cdots b_m)^k}\qquad(1.2)$$

by (1.1), where the sum $\sum'$ is over $1\le n\le N$ such that $\{\mathbf{b}^n\alpha\}\in\mathcal{B}$ and we are choosing

$$N>\max_{\mathbf{g}}N(\varepsilon,\mathcal{B}_k(\mathbf{g})),$$

$\max_{\mathbf{g}}$ being over $g_i=c_i-1,\ldots,d_i$ for each i. The expression (1.2) is

$$<\frac{\prod_{i=1}^{m}(d_i-c_i+2)-\prod_{i=1}^{m}(d_i-c_i)}{(b_1\cdots b_m)^k}+\varepsilon/2\qquad(1.3)$$

since $d_i-c_i+2<b_i^k$ for all i.

The first term in (1.3) is equal to

$$\left(2\sum_{j=1}^{m}\prod_{\substack{i=1\\i\neq j}}^{m}(d_i-c_i+\mu_i)\right)/(b_1\cdots b_m)^k$$

for some μ_is with $0\le\mu_i\le 2$ by the mean value theorem. Thus (1.3) is

$$<\frac{2m(d_l-c_l+\mu_l)^{m-1}}{(b_l)^{mk}}+\varepsilon/2$$

for some l,

$$<2m/b_l^k+\varepsilon/2<\varepsilon/2+\varepsilon/2=\varepsilon.$$

as required.

Similarly

$$\frac{N\lambda(\mathcal{B})-\sum' 1}{N}<\varepsilon$$

follows by using the upper bounds given in place of lower bounds and vice versa which leads to the same expression (1.3), hence

$$\lim_{N\to\infty}\frac{D_N(b_1^n\alpha_1,\ldots,b_m^n\alpha_m)}{N}=0$$

i.e. $(b_1^n\alpha_1,\ldots,b_m^n\alpha_m)$ is uniformly distributed modulo 1 in $[0,1)^m$.

Now instead suppose that $(b_1^n\alpha_1,\ldots,b_m^n\alpha_m)$ is uniformly distributed modulo 1 in $[0,1)^m$. Then

$$A(\mathcal{B}_k(\mathbf{g}),\mathbf{b},N)=\sum_{\substack{n=1\\\{\mathbf{b}^n\alpha\}\in\mathcal{B}}}^{N}1$$

where

$$\mathcal{B}=\left[\frac{g_1}{b_1^k},\frac{g_1+1}{b_1^k}\right)\times\cdots\times\left[\frac{g_m}{b_m^k},\frac{g_m+1}{b_m^k}\right).$$

Hence, with $\sum'$ as before, we have

$$\left|\frac{A(\mathcal{B}_k,\mathbf{b},N)}{N}-\frac{1}{(b_1\cdots b_m)^k}\right|=\left|\frac{\sum' 1-N\lambda(\mathcal{B})}{N}\right|$$

$$\leq \frac{D_N(b_1^n\alpha_1, \ldots, b_m^n\alpha_m)}{N}$$

which tends to zero as $N \to \infty$ by assumption. Hence $(\alpha_1, \ldots, \alpha_m)$ is a normal point.

For the main theorem below we need the following lemma ([3], page 90):

Lemma 1.1. *Suppose that $f(x)$ is a real-valued function with a monotonic kth-derivative for $x \in [a, b]$, which satisfies $|f^{(k)}(x)| \geq \lambda > 0$ Then*

$$\left| \int_a^b e(f(\alpha))d\alpha \right| \ll \lambda^{-1/k}.$$

We next state a lemma which provides an upper bound for a function $F(N, \alpha)$ for almost all α, when the integral over α of the square of the function is given. It is proved using a variance argument. It is given in [2] (Lemma 5.4).

Lemma 1.2. *Let Y be a measure space with measure μ such that $0 < \mu(Y) < \infty$. Let $F(n, m, \alpha)$ for $n \geq 1$ and $m \geq 0$ be a double sequence of μ-measurable functions and let x_n be a sequence of non-negative real numbers such that*

$$|F(n, n-1, \alpha)| \leq x_n$$

for all n.

Let

$$X(N) = \sum_{n=1}^{N} x_n$$

and suppose that $X(\infty)$ diverges. Also suppose that for any u, v with $0 \leq u < v$ we have

$$\int_Y |F(u, v, \alpha)|^2 d\mu < K \sum_{n=u}^{v} x_n$$

where K is a constant.

Then for almost all α and any $\varepsilon > 0$, we have

$$F(N, 0, \alpha) = O(X^{1/2}(N)(\log(X(N) + 2))^{\frac{3}{2}+\varepsilon} + \max_{n \le N} x_n).$$

There are many variations of this theorem which give different bounds. We next give a variation which is needed for the proof of the main theorem. It is obtained by the method of Lemma 1.4 in [2].

Lemma 1.3. *Let X be a measure space with finite measure μ. Suppose $F_N(\alpha)$ is a measurable function which satisfies*

$$|F_N(\alpha) - F_{N-1}(\alpha)| \ll 1.$$

Suppose also that

$$\int_X |F_N(\alpha)|^2 d(\mu) \ll N^{1+\gamma}$$

for all N, where $0 \le \gamma < 1$. Then for almost all α,

$$F_N(\alpha) \ll N^{\frac{2+\gamma}{3}} (\log N)^{\frac{1}{2}+\varepsilon}$$

for any $\varepsilon > 0$.

We also need the following theorem [4]:

Theorem 1.5. *(The Weyl Criterion)*

A sequence $(x_{n1}, x_{n2}, \dots, x_{nm})$ is uniformly distributed modulo 1 in $[0, 1)^m$ if and only if

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} e(l_1 x_{n1} + \cdots + l_m x_{nm}) = 0$$

for all sets of integers $l_1, \dots l_m$, not all zero.

We are now in a position to be able to prove the main theorem of the chapter.

Theorem 1.6. $(\alpha, \alpha^2, \dots, \alpha^m)$ *is a normal point of $\mathbb{R}^m$ for almost all α.*

Proof. By Theorem 1.4, it suffices to prove that for any $b_1, \ldots, b_m$, the sequence $(\alpha b_1^n, \ldots, \alpha^m b_m^n)$ is uniformly distributed modulo 1 in $[0,1)^m$ for almost all α. We prove this using the Weyl criterion. Let $(l_1, \ldots, l_m) \in \mathbb{Z}^m - \{\mathbf{0}\}$. Consider

$$\int_0^1 \left| \sum_{n=1}^{N} e(l_1 b_1^n \alpha + \cdots + l_m b_m^n \alpha^m) \right|^2 d\alpha.$$

This is equal to

$$\sum_{n_1=1}^{N} \sum_{n_2=1}^{N} \int_0^1 e(l_1(b_1^{n_1} - b_1^{n_2})\alpha + \cdots + l_m(b_m^{n_1} - b_m^{n_2})\alpha^m) d\alpha$$

$$= N + \sum_{\substack{n_1=1 \\ n_2=1 \\ n_1 \neq n_2}}^{N} \int_0^1 e(l_1(b_1^{n_1} - b_1^{n_2})\alpha + \cdots + l_m(b_m^{n_1} - b_m^{n_2})\alpha^m) d\alpha.$$

Let

$$f(\alpha) = l_1(b_1^{n_1} - b_1^{n_2})\alpha + \cdots + l_m(b_m^{n_1} - b_m^{n_2})\alpha^m$$

for $1 \leq n_1, n_2 \leq N, n_1 \neq n_2$, and let k be the greatest integer such that $l_k \neq 0$ so that we have $l_j = 0$ for all $j > k$. Then the k-th derivative of $f(\alpha)$ satisfies

$$f^{(k)}(\alpha) = k! l_k (b_k^{n_1} - b_k^{n_2}).$$

Thus $f^{(k)}(\alpha)$ is certainly monotonic for $\alpha \in [0,1]$ and

$$|f^{(k)}(\alpha)| = |k! l_k (b_k^{n_1} - b_k^{n_2})| > 0$$

since $l_k \neq 0$ and $n_1 \neq n_2$. Therefore by Lemma 1.1:

$$\left| \int_0^1 e(f(\alpha)) d\alpha \right| \ll \frac{1}{(k! |l_k (b_k^{n_1} - b_k^{n_2})|)^{1/k}}$$

and hence

$$\left|\sum_{\substack{n_1=1\\n_2=1\\n_1\neq n_2}}^{N}\int_0^1 e(f(\alpha))d\alpha\right| \ll \left(\frac{1}{|l_k|k!}\right)^{1/k}\sum_{\substack{n_1=1\\n_2=1\\n_1\neq n_2}}^{N}\left(\frac{1}{|b_k^{n_1}-b_k^{n_2}|}\right)^{1/k}$$

$$= 2\left(\frac{1}{|l_k|k!}\right)^{1/k}\sum_{n_2=1}^{N-1}\sum_{n_1>n_2}^{N}\left(\frac{1}{b_k^{n_1}-b_k^{n_2}}\right)^{1/k}$$

$$= 2\left(\frac{1}{|l_k|k!}\right)^{1/k}\sum_{n_2=1}^{N-1}\left(\frac{1}{b_k^{n_2}}\right)^{1/k}\sum_{n_1>n_2}^{N}\left(\frac{1}{b_k^{n_1-n_2}-1}\right)^{1/k}.$$

Now

$$\sum_{n_1>n_2}^{N}\left(\frac{1}{b_k^{n_1-n_2}-1}\right)^{1/k} < \sum_{r=1}^{N-n_1}\left(\frac{2}{b_k^r}\right)^{1/k} < K_1$$

for some constant K_1 , since we have a convergent sum. Therefore

$$\left|\sum_{\substack{n_1=1\\n_2=1\\n_1\neq n_2}}^{N}\int_0^1 e(f(\alpha)d\alpha\right| < K_2$$

for some constant K_2. Therefore

$$\int_0^1\left|\sum_{n=1}^{N} e(l_1b_1^n\alpha+\cdots+l_mb_m^n\alpha^m)\right|^2 d\alpha < N+K_2 = O(N).$$

It follows that for any $\varepsilon > 0$,

$$\sum_{n=1}^{N} e(l_1b_1^n\alpha+\cdots+l_mb_m^n\alpha^m) = O(N^{2/3}(\log N)^{(1/2)+\varepsilon})$$

for almost all α, by applying Lemma 1.3 with $F_N(\alpha)=\sum_{n=1}^{N} e(l_1b_1^n\alpha+\cdots+l_mb_m^n\alpha^m)$.

Now by the Weyl criterion (Theorem 1.5), we obtain the required result that $(\alpha b_1^n,\ldots,\alpha^m b_m^n)$ is uniformly distributed modulo 1 in $[0,1)^m$ for almost all α. Hence $(\alpha, \ldots, \alpha^m)$ is a normal point of $\mathbb{R}^m$ to base $\mathbf{b} = (b_1, \ldots, b_m)$ for almost all α.

Since this result holds for any $b_1, \ldots, b_m$, we can deduce that $(\alpha, \ldots, \alpha^m)$ is an absolutely normal point for almost all α.

Chapter 2

Constructing Normal Numbers

We would like to investigate constructions that yield normal numbers. As examples of these, Champernowne [5] in 1933 proved that the number formed as follows, from the natural numbers written consecutively, 0.1234567891011..., is normal and Besicovitch [6] in 1934 proved that the decimal 0.1491625..., formed from the sequence of square numbers, is normal. It is natural to search for integer sequences that yield normal numbers in this way. In this connection Copeland and Erdős (1946) [7] have proved that if $p_1, p_2, \ldots$ is a sequence of positive integers such that for every $\theta < 1$, the number of p's up to n is greater than n^θ if n is sufficiently large, then the infinite decimal $0.p_1p_2p_3\ldots$ is normal. This includes the result that the decimal formed from the sequence of primes 0.23571113... is normal. Also, Davenport and Erdős (1952) [8] have proved the result, conjectured by Copeland and Erdős, that if $f(x)$ is any polynomial in x whose values for $x = 1, 2, 3, \ldots$ are all positive integers, then the decimal $0.f(1)f(2)f(3)\ldots$ is normal. Nakai and Shiokawa [9] have proved that the same result holds for $0.f(2)f(3)f(5)\cdots$ where the sequence of primes replaces the sequence of integers.

Here we prove a result where we replace polynomial sequences by sequences formed from integer parts of real sequences. The idea of the proof follows that of Davenport and Erdős' theorem. First we need a lemma (this is Lemma 5 in [10]):

Lemma 2.1. *Let F be a real function on $Z \leq z \leq Z_1$ where $Z_1 \leq 2Z$. Suppose that for some k with $2 \leq k \ll 1$, the $k+1$th derivative of F is continuous in this interval and satisfies*

$$M^{-1} \ll \frac{f^{(k+1)}(x)}{(k+1)!} \ll M^{-1},$$

where $Z \ll M \ll Z^2$. Then

$$\sum_{Z<z\leq Z_1} e(F(z)) \ll Z^{1-t}$$

where

$$t = \frac{1}{3k^2 \log(125k)}.$$

We also need the following lemma on finite Fourier series. It is proved in [11] (Lemma 2.7):

Lemma 2.2. *Let $\mathcal{I} \subset [0,1)$ and let $\chi(\alpha)$ be equal to 1 if $\alpha \in \mathcal{I}$ and equal to 0 otherwise. Then for any integer $L \geq 1$ there are trigonometric polynomials $T_i(\alpha)$ for $i = 1, 2$, such that $T_1(\alpha) \leq \chi(\alpha) \leq T_2(\alpha)$ and*

$$T_i(\alpha) = \sum_{t=-L}^{L} \hat{T}_i(t) e(t\alpha)$$

where

$$\hat{T}_1(0) = \lambda(\mathcal{I}) - \frac{1}{L+1}, \; \hat{T}_2(0) = \lambda(\mathcal{I}) + \frac{1}{L+1}$$

and

$$|\hat{T}_i(t)| \leq \min\left(\lambda(\mathcal{I}) + (-1)^i \frac{1}{L+1}, \frac{3}{2|t|}\right).$$

We can now prove the following:

Theorem 2.1. *Let $\alpha > 1$ be real and not an integer. Then the decimal $0.[1^\alpha][2^\alpha][3^\alpha]\cdots$ is normal to base 10.*

Proof. Let $N(u,t)$ be the number of times a particular combination of s digits occurs among the $(u+1)$st to the tth digits of the decimal $0.[1^\alpha][2^\alpha][3^\alpha]\cdots$ and let $N(t) = N(0,t)$. Thus we need to show that

$$\lim_{t\to\infty} \frac{N(t)}{t} = 10^{-s}.$$

We have

$$N(u,t) \le N(t) - N(u) \le N(u,t) + s - 1 \tag{2.1}$$

since the combinations counted by $N(t) - N(u)$ are those in $N(u,t)$ together with any that include the uth and $u+1$st digits. Let x_n be the largest integer for which $[x_n^\alpha]$ has less than n digits. Then if n is sufficiently large (as will be supposed throughout), $[(x_n+1)^\alpha]$ has n digits and so do all of $[(x_n+2)^\alpha], \ldots, [x_{n+1}^\alpha]$.

We have $x_n \sim 10^{-1/\alpha}10^{n/\alpha}$ as $n \to \infty$. Suppose the last digit of $[x_n^\alpha]$ occupies the t_nth place in the decimal $0.[1^\alpha][2^\alpha]\cdots$. Then the number of digits in the block $[(x_n+1)^\alpha][(x_n+2)^\alpha]\cdots[x_{n+1}^\alpha]$ is $t_{n+1} - t_n$ and is also $n(x_{n+1} - x_n)$. Hence $t_n \sim 10^{-1/\alpha} n 10^{n/\alpha}$ as $n \to \infty$.

It suffices to prove that

$$N(t_n, t) = 10^{-s}(t - t_n) + o(t_n) \tag{2.2}$$

as $n \to \infty$, for $t_n < t \le t_{n+1}$. This is since, by (2.1), we have

$$N(t) - N(t_h) = \sum_{r=h}^{n-1} N(t_r, t_{r+1}) + N(t_n, t) + R$$

for a suitable fixed h, where $|R| < ns$, and (2.2) includes the special case

$$N(t_r, t_{r+1}) = 10^{-s}(t_{r+1} - t_r) + o(t_r),$$

hence we obtain

$$N(t) = 10^{-s}t + o(t),$$

which proves the theorem.

To prove (2.2), suppose without loss of generality that t differs from t_n by an exact multiple of n. Putting $t = t_n + nX$, the number $N(t_n, t)$ is the number of times that the given combination of s digits occurs in the block

$$[(x_n + 1)^\alpha][(x_n + 2)^\alpha] \cdots [(x_n + X)^\alpha]$$

where $0 < X \leq x_{n+1} - x_n$. We need only consider combinations occurring entirely in the same $[x^\alpha]$, since the others number at most $(s-1)(x_{n+1} - x_n)$ which is $o(t_n)$.

The number of times that a given combination of digits, $a_1 \cdots a_s$ occurs in a particular $[x^\alpha]$ is the same as the number of values of m with $s \leq m \leq n$ for which the fractional part of $10^{-m}x^\alpha$ begins with the decimal $0.a_1 \cdots a_s$. If we define $\theta(z)$ to be 1 if z is congruent modulo 1 to a number lying in a certain interval of length 10^{-s}, and 0 otherwise, the number of times the given combination occurs in $[x^\alpha]$ is

$$\sum_{m=s}^{n} \theta(10^{-m}x^\alpha).$$

Hence

$$N(t_n, t) = \sum_{x=x_n+1}^{x_n+X} \sum_{m=s}^{n} \theta(10^{-m}x^\alpha) + O(x_{n+1} - x_n),$$

the error term arising from the combinations, already mentioned, that begin in one $[x^\alpha]$ and end in the next.

We shall prove that if δ is any fixed positive number, and $\delta n < m < (1-\delta)n$, then

$$\sum_{x=x_n+1}^{x_n+X} \theta(10^{-m}x^{\alpha}) = 10^{-s}X + o(x_{n+1} - x_n) \tag{2.3}$$

uniformly in m. This implies that

$$\sum_{m=s}^{n} \sum_{x=x_n+1}^{x_n+X} \theta(10^{-m}x^{\alpha}) = 10^{-s}nX + o(n(x_{n+1} - x_n))$$

since the contribution of the remaining values of m is at most $2\delta nX$ where δ is arbitrarily small. This will prove (2.2).

Using Lemma 2.2 we can construct for any $\eta > 0$, functions $\theta_1(z)$ and $\theta_2(z)$, periodic in z with period 1, such that $\theta_1(z) \leq \theta(z) \leq \theta_2(z)$, having Fourier expansions of the form

$$\theta_1(z) = 10^{-s} - \eta + \sum_{\nu} \hat{\theta}_1(\nu)e(\nu z),$$

$$\theta_2(z) = 10^{-s} + \eta + \sum_{\nu} \hat{\theta}_2(\nu)e(\nu z).$$

where the summation is over all integers ν with $|\nu| < \eta^{-1}$, $\nu \neq 0$, and

$$|\hat{\theta}_i(\nu)| \leq \frac{1}{|\nu|}.$$

Using these functions to approximate $\theta(10^{-m}x^{\alpha})$ in (2.3), we see that it is sufficient to estimate the sum

$$S_{n,m,\nu} = \sum_{x=x_n+1}^{x_n+X} e(10^{-m}\nu x^{\alpha}).$$

We shall prove

$$|S_{n,m,\nu}| = o(x_{n+1} - x_n)$$

for all m and ν such that $\delta n < m < (1-\delta)n$ and $1 \le \nu < \eta^{-1}$. This will prove (2.3).

Let $m = ln$ where $l \in (\delta, 1-\delta)$.

Case 1. Suppose $\alpha(1-l) \ge 1$. There exists an r such that

$$\sum_{x=x_n+1}^{x_n+X} e(10^{-m}\nu x^{\alpha})$$

$$= \left(\sum_{x=x_n+1}^{2x_n} + \sum_{x=2x_n+1}^{4x_n} + \cdots + \sum_{x=2^{r-1}x_n+1}^{2^r x_n} + \sum_{x=2^r x_n+1}^{x_n+X} \right) e(10^{-m}\nu x^{\alpha})$$

where $(2^r - 1)x_n < X \le (2^{r+1}-1)x_n$.

Let $f(x) = 10^{-m}\nu x^{\alpha}$. Then for each $j \in \{1, \ldots, r+1\}$, f is a real function on $[2^{j-1}x_n + 1, 2^j x_n]$ and $2^j x_n \le 2(2^{j-1}x_n + 1)$. Hence by Lemma 2.1, if for some k with $2 \le k \ll 1$, the $(k+1)$-th derivative of f is continuous in this interval and satisfies

$$M^{-1} \ll \frac{f^{(k+1)}(x)}{(k+1)!} \ll M^{-1},$$

where $2^{j-1}x_n + 1 \ll M \ll (2^{j-1}x_n + 1)^2$, then

$$\sum_{2^{j-1}x_n+1}^{2^j x_n} e(f(x)) \ll (2^{j-1}x_n + 1)^{1-\sigma}$$

where

$$\sigma = \frac{1}{3k^2 \log(125k)}.$$

If this holds then we will have

$$\left| \sum_{x=x_n+1}^{x_n+X} e(f(x)) \right| \le \sum_{j=1}^{r+1} \left| \sum_{2^{j-1}x_n+1}^{2^j x_n} e(f(x)) \right|$$

and hence

$$\sum_{x=x_n+1}^{x_n+X} e(f(x)) \ll \sum_{j=1}^{r+1} (2^{j-1}x_n)^{1-\sigma} \ll (r+1)(2^r x_n)^{1-\sigma}.$$

Since $2^r x_n < X + x_n$, we will have

$$2^r < \frac{X+x_n}{x_n} \leq \frac{x_{n+1}}{x_n} = O(1)$$

and thus

$$\sum_{x=x_n+1}^{x_n+X} e(f(x)) \ll x_n^{1-\sigma} \ll (x_{n+1}-x_n)^{1-\sigma} = o(x_{n+1}-x_n)$$

which will prove the theorem.

It remains to show that Lemma 2.1 applies. Let $k = [\alpha(1-l)]+1$. Then $k \geq 2$ by our assumption. We have

$$\frac{f^{(k+1)}(x)}{(k+1)!} = \frac{10^{-m}\nu\alpha(\alpha-1)\cdots(\alpha-k)x^{\alpha-(k+1)}}{(k+1)!}$$

and since $x_n \sim 10^{-1/\alpha}10^{n/\alpha}$ as $n \to \infty$, we have

$$\frac{f^{(k+1)}(x)}{(k+1)!} \sim \frac{10^{-ln}\nu\alpha(\alpha-1)\cdots(\alpha-k)10^{-1/\alpha}10^{n-(k+1)n/\alpha}}{(k+1)!}$$

$$= 10^{n(1-l)-([\alpha(1-l)]+2)n/\alpha}V_k$$

where

$$V_k = \frac{\nu\alpha(\alpha-1)\cdots(\alpha-k)10^{-1/\alpha}}{(k+1)!}$$

is a constant depending only on ν, α and k.

Since

$$\alpha(1-l)+1 \leq [\alpha(1-l)]+2 \leq \alpha(1-l)+2$$

we have that

$$10^{-2n/\alpha} \leq 10^{(1-l)n-([\alpha(1-l)]+2)n/\alpha} \leq 10^{-n/\alpha}.$$

Since $2^{j-1}x_n+1 \sim 2^{j-1}10^{-1/\alpha}10^{n/\alpha}$, it follows that

$$\frac{1}{(2^{j-1}x_n+1)^2} \ll \frac{f^{(k+1)}(x)}{(k+1)!} \ll \frac{1}{2^{j-1}x_n+1}$$

as required.

Case 2. Suppose $\alpha(1-l) < 1$. By Lemmas 4.8 and 4.2 of [3] we obtain

$$\left|\sum_{x=a}^{b} e(f(x))\right| \leq 8\pi/Y$$

if $|f'(x)| \geq Y > 0$, provided that $f'(x)$ is monotonic on the interval (a, b) and $|f'(x)| \leq \mu < 1$ for some Y and μ.

We have $f'(x) = 10^{-m}\nu\alpha x^{\alpha-1}$, which is monotonic on $(x_n + 1, x_n + X)$. Also

$$|f'(x)| \leq 10^{-m}\nu\alpha 10^{n(\alpha-1)/\alpha}$$

which is less than 1 if $m > n(\alpha-1)/\alpha + \log_{10}\nu\alpha$. So we need

$$l > \frac{\alpha-1}{\alpha} + \frac{\log_{10}\nu\alpha}{n}$$

but this holds for n sufficiently large, by our assumption. We also have

$$|f'(x)| > \nu\alpha 10^{n(\alpha-1)/\alpha - ln} > \nu\alpha 10^{n(\alpha-1)/\alpha-(1-\delta)n} = \nu\alpha 10^{n(\delta-1/\alpha)} > 0.$$

Thus

$$\left|\sum_{x=x_n+1}^{x_n+X} e(f(x))\right| \leq \frac{8\pi}{\nu\alpha 10^{-n/\alpha+n\delta}} = o(x_n) = o(x_{n+1} - x_n)$$

since $\delta > 0$. This completes the proof.

The following definition was introduced by Besicovitch [6]:

Definition A positive integer q is said to be (ε, s)-normal if the number of times any particular sequence $a_1 \ldots a_l$ of l digits, where $l \leq s$, occurs in q lies between $(1-\varepsilon)10^{-l}q'$ and $(1+\varepsilon)10^{-l}q'$ where q' is the number of digits in q.

We say that almost all numbers are (ε, s)-normal for fixed ε and s if the number of numbers $n \leq x$ not (ε, s)-normal is $o(x)$ as $x \to \infty$.

Davenport and Erdős [8] proved that for any ε and s, almost all the numbers $f(1), f(2), \ldots$ are (ε, s)-normal where $f(x)$ is any polynomial with positive integer values for $x = 1, 2, 3, \ldots$. In this connection we can prove the following theorem:

Theorem 2.2. *For any* ε *and* s*, almost all the numbers* $[1^\alpha], [2^\alpha], [3^\alpha], \ldots$ *are* (ε, s)*- normal.*

Proof. As in the previous proof, we look at the values of x such that $x_n < x \leq x_{n+1}$ i.e. the values for which $[x^\alpha]$ has exactly n digits. Let $T(x)$ denote the number of times that a particular combination of digits $a_1 a_2 \cdots a_h$ (where $h \leq s$) occurs in $[x^\alpha]$. Then with the previous notation,

$$T(x) = \sum_{m=h}^{n} \theta(10^{-m} x^\alpha).$$

We have proved previously that

$$\sum_{x=x_n+1}^{x_n+X} T(x) \sim 10^{-h} n X$$

as $n \to \infty$ where $X = x_{n+1} - x_n$. Our object this time is to estimate the number of values of x for which $T(x)$ deviates appreciably from its average value of $10^{-h} n$. We shall prove that

$$\sum_{x=x_n+1}^{x_n+X} T^2(x) \sim 10^{-2h} n^2 X \tag{2.4}$$

as $n \to \infty$. This will imply that

$$\sum_{x=x_n+1}^{x_n+X} (T(x) - 10^{-h} n)^2 = \sum_{x=x_n+1}^{x_n+X} T^2(x) - 2(10^{-h} n) \sum_{x=x_n+1}^{x_n+X} T(x) + 10^{-2h} n^2 X$$

$$= o(10^{-2h} n^2 X) \text{ as } n \to \infty.$$

Hence the number of values of x with $x_n < x \leq x_{n+1}$ for which the combination $a_1 a_2 \cdots a_h$ does not occur between $(1-\varepsilon)10^{-h}n$ and $(1+\varepsilon)10^{-h}n$ times, must be $o(x_{n+1} - x_n)$ for any fixed ε. Since this is true for each combination of at most s digits, it follows that $[x^\alpha]$ is (ε, s) normal for almost all x.

To prove (2.4) we write the left-hand side as

$$\sum_{x_n+1}^{x_n+X} \sum_{m_1=h}^{n} \sum_{m_2=h}^{n} \theta(10^{-m_1}x^\alpha)\theta(10^{-m_2}x^\alpha).$$

Again we restrict ourselves to values of m_1 and m_2 which satisfy

$$\delta n < m_1 < (1-\delta)n,\ \delta n < m_2 < (1-\delta)n$$

for some δ sufficiently small, since then the contribution of the remaining terms is small compared with the right-hand side of (2.4). For a similar reason we can impose the restriction $m_2 - m_1 > \delta n$. Proceeding as before by using Lemma 2.2 to approximate θ with the functions θ_1 and θ_2, we find that it suffices to estimate the sum

$$S(n, m_1, m_2, \nu_1, \nu_2) = \sum_{x_n+1}^{x_n+X} e((10^{-m_1}\nu_1 + 10^{-m_2}\nu_2)x^\alpha)$$

where $|\nu_1| < \eta^{-1}$ and $|\nu_2| < \eta^{-1}$. The derivation of this follows the proof of the previous theorem closely. We let

$$f(x) = (10^{-m_1}\nu_1 + 10^{-m_2}\nu_2)x^\alpha$$

and we let $m_1 = l_1 n$ and $m_2 = l_2 n$ where $l_1, l_2 \in (\delta, 1-\delta)$. We also choose $k = [\alpha(1-l)] + 1$ where $l = \min(l_1, l_2)$. We have the same two cases as before:

Case 1. We suppose $\alpha(1-l) \geq 1$. Without loss of generality let $l_2 \geq l_1$.

We have

$$\frac{f^{(k+1)}(x)}{(k+1)!} = \frac{(10^{-m_1}\nu_1 + 10^{-m_2}\nu_2)\alpha(\alpha-1)\cdots(\alpha-k)x^{\alpha-(k+1)}}{(k+1)!}$$

$$\sim \frac{(10^{-l_1 n}\nu_1 + 10^{-l_2 n}\nu_2)\alpha(\alpha-1)\cdots(\alpha-k)10^{(k+1)/\alpha-1}10^{n-(k+1)n/\alpha}}{(k+1)!}$$

$$= (10^{n(1-l_1)-([\alpha(1-l_1)]+2)n/\alpha}\nu_1 + 10^{n(1-l_2)-([\alpha(1-l_1)]+2)n/\alpha}\nu_2)W_k$$

where $W_k = \frac{\alpha\cdots(\alpha-k)10^{(k+1)/\alpha-1}}{(k+1)!}$ is a constant depending only on α and k.

We have

$$10^{-2n/\alpha} \le 10^{n(1-l_1)-([\alpha(1-l_1)]+2)n/\alpha} \le 10^{-n/\alpha}$$

as before. We also have

$$10^{-2n/\alpha+(l_1-l_2)n} \le 10^{n(1-l_2)-([\alpha(1-l_1)]+2)n/\alpha} \le 10^{-n/\alpha},$$

and since $l_2 \ge l_1$ we deduce from these inequalities that

$$10^{-2n/\alpha} \ll (10^{n(1-l_1)-([\alpha(1-l_1)]+2)n/\alpha}\nu_1 + 10^{n(1-l_2)-([\alpha(1-l_1)]+2)n/\alpha}\nu_2)W_k$$

$$\ll 10^{-n/\alpha}$$

and hence the result follows as before by Lemma 2.1.

Case 2. We suppose $\alpha(1-l) < 1$ and so it follows that both $\alpha(1-l_1) < 1$ and $\alpha(1-l_2) < 1$.

We need to show that $|f'(x)| \le \mu < 1$. We have

$$|f'(x)| = |(10^{-m_1}\nu_1 + 10^{-m_2}\nu_2)\alpha x^{\alpha-1}|$$

$$\le (10^{n(\alpha-1)/\alpha - l_1 n}|\nu_1| + 10^{n(\alpha-1)/\alpha - l_2 n}|\nu_2|)\alpha$$

since $x^{\alpha-1} \sim 10^{(n-1)\alpha-1/\alpha}$. This is equal to

$$(10^{n(1-l_1-1/\alpha)}|\nu_1| + 10^{n(1-l_2-1/\alpha)}|\nu_2|)\alpha < 1$$

provided that n is sufficiently large, by our assumption.

We also have

$$|f'(x)| \sim |10^{(1-l_1)n-n/\alpha}\nu_1\alpha + 10^{(1-l_2)n-n/\alpha}\nu_2\alpha|$$

$$\geq |10^{(1-l_1)n-n/\alpha}|\nu_1|\alpha - 10^{(1-l_2)n-n/\alpha}|\nu_2|\alpha|.$$

We show this expression is never zero. If we assume to the contrary that it can equal zero then we would have

$$10^{(1-l_1)n-n/\alpha}|\nu_1|\alpha = 10^{(1-l_2)n-n/\alpha}|\nu_2|\alpha$$

and hence

$$10^{(l_2-l_1)n} = |\nu_2/\nu_1|.$$

We have though, that $(l_2-l_1)n = m_2-m_1 > \delta$ and therefore $10^{(l_2-l_1)n} > 10^{\delta n}$ but also $|\nu_2/\nu_1| \leq \eta^{-1} < 10^{\delta n}$ for n sufficiently large. This contradiction implies that $|f'(x)|$ is bounded above zero, as required.

As before, by Lemmas 4.8 and 4.2 of [3] we obtain

$$\left|\sum_{x_n+1}^{x_n+X} e(f(x))\right| \leq \frac{8\pi}{|10^{(1-l_1)n-n/\alpha}\nu_1\alpha + 10^{(1-l_2)n-n/\alpha}\nu_2\alpha|}. \qquad (2.5)$$

The denominator is

$$O(10^{(1-l)n-n/\alpha})$$

and so (2.5) is

$$o(x_n) = o(x_{n+1} - x_n).$$

Thus (2.4) follows and the result holds.

It is clear from the proofs of these two results that they could easily be generalised by, for example, replacing $[x^\alpha]$ by $[a_1x^{\alpha_1} + a_2x^{\alpha_2} + \cdots + a_gx^{\alpha_g}]$ where $\alpha_1 > \alpha_2 \ldots$ and the a_i's are positive constants.

A different construction problem not considered here, is to look for examples of α with α normal but α^2 not normal and vice versa.

Chapter 3

Integer Parts of Sequences 1: Square-free Numbers

We now would like to look at integer parts of sequences and investigate what can be proved about the terms simultaneously for almost all α. For a different example in this area, Harman [12] has proved that for p prime, for almost all $\alpha > 0$ both,

1. $p, [p\alpha], [p^2\alpha], \ldots, [p^{20}\alpha]$ are all prime infinitely often and
2. $p, [p\alpha], [(p\alpha)^2], \ldots, [(p\alpha)^{20}]$ are all prime infinitely often.

Here we consider square-free numbers, which can be counted by a simple formula, instead of primes (although we look at integer parts of sequences being prime in chapter 5). We want to consider the sequence $[10^n\alpha]$, $[10^n\alpha^2]$, $\ldots [10^n\alpha^k]$. It should be expected that we can obtain an asymptotic formula for the number of times that the terms in this sequence are all square-free and indeed it turns out that this is the case.

First we need a well-known elementary lemma.

Lemma 3.1. *Let $\mathcal{A}$ be a subset of the set of integers from* 1 *to* X. *Then the number of square-free integers in* $\mathcal{A}$ *is equal to*

$$\sum_{d=1}^{\sqrt{X}} \mu(d)|\mathcal{A}_{d^2}|$$

where

$$\mathcal{A}_{d^2} = \{ld^2 : ld^2 \in \mathcal{A}\}, \qquad l \in \mathbb{Z}.$$

The next lemma is also needed for the proof of the main theorem below. This is Weyl's inequality (see [13], lemmas 2.1, 2.2 and 2.4): a proof is given for completeness and because a variant will also be used in the proof of the main theorem.

Lemma 3.2.

$$\sum_{t=1}^{L} \left| \sum_{l=K}^{2K} e(tl^2 a/q) \right| = O\left((LK)^{1+(\eta/2)} \left(\frac{1}{q} + \frac{\log q}{K} + \frac{q \log q}{LK^2} \right)^{1/2} \right)$$

for any $\eta > 0$, *where* $K = 10^n/d^2$.

Proof. Consider

$$\left| \sum_{l=K}^{2K} e(tl^2 a/q) \right|^2.$$

This is equal to

$$\sum_{l,l'=K}^{2K} e\left(\frac{l^2 ta}{q} - \frac{l'^2 ta}{q} \right) = \sum_{l,l'=K}^{2K} e\left((l+l')(l-l') \frac{ta}{q} \right).$$

Letting $l - l' = u$ and $l + l' = v$, this becomes:

$$K + 2\Re \sum_{v=2K+1}^{4K-1} \sum_{u=1}^{\min(v-2K, 4K-v)} e(uvta/q)$$

summed over $u \equiv v \bmod 2$. The first term in the expression is the contribution from $u = 0$ and the second term is twice the real part of the sum over the remaining positive values of u, since the substitution $u \to -u$ changes each summand into its complex conjugate.

We observe that

$$\left|\sum_{x=1}^{X} e(bx)\right| = \frac{|e(b(X+1)) - e(b)|}{|e(b)-1|} \leq \frac{2}{|e(b)-1|} = \frac{1}{|\sin \pi b|} \leq \frac{1}{2 \parallel b \parallel}.$$

For fixed v, the terms $uvta/q$ form an arithmetic progression with common difference $2vta/q$ and length at most $10^n/d^2$. Therefore the sum of $e(uvta/q)$ over this progression is

$$\ll \min(K, \parallel 2tva/q \parallel^{-1})$$

where K is the trivial bound.

Now summing over t and v we have

$$\sum_{t=1}^{L}\left|\sum_{l=K}^{2K} e(tl^2a/q)\right|^2 \ll 2LK + 2\sum_{t=1}^{L}\sum_{2K+1}^{4K-1} \min(K, \parallel 2tva/q \parallel^{-1})$$

$$\leq 2LK + 2\sum_{t=1}^{8LK} \tau(t)\min(K, \parallel ta/q \parallel^{-1}).$$

We now split the range for t into blocks of length q. In each block we expect $\parallel ta/q \parallel$ to take on each value $0, 1/q, 2/q, \ldots, c/q$ approximately twice, where $c/q \leq 1/2, (c+1)/q > 1/2$. We also have $\tau(t) \ll t^\eta$ for any $\eta > 0$. Hence

$$\sum_{t=1}^{L}\left|\sum_{l=\frac{10^n}{d^2}}^{\frac{2.10^n}{d^2}} e(tl^2a/q)\right|^2 \ll LK + (LK)^\eta\left(\frac{LK}{q}+1\right)(K + q\log q)$$

since the number of blocks is $O(\frac{LK}{q}+1)$ and

$$\sum_{h=1}^{q}\frac{q}{h} = O(q\log q)$$

and K is the value when $\| ta/q \| = 0$.

The right-hand side of our bound is

$$O\left((LK)^\eta\left(\frac{LK^2}{q} + LK\log q + q\log q\right)\right).$$

Hence by Cauchy's inequality

$$\sum_{t=1}^{L}\left|\sum_{l=\frac{10^n}{d^2}}^{\frac{2.10^n}{d^2}} e(tl^2a/q)\right| \leq O\left(L^{1/2}(LK)^{\eta/2}\left(\frac{LK^2}{q} + LK\log q + q\log q\right)^{1/2}\right)$$

$$= O\left((LK)^{1+(\eta/2)}\left(\frac{1}{q} + \frac{\log q}{K} + \frac{q\log q}{LK^2}\right)^{1/2}\right)$$

and the lemma is proved.

Now we are ready to prove our main result. We first give a direct proof which involves finding the intersections of intervals. This method gives an error term with a saving of a power of N. Later we give a more general version of the theorem, proved by the use of Fourier series and Lemma 1.1, which has a far weaker error term.

Theorem 3.1. *For almost all α, the number of solutions in n to $[10^n\alpha]$ and $[10^n\alpha^2]$ simultaneously square-free, for $n \leq N$, is*

$$\left(\frac{6}{\pi^2}\right)^2 N + O\left(N^{\frac{2+c}{3}}(\log N)^{\frac{1}{2}+\gamma}\right)$$

as $N \to \infty$, where γ is arbitrarily small and $0 < c < 1$ is a constant.

Proof. Let the intervals $\mathcal{I}_n$ for each natural number n be defined by

$$\mathcal{I}_n = \bigcup_{\substack{r \text{ s.f.} \\ 10^n \leq r < 4.10^n}} \left[\frac{r}{10^n}, \frac{r+1}{10^n}\right) \bigcap \bigcup_{\substack{s \text{ s.f.} \\ 10^n \leq s < 2.10^n}} \left[\frac{s^2}{10^{2n}}, \frac{(s+1)^2}{10^{2n}}\right)$$

where $s.f.$ is used as an abbreviation of square-free. Then $\alpha^2 \in \mathcal{I}_n$ exactly when $[10^n\alpha]$ and $[10^n\alpha^2]$ are square-free for $1 < \alpha < 2$. Therefore the number of solutions to $[10^n\alpha]$ and $[10^n\alpha^2]$ simultaneously square-free, for $n \leq N$, for a particular $\alpha \in (1, 2)$, is

$$\sum_{n=1}^{N} \chi(\mathcal{I}_n) \qquad \text{where} \qquad \chi(\mathcal{I}_n) = \begin{cases} 1 & \text{if } \alpha^2 \in \mathcal{I}_n \\ 0 & \text{otherwise} \end{cases}$$

We will establish an upper bound for

$$\int_1^4 \left(\sum_{n=1}^{N} \chi(\mathcal{I}_n) - \left(\frac{6}{\pi^2}\right)^2 N \right)^2 d\alpha^2.$$

The required result will then follow from Lemma 1.3. When expanded the expression becomes:

$$\int_1^4 \left(\sum_{n=1}^{N} \chi(\mathcal{I}_n) \sum_{m=1}^{N} \chi(\mathcal{I}_m) + \left(\frac{6}{\pi^2}\right)^4 N^2 - 2\left(\frac{6}{\pi^2}\right)^2 N \sum_{n=1}^{N} \chi(\mathcal{I}_n) \right) d\alpha^2.$$

We have

$$\int_1^4 \sum_{n=1}^{N} \chi(\mathcal{I}_n) d\alpha^2 = \sum_{n=1}^{N} \int_1^4 \chi(\mathcal{I}_n) d\alpha^2 = \sum_{n=1}^{N} \lambda(\mathcal{I}_n)$$

and

$$\int_1^4 \sum_{n=1}^{N} \chi(\mathcal{I}_n) \sum_{m=1}^{N} \chi(\mathcal{I}_m) d\alpha^2 = \sum_{n,m=1}^{N} \lambda(\mathcal{I}_n \cap \mathcal{I}_m)$$

so we need an asymptotic formula for:

$$\sum_{n,m=1}^{N} \lambda(\mathcal{I}_n \cap \mathcal{I}_m) + 3\left(\frac{6}{\pi^2}\right)^4 N^2 - 2\left(\frac{6}{\pi^2}\right)^2 N \sum_{n=1}^{N} \lambda(\mathcal{I}_n). \tag{3.1}$$

We will start by considering $\lambda(\mathcal{I}_n)$: The length of $\mathcal{I}_n$ is the sum of the lengths of each possible intersection of two intervals, one being of the form

$$\left[\frac{r}{10^n}, \frac{r+1}{10^n}\right)$$

and the other of the form

$$\left[\frac{s^2}{10^{2n}}, \frac{(s+1)^2}{10^{2n}}\right).$$

Since an interval of the latter type is always longer than one of the former, each intersection of two intervals is one of three types. Either only a part of the shorter interval, which includes the infimum, intersects with the longer interval or only a part of the shorter interval, including the supremum, intersects with the longer interval or else the shorter interval is wholly contained in the longer one. We consider these cases separately:

Case 1. Suppose we have

$$\frac{r}{10^n} \leq \frac{s^2}{10^{2n}}.$$

The condition required for a non-empty intersection between the two intervals is thus

$$\frac{r}{10^n} \leq \frac{s^2}{10^{2n}} < \frac{r+1}{10^n}.$$

which is equivalent to

$$\left[\frac{s^2}{10^n}\right] = r.$$

Since $10^n < 2s+1$, we have that

$$\frac{1}{10^n} < \frac{2s+1}{10^{2n}} \qquad \text{and} \qquad \frac{r}{10^n} \leq \frac{s^2}{10^{2n}}$$

together imply that

$$\frac{r+1}{10^n} < \frac{(s+1)^2}{10^{2n}}.$$

Therefore the size of the intersection is

$$\frac{r+1}{10^n} - \frac{s^2}{10^{2n}} = \frac{[s^2/10^n]+1}{10^n} - \frac{s^2/10^n}{10^n} = \frac{1-\{s^2/10^n\}}{10^n}.$$

It follows that the length of $\mathcal{I}_n$ is

$$\frac{1}{10^n}\left(\sum_{\substack{s \text{ s.f.} \\ r \text{ s.f.} \\ 10^n<s<2.10^n}} \left(1-\left\{\frac{s^2}{10^n}\right\}\right)\right) \qquad \text{where} \qquad r=\left[\frac{s^2}{10^n}\right].$$

Let $(10^n, 2.10^n) = \mathcal{A}$ and $\{s : s \in \mathcal{A}\,,\, r \text{ s.f.}\} = \mathcal{B}$, then, considering only the first term in the sum above for the time being, we have:

$$\sum_{\substack{s \text{ s.f.} \\ r \text{ s.f.} \\ s\in\mathcal{A}}} 1 = \sum_{\substack{s \text{ s.f.} \\ s\in\mathcal{B}}} 1 = \sum_{d=1}^{\sqrt{(2.10^n)}} \mu(d)\,|\mathcal{B}_{d^2}|$$

where

$$\mathcal{B}_{d^2} = \{l : ld^2 \in \mathcal{B}\}, \qquad l \in \mathbb{Z},$$

using Lemma 3.1. Therefore

$$\begin{aligned}
|\mathcal{B}_{d^2}| &= \left|\left\{ld^2 : ld^2 \in \mathcal{A}\,,\, \left[\frac{(ld^2)^2}{10^n}\right] \text{ s.f.}\right\}\right| \\
&= \left|\left\{\left[\frac{(ld^2)^2}{10^n}\right] : ld^2 \in \mathcal{A}\,,\, \left[\frac{(ld^2)^2}{10^n}\right] \text{ s.f.}\right\}\right| \\
&= \sum_{e=1}^{\sqrt{(4.10^n)}} \mu(e)\,|\mathcal{C}_{e^2}|
\end{aligned}$$

where $\mathcal{C} = \left\{\left[\frac{(ld^2)^2}{10^n}\right] : ld^2 \in \mathcal{A}\right\}$. Therefore

$$|\mathcal{C}_{e^2}| = \left|\left\{m : me^2 = \left[\frac{(ld^2)^2}{10^n}\right]\,, ld^2 \in \mathcal{A}\right\}\right|, \qquad m \in \mathbb{Z}$$

and thus we have:

$$\sum_{\substack{s \text{ s.f.} \\ r \text{ s.f.} \\ s\in\mathcal{A}}} 1 = \sum_{d=1}^{\sqrt{(2.10^n)}} \mu(d) \sum_{e=1}^{\sqrt{(4.10^n)}} \mu(e) \sum_{\substack{me^2=\left[\frac{(ld^2)^2}{10^n}\right] \\ ld^2\in\mathcal{A}}} 1. \tag{3.2}$$

The equation

$$me^2 = \left[\frac{(ld^2)^2}{10^n}\right]$$

can be rewritten as

$$me^2 \le \frac{l^2d^4}{10^n} < me^2 + 1,$$

which is equivalent to

$$0 \le \left\{\frac{l^2d^4}{10^n e^2}\right\} < \frac{1}{e^2}.$$

We can now use the lemma on Fourier series to estimate the number of these fractional parts. We write

$$\frac{d^4}{e^2 10^n} = a/q$$

where $(a, q) = 1$ and let

$$\phi(l) = \left\{\frac{l^2 a}{q}\right\}.$$

We define

$$\chi(\phi(l)) = \begin{cases} 1 & \text{if } 0 \le \phi(l) < \frac{1}{e^2} \\ 0 & \text{otherwise} \end{cases}.$$

Then by Lemma 2.2, for any integer $L \ge 1$ there are trigonometric polynomials $T_i(\phi(l))$ for $i = 1, 2$, such that

$$T_1(\phi(l)) \le \chi(\phi(l)) \le T_2(\phi(l))$$

and

$$T_i(\phi(l)) = \sum_{t=-L}^{L} \hat{T}_i(t) e(t\phi(l))$$

where

$$\hat{T}_1(0) = \frac{1}{e^2} - \frac{1}{L+1},\ \hat{T}_2(0) = \frac{1}{e^2} + \frac{1}{L+1}$$

and

$$|\hat{T}_i(t)| \le \min\left(\frac{1}{e^2} \mp \frac{1}{L+1}, \frac{3}{2|t|}\right).$$

Suppose we disregard for the moment all values of d and e other than $d \leq 10^{\delta n}$ and $e \leq 10^{\delta n}$, for some small δ. Then using the above Fourier series for $\chi(\phi(l))$ (and noting that $e(t\{\frac{l^2 a}{q}\}) = e(\frac{tl^2 a}{q})$), we see that the right-hand side of (3.2), for this range of d and e, is equal to:

$$\sum_{d=1}^{10^{\delta n}} \mu(d) \sum_{e=1}^{10^{\delta n}} \mu(e) \sum_{l=\frac{10^n}{d^2}}^{\frac{2.10^n}{d^2}} \left(\frac{1}{e^2} + \frac{x}{L+1} + \sum_{t=1}^{L} \hat{T}_i(t) e(tl^2 a/q) \right)$$

where $-1 \leq x \leq 1$. The first term in this sum, which will give a contribution to the main term, is equal to

$$10^n \sum_{d=1}^{10^{\delta n}} \mu(d) \sum_{e=1}^{10^{\delta n}} \mu(e) \frac{1}{d^2 e^2} = 10^n \sum_{d=1}^{10^{\delta n}} \frac{\mu(d)}{d^2} \sum_{e=1}^{10^{\delta n}} \frac{\mu(e)}{e^2}$$

$$= 10^n \left(\frac{6}{\pi^2} + O\left(\sum_{d>10^{\delta n}} \frac{1}{d^2} \right) \right) \left(\frac{6}{\pi^2} + O\left(\sum_{e>10^{\delta n}} \frac{1}{e^2} \right) \right)$$

by a well-known elementary lemma (see [14], page 27). This is

$$10^n \left(\frac{6}{\pi^2} \right)^2 + O(10^{n-\delta n})$$

since $\sum_{d>k} 1/d^2 = O(1/k)$.

Now if we consider the case $d > 10^{\delta n}$ we have:

$$\left| \sum_{\substack{s \text{ s.f.} \\ r \text{ s.f.} \\ s \in \mathcal{A}}} 1 \right| \leq \left| \sum_{d>10^{\delta n}} \mu(d) \left| \mathcal{B}_{d^2} \right| \right| \leq \left| \sum_{d>10^{\delta n}} \left(\frac{10^n}{d^2} + O(1) \right) \right|$$

since $\mathcal{B}$ is a set of integers between 10^n and 2.10^n. Therefore this error is

$$O(10^{n-\delta n}) + O(10^{\frac{n}{2}}) = O(10^{n-\delta n})$$

assuming δ will be chosen to be less than $1/2$. The case when $e > 10^{\delta n}$ is similar.

Considering the second term in the sum, there will be values of d and e which give a positive value of $\mu(d)\mu(e)$ and so we need an upper bound for the magnitude of the error of the sum over these values of d and e. The other values of d and e give a negative value of $\mu(d)\mu(e)$ so we need a lower bound for the sum over these values of d and e. It will thus suffice to consider the expression

$$10^n \left(\frac{6}{\pi^2}\right)^2 + O\left(\sum_{d=1}^{10^{\delta n}} \sum_{e=1}^{10^{\delta n}} \left(\frac{10^n}{d^2(L+1)} + \sum_{t=1}^{L} |\hat{T}_i(t)| \left| \sum_{l=\frac{10^n}{d^2}}^{\frac{2.10^n}{d^2}} e(tl^2 a/q) \right| \right)\right) + O(10^{n-\delta n}). \quad (3.3)$$

for $i = 1$ and $i = 2$ where the first term is the main term and the third arises from the errors already calculated.

By Lemma 3.2 the second term in (3.3) becomes

$$O\left(\sum_{d=1}^{10^{\delta n}} \sum_{e=1}^{10^{\delta n}} \left(\frac{10^n}{d^2(L+1)} + \max\left(\frac{1}{e^2} \mp \frac{1}{L+1}, \frac{3}{2L}\right) (LK)^{1+(\eta/2)} \left(\frac{1}{q} + \frac{\log q}{K} + \frac{q \log q}{LK^2}\right)^{1/2}\right)\right).$$

where $K = 10^n/d^2$. We have

$$\sum_{d=1}^{10^{\delta n}} \sum_{e=1}^{10^{\delta n}} \frac{10^n}{d^2(L+1)} = O\left(\frac{10^{n+\delta n}}{L+1}\right)$$

for the first part of this error term. By our choice of q we have $q \geq 10^n/d^4$ and $q \leq 10^n e^2$ and since $d, e \leq 10^{\delta n}$ and $K = 10^n/d^2$ we have

$$\frac{1}{q} + \frac{\log q}{K} + \frac{q \log q}{LK^2} \leq 10^{4\delta n - n} + 10^{2\delta n - n}(n + 2\delta n) + \frac{10^{6\delta n - n}(n + 2\delta n)}{L}.$$

We now choose $L = 10^{3\delta n}$. Therefore

$$\frac{1}{e^2} + \frac{1}{L+1} \geq 10^{-2\delta n} + \frac{1}{2L} = 10^{-2\delta n} + \frac{1}{2}10^{-3\delta n} > \frac{3}{2}10^{-3\delta n} = \frac{3}{2L}.$$

Also

$$\frac{1}{e^2} - \frac{1}{L+1} \geq 10^{-2\delta n} + \frac{1}{L} = 10^{-2\delta n} - 10^{-3\delta n} > \frac{3}{2}10^{-3\delta n} = \frac{3}{2L}$$

and hence the remaining error term from (3.3) is

$$O(10^n 10^{3\delta n} 10^{(n+3\delta n)\eta}(10^{4\delta n-n} + 10^{2\delta n-n}(n+2\delta n) + 10^{3\delta n-n}(n+2\delta n))^{1/2})$$

$$= O(10^{n+3\delta n+2\delta n-(n/2)+\gamma}) = O(10^{(n/2)+5\delta n+\gamma})$$

for any $\gamma > 0$. We have, therefore that the sum (3.2) is equal to

$$10^n \left(\frac{6}{\pi^2}\right)^2 + O(10^{n-\delta n}) + O(10^{n+\delta n-3\delta n}) + O(10^{(n/2)+5\delta n+\gamma})$$

$$= 10^n \left(\frac{6}{\pi^2}\right)^2 + O(10^{n-\delta n}) + O(10^{(n/2)+5\delta n+\gamma}).$$

We still need to take account of the fact that the length of $\mathcal{I}_n$ is actually

$$\frac{1}{10^n}\left(\sum_{\substack{s \text{ sf.} \\ r \text{ sf.} \\ 10^n < s < 2.10^n}} 1 - \left\{\frac{s^2}{10^n}\right\}\right). \tag{3.4}$$

To determine the second term in the bracket in this sum, we divide the range, $[0,1)$, for $\{\frac{s^2}{10^n}\}$, into intervals of length ε and approximate $\{\frac{s^2}{10^n}\}$ in the interval $[A, A+\varepsilon)$ by $A+\varepsilon$. This gives a total error of at most $\varepsilon 10^n$.

Working as before, the sum (3.2) in the range $A \leq \{\frac{s^2}{10^n}\} < A+\varepsilon$ is the same as before except that the range of $\phi(l)$ needs to be modified. We have

$A \leq \{\frac{l^2d^4}{10^n}\} < A+\varepsilon$, but since $[\frac{l^2d^4}{10^n}]/e^2$ is an integer, m, it must be equal to $[\frac{l^2d^4}{e^2 10^n}]$ and hence $\{\frac{l^2d^4}{10^n}\}/e^2 = \{\frac{l^2d^4}{e^2 10^n}\}$ and so we have the condition

$$\frac{A}{e^2} \leq \phi(l) < \frac{A+\varepsilon}{e^2}.$$

Thus the Fourier series is the same as before except that the main term is ε/e^2 and $|\hat{T}_2(t)| \leq \min(\frac{\varepsilon}{e^2} + \frac{1}{L+1}, \frac{3}{2|t|})$.

We work as before and choose $L = 10^{3\delta n}$ and next make the choice $\varepsilon = 10^{-\frac{1}{2}\delta n}$. This means that $\frac{\varepsilon}{e^2} + \frac{1}{L+1}$ will always be larger than $\frac{3}{2L}$ as before.

The sum of $A+\varepsilon$ over all values of A is:

$$\sum_{\substack{k=0 \\ A=k\varepsilon}}^{[\frac{1}{\varepsilon}]} A+\varepsilon = \frac{1}{2}\left[\frac{1}{\varepsilon}\right]\left(\left[\frac{1}{\varepsilon}\right]+1\right)\varepsilon + \varepsilon\left(\left[\frac{1}{\varepsilon}\right]+1\right) = \frac{1+\varepsilon}{2\varepsilon} + O(1).$$

Therefore the second term in the bracket in (3.4) is equal to

$$\left(\frac{1}{2\varepsilon}\right)\varepsilon 10^n \left(\frac{6}{\pi^2}\right)^2 + O\left(\left(\frac{1}{2\varepsilon}\right)10^{n-\delta n}\right) + O\left(\left(\frac{1}{2\varepsilon}\right)\varepsilon 10^{(n/2)+5\delta n+\gamma}\right) + O(\varepsilon 10^n)$$

$$= \frac{1}{2}10^n\left(\frac{6}{\pi^2}\right)^2 + O(10^{n-\frac{1}{2}\delta n}) + O(10^{(n/2)+5\delta n+\gamma}).$$

This must be subtracted from the sum (3.2) and hence the result is:

$$\frac{1}{2}10^n\left(\frac{6}{\pi^2}\right)^2 + O(10^{n-\frac{1}{2}\delta n}) + O(10^{(n/2)+5\delta n+\gamma}).$$

We see that the second error term is the smaller if $\delta < 1/11$, otherwise the first is the smaller. Hence the least error is when $\delta = 1/11$ and is thus $O(10^{\frac{21}{22}n+\gamma})$ for any $\gamma > 0$. We must also divide by 10^n and so (3.4) becomes

$$\frac{1}{2}\left(\frac{6}{\pi^2}\right)^2 + O(10^{-\frac{1}{22}n+\gamma}).$$

Case 2. Suppose that

$$\frac{r}{10^n} > \frac{s^2}{10^{2n}} \qquad \text{and} \qquad \frac{r+1}{10^n} \leq \frac{(s+1)^2}{10^{2n}}$$

so that the smaller interval is wholly contained in the larger. Then the length of the intersection is the length of the small interval and hence is $1/10^n$.

Since $s^2/10^n < r$, by dividing the interval up, we have

$$\begin{array}{rcccl} & r & \leq & \frac{s^2}{10^n}+1 & < & r+1 \\ \text{or} & r & \leq & \frac{s^2}{10^n}+2 & < & r+1 \\ & & \vdots & & & \\ \text{or} & r & \leq & \frac{s^2}{10^n}+j & < & r+1 \end{array}$$

for the largest j such that $j < (2s+1)/10^n$, since $r+1 \leq s^2/10^n+(2s+1)/10^n$. There are in fact three subcases as follows:

Case A. If $r \leq \frac{s^2}{10^n}+1 < r+1$, then $[\frac{s^2}{10^n}+1] = r$. We have the full range for s, $10^n \leq s < 2.10^n$ as before. Thus we have $me^2 - 1 \leq l^2d^4/10^n < me^2$ and therefore

$$1 - \frac{1}{e^2} \leq \left\{ \frac{l^2d^4}{e^2 10^n} \right\} < 1.$$

This gives us the same result as from the sum (3.2), giving a contribution of

$$\left(\frac{6}{\pi^2}\right)^2 + O(10^{-\frac{1}{22}n+\gamma}).$$

Case B. If $r \leq \frac{s^2}{10^n}+2 < r+1$ then $[\frac{s^2}{10^n}+2] = r$. The range for s is now restricted by the condition $r+1 < s^2/10^n + (2s+1)/10^n$. This gives us

$$\left[\frac{s^2}{10^n}\right] + 3 < \frac{s^2}{10^n} + \frac{2s+1}{10^n}$$

and therefore

$$s > \frac{(3 - \{s^2/10^n\})10^n - 1}{2}.$$

We deal with this by dividing $\{s^2/10^n\}$ into ranges of length ε as we did previously. We have $[\frac{l^2d^4}{10^n}+2] = me^2$ and $A < \{l^2d^4/10^n\} < A+\varepsilon$. This gives us

$$\frac{A}{e^2} < \left\{ \frac{l^2d^4}{e^2 10^n} + \frac{2}{e^2} \right\} < \frac{A+\varepsilon}{e^2}.$$

The range of s is thus from $\frac{(3-A)10^n-1}{2}$ to 2.10^n and so is of length $10^n(1/2 + A/2)+1/2$. Summing over A this gives

$$\sum_{\substack{k=0\\ A=k\varepsilon}}^{[\frac{1}{\varepsilon}]} 10^n(1/2+A/2)+1/2 = \frac{1}{2\varepsilon}10^n + \frac{1-\varepsilon}{4\varepsilon}10^n + \frac{1}{2\varepsilon} + O(10^n).$$

This is multiplied by ε from the range of $\{\frac{l^2d^4}{e^2 10^n}+\frac{2}{e^2}\}$ to give

$$\frac{3}{4}10^n + O\left(10^{n-\frac{1}{2}\delta n}\right).$$

Thus the contribution from case B is

$$\frac{3}{4}\left(\frac{6}{\pi^2}\right)^2 + O(10^{-\frac{1}{22}n+\gamma}).$$

Case C. The only other possibility is that $r \le \frac{s^2}{10^n}+3 < r+1$. In this case we have

$$s > \frac{(4-\{s^2/10^n\})10^n-1}{2}.$$

Proceeding exactly as in case B we obtain the contribution

$$\frac{1}{4}\left(\frac{6}{\pi^2}\right)^2 + O(10^{-\frac{1}{22}n+\gamma}).$$

Therefore the total contribution from Case 2 is

$$2\left(\frac{6}{\pi^2}\right)^2 + O(10^{-\frac{1}{22}n+\gamma}).$$

Case 3. The remaining case is that when

$$\frac{r}{10^n} > \frac{s^2}{10^{2n}} \qquad \text{and} \qquad \frac{r+1}{10^n} > \frac{(s+1)^2}{10^{2n}}$$

and it is clear that this is similar to case 1, again giving a contribution of

$$\frac{1}{2}\left(\frac{6}{\pi^2}\right)^2 + O(10^{-\frac{1}{22}n+\gamma}).$$

Combining the above cases we obtain:

$$\lambda(\mathcal{I}_n) = 3\left(\frac{6}{\pi^2}\right)^2 + O(10^{-\frac{1}{22}n+\gamma}).$$

We sum over $n \leq N$ which produces the result

$$\int_1^4 \sum_{n=1}^{N} \chi(\mathcal{I}_n) d\alpha^2 = 3\left(\frac{6}{\pi^2}\right)^2 N + O(1).$$

We now look at the first term in (3.1). We require an asymptotic formula for

$$\int_1^4 \sum_{n=1}^{N} \chi(\mathcal{I}_n) \sum_{m=1}^{N} \chi(\mathcal{I}_m) d\alpha^2$$

which is equal to

$$\sum_{n,m \leq N} \lambda\left(\mathcal{I}_n \cap \mathcal{I}_m\right).$$

The length of $(\mathcal{I}_n \cap \mathcal{I}_m)$ is the length of

$$\left[\frac{r_1}{10^n}, \frac{r_1+1}{10^n}\right) \bigcap \left[\frac{s_1^2}{10^{2n}}, \frac{(s_1+1)^2}{10^{2n}}\right) \bigcap \left[\frac{r_2}{10^m}, \frac{r_2+1}{10^m}\right) \bigcap \left[\frac{s_2^2}{10^{2m}}, \frac{(s_2+1)^2}{10^{2m}}\right)$$

for s_1, s_2, r_1 and r_2 square-free and $10^n \leq s_1 < 2.10^n$, $10^m \leq s_2 < 2.10^m$, $10^n \leq r_1 < 4.10^n$ and $10^m \leq r_2 < 4.10^m$. Suppose first that we have both

$$\frac{r_1}{10^n} \leq \frac{s_1^2}{10^{2n}} \qquad \text{and} \qquad \frac{r_2}{10^m} \leq \frac{s_2^2}{10^{2m}}$$

and that $n < m$. Suppose further that $m^c < n < m - m^c$ where $c > 0$ is suitably small. We must therefore find the total length of all intervals

$$\left[\frac{s_2^2}{10^{2m}}, \frac{r_2+1}{10^m}\right) \bigcap \left[\frac{s_1^2}{10^{2n}}, \frac{r_1+1}{10^n}\right)$$

where

$$r_1 = \left[\frac{s_1^2}{10^n}\right] \text{ and } r_2 = \left[\frac{s_2^2}{10^m}\right].$$

Since $n < m$, we expect the interval

$$\left[\frac{s_2^2}{10^{2m}}, \frac{r_2+1}{10^m}\right)$$

to be the shorter one. We assume that if there is an intersection then the shorter interval lies wholly inside the longer one, since we only need an upper bound, so that the length of intersection is

$$\frac{1-\{\frac{s_2^2}{10^m}\}}{10^m}.$$

Suppose that

$$q_i < s_2 \leq q_i + S,$$

where the q_i take values between 10^m and 2.10^m, each one separated by at least S where $S = 10^{m-n}$. For an intersection it is necessary that

$$\frac{s_1^2}{10^{2n}} \leq \frac{s_2^2}{10^{2m}}$$

and thus $s_2 \geq s_1 S$. Define $q = q_i = s_1 S$. Therefore we have

$$s_1 S < s_2 \leq s_1 S + S.$$

Proceeding as before and letting $s_1 = ld^2, r_1 = me^2, s_2 = hf^2$ and $r_2 = jg^2$ for l, m, h and $j \in \mathbb{Z}$, we see that for fixed s_1 we need to count the number of times that

$$jg^2 = \left[\frac{h^2 f^4}{10^m}\right]$$

for $q < hf^2 \leq q + S$.

We divide the range of $\{s_2^2/10^m\}$ again into intervals of length ε, approximating the whole summand $1 - \{s_2^2/10^m\}$, by $1 - A$ so that we can sum 1 over the appropriate values and then multiply by

$$\sum_{\substack{k=0 \\ A=k\varepsilon}}^{[\frac{1}{\varepsilon}]} 1 - A = \frac{1}{2\varepsilon} + O(1).$$

We have

$$\frac{A}{g^2} \leq \left\{ \frac{h^2 f^4}{g^2 10^m} \right\} \leq \frac{A+\varepsilon}{g^2}.$$

Letting $s = s_2 - q$ so that $1 \leq s \leq S$, and using the Fourier series for $\chi(\{\frac{(q+s)^2 t}{g^2 10^m}\})$, for $f, g \leq 10^{\delta m}$, we obtain the main error term:

$$O\left(\sum_{f=1}^{10^{\delta m}} \sum_{g=1}^{10^{\delta m}} \sum_{t=1}^{L} |\hat{T}_2(t)| \sum_q \left| \sum_s e\left(\frac{(q+s)^2 t}{g^2 10^m} \right) \right| \right).$$

We use a variant of Lemma 3.2 and consider the square of the modulus:

$$\left| \sum_s e\left(\frac{(q+s)^2 t}{g^2 10^m} \right) \right|^2 = \sum_{s,s'=1}^{S} e\left(\frac{(q+s)^2 t}{g^2 10^m} - \frac{(q+s')^2 t}{g^2 10^m} \right)$$

$$= \sum_{s,s'} e\left(\frac{(2q+s+s')(s-s')t}{g^2 10^m} \right) = \sum_{u,v} e\left(\frac{(2q+v)ut}{g^2 10^m} \right)$$

where $u = s - s', v = s + s'$. Again, by separating off the terms with $u = 0$ and by means of the substitution $u \to -u$, we can write this as

$$S + 2\Re \sum_{u,v} e\left(\frac{(2q+v)ut}{g^2 10^m} \right)$$

where the sum is over $3 \leq v \leq 2S - 1, 1 \leq u \leq \min(v - 2, 2S - v)$ and $u \equiv v \bmod 2$.

For fixed v, the terms $\frac{(2q+v)ut}{g^2 10^m}$ form an arithmetic progression with common difference $\frac{(2q+v)2t}{g^2 10^m}$ and length at most $S - 1$. Therefore

$$\left| \sum_s e\left(\frac{(q+s)^2 t}{g^2 10^m} \right) \right|^2 \ll S + \sum_{v=1}^{2S} \min\left(S, \left\| \frac{(2q+v)2t}{g^2 10^m} \right\|^{-1} \right)$$

$$\ll \sum_{v=1}^{2S} \min\left(S, \left\| \frac{(2q+v)2t}{g^2 10^m} \right\|^{-1}\right).$$

Let $2q+v=w$ so that $2q_i+1 \le w \le 2q_i+2S$ for each q_i. We therefore now consider

$$\sum_{t=1}^{L} \sum_{w} \min\left(S, \left\| \frac{2tw}{g^2 10^m} \right\|^{-1}\right).$$

Fix $t \le L$ and divide the range for w into blocks of length p where $a/p = \frac{2t}{g^2 10^m}$ and $(a,p)=1$. The number of values taken by w is bounded above by $2S(10^m/S)(1/f^2)$ since the number of q_is is at most $10^m/S$ and $s_2 = hf^2$. Hence

$$\sum_{w} \min\left(S, \| aw/p \|^{-1}\right) \ll \left(\frac{10^m}{pf^2}+1\right)(S+p\log p)$$

$$\ll \frac{10^m S}{pf^2} + \frac{10^m \log p}{f^2} + p\log p.$$

This is largest when $t|10^m$ which is when $p = g^2 10^m/t$. Therefore it is

$$\ll \frac{St}{g^2 f^2} + \frac{10^m m}{f^2} + \frac{g^2 10^m m}{t}$$

Hence, summing over t we have:

$$\sum_{t=1}^{L} \sum_{q} \left| \sum_{s} e\left(\frac{(q+s)^2 t}{g^2 10^m}\right) \right| \ll L^{1/2} \left(\frac{SL^2}{g^2 f^2} + \frac{10^m L m}{f^2} + g^2 10^m m \log L\right)^{\frac{1}{2}}$$

using Cauchy's inequality. The right-hand side is equal to

$$L^{1/2} \frac{S}{f^2} \left(\frac{L^2 f^2}{g^2 S} + \frac{10^m L m f^2}{S^2} + \frac{g^2 10^m m f^4 \log L}{S^2}\right)^{\frac{1}{2}}.$$

Choosing $L = 10^{3\delta m}$, and using $f, g \le 10^{\delta m}$, this is

$$O\left(10^{\frac{3}{2}\delta m} \left(\frac{10^{m-n}}{f^2}\right) (10^{8\delta m - m + n} + 10^{5\delta m - m + 2n} m + 10^{6\delta m - m + 2n} m 3\delta m)^{\frac{1}{2}}\right).$$

If $n \ge 2\delta m$ then this is

$$O\left(10^{\frac{m}{2} + \frac{9}{2}\delta m} m/f^2\right).$$

If $n < 2\delta m$ then it is

$$O\left(10^{\frac{m}{2}-\frac{n}{2}+\frac{11}{2}\delta m}(1/f^2)\right).$$

In a similar way as before we let $\varepsilon = 10^{-\frac{1}{2}\delta m}$ so that $\frac{\varepsilon}{g^2} + \frac{1}{L+1} > \frac{3}{2L}$ and hence the error term is

$$O\left(\left(\frac{1}{2\varepsilon}\right)\varepsilon \sum_{f=1}^{10^{\delta m}} \frac{1}{f^2} \sum_{g=1}^{10^{\delta m}} \frac{1}{g^2} 10^{\frac{m}{2}+\frac{9}{2}\delta m} m\right) = O(10^{\frac{m}{2}+\frac{9}{2}\delta m} m)$$

if $n \geq 2\delta m$ etc.

Considering the main term now, this is

$$\left(\frac{1}{2\varepsilon}\right) \sum_{f=1}^{10^{\delta m}} \mu(f) \sum_{g=1}^{10^{\delta m}} \mu(g) \sum_{q} \sum_{s} \frac{\varepsilon}{g^2}.$$

The number of s is equal to the number of values of hf^2 which is $\psi S/f^2$ where $\psi = (s_1^2 + \theta 10^n)^{1/2} - s_1$ where $\theta 10^n$ is the length of the longer interval. This is since we have

$$\frac{s_1^2}{10^{2n}} \leq \frac{s_2^2}{10^{2m}} < \frac{s_1^2}{10^{2n}} + \frac{\theta}{10^n}.$$

Here we are ignoring the length of the shorter interval in determining the range of s_2. This is a small order error.

We divide the range of $\{s_1^2/10^n\}$ into intervals of length ε as before. Approximating $\{s_1^2/10^n\}$ by B we have that

$$\psi = (s_1^2 + (1-B)10^n)^{1/2} - s_1.$$

Including the sum over n, and changing the sum over q to a sum over s_1, since $q = s_1 S$, the main term thus becomes

$$\sum_{B} \left(\frac{1}{2}\right) \sum_{f=1}^{10^{\delta m}} \frac{\mu(f)}{f^2} \sum_{g=1}^{10^{\delta m}} \frac{\mu(g)}{g^2} \sum_{m^c < n < m - m^c} S \sum_{*} (s_1^2 + (1-B)10^n)^{1/2} - s_1$$

where the sum * is over $10^n \le s_1 < 2.10^n$ with s_1, r_1 square-free and $B < \{s_1^2/10^n\} < B + \varepsilon$. The sum * is therefore equal to

$$\left(\varepsilon \left(\frac{6}{\pi^2} \right)^2 + O(10^{-\frac{n}{22}+\gamma}) \right) \sum_{10^n \le s_1 < 2.10^n} \psi$$

using the result already proved.

Expanding ψ by means of the binomial theorem we obtain

$$\psi = -s_1 + s_1 \left(1 + \frac{(1-B)10^n}{s_1^2} \right)^{1/2} = \frac{(1-B)10^n}{2s_1} - \frac{(1-B)^2 10^{2n}}{8s_1^3} + \cdots .$$

This is

$$\frac{(1-B)10^n}{2s_1} + O(10^{-n}).$$

We have

$$\sum_{10^n}^{2.10^n} \frac{1}{s_1} = \log 2 + O(10^{-n}).$$

Since the sum over B of $1 - B$ gives a factor of $1/2$, the main term becomes

$$\frac{1}{8} \log 2 \left(\frac{6}{\pi^2} \right)^4 10^m (m - 2m^c).$$

The other error terms are

$$O(10^{m-\delta m}), O(10^{m-\frac{n}{22}+\gamma})$$

(both from the main term),

$$O\left(\left(\frac{1}{2\varepsilon} \right) \frac{10^{m+\delta m}}{L+1} \right), O\left(\left(\frac{1}{2\varepsilon} \right) \frac{10^{m+\delta m-\frac{n}{22}+\gamma}}{L+1} \right),$$

another

$$O(10^{m-\delta m})$$

(from the cases $f, g > 10^{\delta m}$) and

$$O(\varepsilon 10^m)$$

(from the approximation of $\{s_2^2/10^m\}$). Putting $L = 10^{3\delta m}$ and $\varepsilon = 10^{-\frac{1}{2}\delta m}$ into these terms, the error we are left with is

$$O\left(10^{\frac{m}{2}+\frac{9}{2}\delta m}m\right) + O\left(10^{m-\frac{n}{22}+\gamma}\right) + O\left(10^{m-\frac{1}{2}\delta m}\right)$$

if $n \geq 2\delta m$, or

$$O\left(10^{\frac{m}{2}-\frac{n}{2}+\frac{11}{2}\delta m}\right) + O\left(10^{m-\frac{n}{22}+\gamma}\right)$$

if $n < 2\delta m$. Therefore if $n > 11\delta m$, then $\delta < 1/11$ and the least possible error is greater than $O(10^{\frac{21}{22}m})$. If $2\delta m \leq n \leq 11\delta m$, then we must have $\delta < 1/9$. If instead $n < 2\delta m$, then we see that we must have $\delta \leq 1/11$. If $n = m^c$, with c sufficiently small, then the error is no more than $O(10^{m-\frac{m^c}{22}+\gamma})$.

We must now consider the other cases that arise from the different ways the intervals can intersect. Leaving the longer interval the same we first consider the different possibilities for the shorter interval. These correspond to the cases 1, 2 and 3 in the first part of the proof: we have already done case 1. It is clear from the earlier work that case 2 will give a factor of 2 instead of the factor 1/2 we had in case 1, and case 3 will give a factor of 1/2. Thus the total for all possible shorter intervals that could intersect with this longer interval is

$$3.\frac{1}{4}\log 2\left(\frac{6}{\pi^2}\right)^4 10^m(m - 2m^c).$$

Now we consider the three possible types of longer interval. We have done the first case. It is only necessary, by the above, to do the calculation in each case when the shorter interval is

$$\left[\frac{s_2^2}{10^{2m}}, \frac{r_2+1}{10^m}\right).$$

Case 2. This is the case that arises when the interval

$$\left[\frac{r_1}{10^n}, \frac{r_1+1}{10^n}\right)$$

lies completely inside the interval

$$\left[\frac{s_1^2}{10^{2n}}, \frac{(s_1+1)^2}{10^{2n}}\right).$$

As before this can be divided into 3 subcases:

Case A. We have $r_1/10^n < s_2^2/10^{2m} < (r_1+1)/10^n$ and $r_1 = [\frac{s_1^2}{10^n}+1]$. The length of the intersection is $1/10^n$ and therefore the range of s_2 is

$$\frac{S}{f^2}\left(\left(\left[\frac{s_1^2}{10^n}+1\right]10^n + 10^n\right)^{\frac{1}{2}} - \left(\left[\frac{s_1^2}{10^n}+1\right]10^n\right)^{\frac{1}{2}}\right)$$

$$= \frac{S}{f^2}\left((s_1^2+(2-B)10^n)^{\frac{1}{2}} - (s_1^2+(1-B)10^n)^{\frac{1}{2}}\right) = \frac{\psi S}{f^2}$$

where $B = \{\frac{s_1^2}{10^n}\}$ and we are redefining ψ. Therefore by the binomial theorem, ψ in this case is equal to:

$$\left(s_1 + \frac{(2-B)10^n}{2s_1}\right) - \left(s_1 + \frac{(1-B)10^n}{2s_1}\right) + O(10^{-n}).$$

Thus we need to sum $\frac{10^n}{2s_1}$ over $10^n \le s_1 < 2.10^n$ and this gives $\frac{1}{2}10^n \log 2$. Thus the factor obtained in this case is $\frac{1}{2}\log 2$ in the place where we had $\frac{1}{4}\log 2$ in case 1.

Case B. Here we have $r_1 = [\frac{s_1^2}{10^n}+2]$ but this evidently makes no difference to the result. The difference is that now the range of s is restricted to

$$\frac{3-B}{2}10^n < s_1 < 2.10^n$$

and thus we obtain $\frac{1}{2}\log\frac{4}{3-B}$ which must be summed over B. Since B is uniformly distributed between 0 and 1, we use Koksma's inequality (Theorem

5.4 of [2]) to replace this sum by an integral over $[0,1)$, which gives an error of

$$O\left(\frac{D_N(B)}{N}\right).$$

This leads to an error term of the same size as previous ones using the Erdős-Turàn theorem ([2], Theorem 5.5) and the exponential sum estimates already derived. We have:

$$\int_0^1 \log(3-B)dB = \int_2^3 \log x dx = [-x + x\log x]_2^3.$$

We will combine this case with case C.

Case C. Here $\frac{4-B}{2}10^n < s_1 < 2.10^n$ and therefore the contribution is $\frac{1}{2}\log\frac{4}{4-B}$ summed over B, which gives

$$\frac{1}{2}(\log 4 - [-x + x\log x]_3^4).$$

Therefore cases B and C together give the factor

$$\log 4 - \frac{1}{2}\left[-x + x\log x\right]_2^4$$

$$= \log 4 + \frac{1}{2}(4 - 4\log 4 - 2 + 2\log 2) = 1 - \log 2.$$

Case 3. In this case the longer interval is $(r_1/10^n, (s_1+1)^2/10^{2n})$ where $r_1 = [\frac{(s_1+1)^2}{10^n}]$, and so the length of the intersection is

$$\frac{\left\{\frac{(s_1+1)^2}{10^n}\right\}}{10^n}.$$

Therefore the range of s_2 is $S\psi/f^2$ where

$$\psi = \left(\left[\frac{(s_1+1)^2}{10^n}\right]10^n + B10^n\right)^{\frac{1}{2}} - \left(\left[\frac{(s_1+1)^2}{10^n}\right]10^n\right)^{\frac{1}{2}}$$

where $B = \{\frac{(s_1+1)^2}{10^n}\}$. This is equal to

$$s_1 + 1 - \left(s_1 + 1 - \frac{B10^n}{2(s_1+1)}\right) + O(10^{-n})$$

by the binomial theorem and therefore the contribution for this case is $\frac{1}{4}\log 2$ as in case 1.

Thus the total contribution to the main term is

$$3\left(2\left(\frac{1}{4}\log 2\right) + \frac{1}{2}\log 2 + 1 - \log 2\right) = 3.$$

When we sum over $m < N$ and divide by 10^m, the main term becomes

$$\frac{3}{2}\left(\frac{6}{\pi^2}\right)^4 (N^2 + N),$$

the sum $\sum_{m<N} 2m^c$ being $< O(N^{1+c})$.

Because we made the assumption $n < m$, we need to multiply the main term by 2. Hence

$$\int_1^4 \sum_{n=1}^{N} \chi \mathcal{I}_n) \sum_{m=1}^{N} \chi(\mathcal{I}_m) d\alpha^2 = 3\left(\frac{6}{\pi^2}\right)^4 N^2 + O(N^{1+c}) + O(N).$$

Therefore using the previous result too, we have

$$\int_1^4 \left(\sum_{n=1}^{N} \chi(\mathcal{I}_n) - \left(\frac{6}{\pi^2}\right)^2 N\right)^2 d\alpha^2$$

$$= (3 + 3 - 6)\left(\frac{6}{\pi^2}\right)^4 N^2 + O(N^{1+c}) = O(N^{1+c}).$$

By Lemma 1.3, since we have

$$\int_1^4 |F_N(\alpha)|^2 d\alpha^2 = O(N^{1+c})$$

and

$$|F_N(\alpha) - F_{N-1}(\alpha)| = \left|\chi(\mathcal{I}_N) - \left(\frac{6}{\pi^2}\right)^2\right| = O(1)$$

we deduce that for almost all α,

$$F_N(\alpha) = \sum_{n=1}^{N} \chi(\mathcal{I}_n) - \left(\frac{6}{\pi^2}\right)^2 N = O\left(N^{\frac{2+c}{3}} (\log N)^{\frac{1}{2}+\gamma}\right).$$

Hence for almost all α:

$$\sum_{n=1}^{N} \chi(\mathcal{I}_n) = \left(\frac{6}{\pi^2}\right)^2 N + O\left(N^{\frac{2+c}{3}} (\log N)^{\frac{1}{2}+\gamma}\right)$$

and the result is proved.

We now prove a more general result. The method of proof that we have used in the two dimensional result above, which involves calculating the lengths of many intersections of intervals, would be much more complicated here because of the great number of cases that would need to be considered. We therefore approach the problem in a different way. This leads to a larger error term: we can only say the error is $O(N(\log N)^{-1/2})$.

Theorem 3.2. *For almost all α the number of solutions to $[10^n\alpha^{a_1}]$, $[10^n\alpha^{a_2}]$, $\cdots [10^n\alpha^{a_k}]$ simultaneously square-free, for $n \le N$, is*

$$\left(\frac{6}{\pi^2}\right)^k N + O(N(\log N)^{-1/2})$$

as $N \to \infty$ where the a_i are a strictly increasing sequence of natural numbers.

Proof. Assume $1 \le \alpha \le 2$. The number of solutions to $[10^n\alpha^{a_1}]$, $[10^n\alpha^{a_2}]$, $\cdots, [10^n\alpha^{a_k}]$ simultaneously square-free for $n \le N$ is

$$\sum_{n=1}^{N} \chi_n(\alpha)$$

where $\mathcal{I}_n = \{\alpha : [10^n\alpha^{a_1}], [10^n\alpha^{a_2}], \cdots, [10^n\alpha^{a_k}] \text{ square-free }\}$ and

$$\chi_n(\alpha) = \begin{cases} 1 & \text{if } \alpha \in \mathcal{I}_n \\ 0 & \text{otherwise} \end{cases}.$$

We have

$$\sum_{n=1}^{N}\chi(\mathcal{I}_n)=\sum_{n=1}^{N}\sum_{d_1=1}^{(2^{a_1}.10^n)^{1/2}}\mu(d_1)\cdots\sum_{d_k=1}^{(2^{a_k}.10^n)^{1/2}}\mu(d_k)\sum_{**}1$$

where the sum $**$ is over $l_1d_1^2=[10^n\alpha^{a_1}]$ such that $l_2d_2^2=[10^n\alpha^{a_2}],\cdots l_kd_k^2=[10^n\alpha^{a_k}]$. This sum can be divided into two parts. Firstly the main sum in which each d_i is only summed between 1 and $\log^2 N$. Secondly, error terms of the type

$$\sum_{n=1}^{N}\sum_{\substack{d_1>\log^2 N\\ l_1d_1^2=[10^n\alpha^{a_1}]}}1.$$

These error terms we now show to be $o(N)$.

We need to find an upper bound for the size of

$$\sum_{n=1}^{N}\sum_{\substack{d_1>\log^2 N\\ l_1d_1^2=[10^n\alpha^{a_1}]}}1.$$

We denote this sum by $S_{N,\alpha}$. Consider $\frac{S_{N,\alpha}}{N}$. This is

$$\leq\frac{1}{N}\sum_{n=1}^{N}\sum_{d_1>\log^2 N}\frac{1}{d_1^2}=O\left(\frac{1}{\log^2 N}\right).$$

We have that

$$\sum_{N=2^k}\frac{1}{\log^2 N}=O\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)$$

which converges. If we let

$$T_{k,\alpha}=\sum_{n=1}^{2^k}\sum_{\substack{d_1>k^2/4\\ l_1d_1^2=[10^n\alpha^{a_1}]}}1$$

then we have that

$$S_{N,\alpha}\leq T_{k,\alpha}\qquad\text{when}\qquad 2^{k-1}\leq N\leq 2^k$$

for large enough k. This is since

$$\log^2 N \geq (k-1)^2 \log^2 2 > k^2/4$$

in this range.

We have

$$\int_1^2 T_{k,\alpha} d\alpha \leq \int_1^2 \sum_{n=1}^{2^k} \sum_{d_1 > \frac{k^2}{4}} \frac{1}{d_1^2} d\alpha \ll \frac{2^k}{k^2},$$

hence if we define

$$V_k = \left\{ \alpha : T_{k,\alpha} > \frac{2^k}{k^{\frac{1}{2}}} \right\}$$

then

$$\mu(V_k) \ll k^{-\frac{3}{2}}.$$

This is since, if we suppose for a contradiction that

$$\mu(V_k) > O(k^{-\frac{3}{2}}),$$

then it follows that

$$\int_1^2 T_{k,\alpha} d\alpha > O\left(\frac{2^k}{k^{\frac{1}{2}}} k^{-\frac{3}{2}} \right) = O\left(\frac{2^k}{k^2} \right),$$

which is false. Therefore

$$\sum_{k=1}^{\infty} \mu(V_k)$$

converges. This implies that almost all α belong to only finitely many V_k, by the first Borel-Cantelli lemma ([2], page 8). Hence for almost all α,

$$T_{k,\alpha} < \frac{2^k}{k^{\frac{1}{2}}}$$

for all large enough k.

We obtain that

$$S_{N,\alpha} \ll \frac{N}{(\log N)^{1/2}}$$

for almost all α, and hence $S_{N,\alpha} = o(N)$ as required.

The other cases when $d_i > \log^2 N$ and $e_i > \log^2 N$ are exactly the same.

We now consider the main sum. As before we can rewrite the conditions of the sum as

$$\left\{\frac{10^n \alpha^{a_1}}{d_1^2}\right\} < \frac{1}{d_1^2}, \ldots, \left\{\frac{10^n \alpha^{a_k}}{d_k^2}\right\} < \frac{1}{d_k^2}.$$

We now use a Fourier series to approximate the number of times these fractional parts lie inside the given intervals. Thus the main error term of the sum is

$$\sum_{n=1}^{N} \sum_{d_1=1}^{\log^2 N} \cdots \sum_{d_k=1}^{\log^2 N} \sum_{t_1,\ldots t_k=-L}^{L} \hat{T}_i(t) e\left(\frac{10^n \alpha^{a_1} t_1}{d_1^2} + \cdots + \frac{10^n \alpha^{a_k} t_k}{d_k^2}\right)$$

where the t_is are not all zero and

$$|\hat{T}_i(t)| \le \prod_i \min\left(\frac{1}{d_i^2} \mp \frac{1}{L+1}, \frac{3}{2|t_i|}\right).$$

The other error terms arising from the Fourier series include the error obtained in calculating the main term, which we now determine:

$$\sum_{n=1}^{N} \sum_{d_1=1}^{\log^2 N} \mu(d_1) \cdots \sum_{d_k=1}^{\log^2 N} \mu(d_k) \frac{1}{(d_1 \cdots d_k)^2}$$

$$= \left(\frac{6}{\pi^2}\right)^k N + O\left(\sum_{n=1}^{N} \frac{1}{\log^2 N}\right).$$

This error is thus $O(N/\log^2 N)$.

Also arising from the Fourier series, we have error terms similar to the following example

$$\sum_{n=1}^{N} \sum_{d_1=1}^{\log^2 N} \cdots \sum_{d_k=1}^{\log^2 N} \frac{1}{(d_1 \cdots d_{k-1})^2 (L+1)} = O\left(\sum_{n=1}^{N} \frac{\log^2 N}{L}\right)$$

$$= O\left(\frac{N}{\log^2 N}\right)$$

if we choose $L = \log^4 N$. There are also smaller order terms.

We need to prove that

$$\int_1^2 \left(\sum_{n=1}^{N} \chi(\mathcal{I}_n)' - \left(\frac{6}{\pi^2}\right)^k N \right)^2 d\alpha$$

is of sufficiently small magnitude, where $\sum_{n=1}^{N} \chi(\mathcal{I}_n)'$ denotes the sum of only the main term and the main error term of the sum $\sum_{n=1}^{N} \chi(\mathcal{I}_n)$ summed over d_i between 1 and $\log^2 N$. Expanding this we find that we must consider

$$\int_1^2 \left(\sum_{n=1}^{N} \chi(\mathcal{I}_n)' \sum_{m=1}^{N} \chi(\mathcal{I}_m)' + \left(\frac{6}{\pi^2}\right)^{2k} N^2 - 2\left(\frac{6}{\pi^2}\right)^k N \sum_{n=1}^{N} \chi(\mathcal{I}_n)' \right) d\alpha.$$

Consider the integral of the main error term of the sum

$$\sum_{n=1}^{N} \chi(\mathcal{I}_n)'.$$

We need to look at

$$\int_1^2 \sum_{t_1,\ldots,t_k=-L}^{L} e\left(\frac{10^n \alpha^{a_1} t_1}{d_1^2} + \cdots + \frac{10^n \alpha^{a_k} t_k}{d_k^2} \right) d\alpha \tag{3.5}$$

Let k' be the largest i such that $t_i \neq 0$. Let

$$f(\alpha) = \frac{10^n \alpha^{a_1} t_1}{d_1^2} + \cdots + \frac{10^n \alpha^{a_k} t_k}{d_k^2}.$$

Then

$$f^{(a_{k'})}(\alpha) = \frac{10^n a_{k'}! t_{k'}}{d_{k'}^2}.$$

It follows that

$$|f^{(a_{k'})}(\alpha)| = \left| \frac{10^n (a_{k'})! t_{k'}}{d_{k'}^2} \right| > 0$$

and hence by Lemma 1.1 we have

$$\left| \int_1^2 e(f(\alpha)) d\alpha \right| \ll \frac{d_{k'}^{2/a_{k'}}}{(10^n (a_{k'})! t_{k'})^{1/a_{k'}}}.$$

Therefore (3.5) is

$$\ll \sum_{k'=1}^{k'=k} \frac{d_{k'}^{2/a_{k'}}}{(10^n(a_{k'})!)^{1/a_{k'}}} \sum_{t_1,\ldots,t_k=-L}^{L} \frac{1}{|t_{k'}|^{1/a_{k'}}}$$

$$\ll \sum_{k'=1}^{k'=k} \frac{d_{k'}^{2/a_{k'}}}{10^{n/a_{k'}}}(2L+1)^{k'}$$

$$\ll \frac{d_{k'}^{2/a_k}}{10^{n/a_k}}(2L+1)^k.$$

Therefore the order of the error is

$$\sum_{n=1}^{N} \sum_{d_1=1}^{\log^2 N} \cdots \sum_{d_k=1}^{\log^2 N} \prod_{d_i} \max\left(\frac{1}{d_i^2} + \frac{1}{L+1}, \frac{3}{2L}\right) \frac{d_k^{2/a_k}}{10^{n/a_k}}(2L+1)^k$$

$$= O\left(\sum_{n=1}^{N} \sum_{d_k=1}^{\log^2 N} \frac{d_k^{(2/a_k)-2}}{10^{n/a_k}}(\log^4 N)^k\right).$$

This error is

$$O\left(\log^{4k} N \sum_{n=1}^{N} 10^{-\frac{n}{a_k}}\right)$$

since

$$\sum_{d_k=1}^{\log^2 N} d_k^{(2/a_k)-2}$$

converges, since we can assume $a_k > 2$ (because the case $a_k = 2$ has been covered by Theorem 3.1). Thus the main error term is

$$O\left(\frac{\log^{4k} N}{10^{\frac{N}{a_k}}}\right).$$

Now we consider

$$\int_1^2 \sum_{n=1}^{N} \chi(\mathcal{I}_n)' \sum_{m=1}^{N} \chi(\mathcal{I}_m)'.$$

We assume that $m < n - N^c$ for $c > 0$ suitably small.

Proceeding exactly as before, we find that for the main error term we need to consider

$$\sum_{t_1,\dots,t_k=-L}^{L}\sum_{u_1,\dots,u_k=-L}^{L}\int_1^2 e\left(\frac{10^n\alpha^{a_1}t_1}{d_1^2}+\cdots+\frac{10^n\alpha^{a_k}t_k}{d_k^2}+\right.$$
$$\left.\frac{10^m\alpha^{a_1}u_1}{e_1^2}+\cdots+\frac{10^m\alpha^{a_k}u_k}{e_k^2}\right)d\alpha \quad (3.6)$$

where not all of the t_is and u_is are zero.

Let k' be the largest i (if such an i exists) such that $t_i \neq 0$ and let k'' be the largest i (if such an i exists) such that $u_i \neq 0$. We denote the expression in the brackets above by $f(\alpha)$ as usual. Suppose that $k' = k''$. Then we have

$$f^{(a_{k'})}(\alpha) = \frac{10^n a_{k'}!t_{k'}}{d_{k'}^2} + \frac{10^m a_{k'}!u_{k'}}{e_{k'}^2}.$$

There are two subcases:

Case 1. $t_{k'}$ and $u_{k'}$ have the same sign. In this case

$$|f^{(a_{k'})}(\alpha)| = \left|\frac{10^n a_{k'}!t_{k'}}{d_{k'}^2} + \frac{10^m a_{k'}!u_{k'}}{e_{k'}^2}\right| > 0$$

and therefore by Lemma 1.1 we have

$$\left|\int_1^2 e(f(\alpha))d\alpha\right| \ll \frac{(d_{k'}e_{k'})^{2/a_{k'}}}{(a_{k'}!)^{1/a_{k'}}(10^n e_{k'}^2 t_{k'} + 10^m d_{k'}^2 u_{k'})^{1/a_{k'}}}.$$

Hence (3.6) is

$$\ll \sum_{k'=1}^{k}(d_{k'}e_{k'})^{2/a_{k'}}\sum_{t_i,u_i=-L}^{L}\frac{1}{(10^n e_{k'}^2 t_{k'} + 10^m d_{k'}^2 u_{k'})^{1/a_{k'}}}$$

$$< \sum_{k'=1}^{k}(d_{k'}e_{k'})^{2/a_{k'}}\sum_{t_i,u_i=-L}^{L}\frac{1}{(10^n e_{k'}^2 t_{k'})^{1/a_{k'}}}$$

$$\ll \sum_{k'=1}^{k}(\log^4 N)^{2/a_{k'}}(2L+1)^{2k'}\frac{1}{10^{n/a_{k'}}}.$$

Therefore the main error term becomes

$$O\left(\sum_{n,m=1}^{N}\sum_{d_1=1}^{\log^2 N}\frac{\mu(d_1)}{d_1^2}\cdots\sum_{e_k=1}^{\log^2 N}\frac{\mu(e_k)}{e_k^2}(\log^4 N)^{2/a_k}(2L+1)^{2k}\frac{1}{10^{n/a_k}}\right).$$

Therefore the contribution from this case is of order

$$\sum_{n,m=1}^{N}(\log N)^{8/a_k}(\log^4 N)^{2k}\frac{1}{10^{n/a_k}}$$

$$= O\left((\log N)^{8k+(8/a_k)}\sum_{n,m=1}^{N}10^{-n/a_k}\right)$$

again choosing $L = \log^4 N$. This is

$$O\left(\frac{N(\log N)^{8k+(8/k)}}{10^{\frac{N}{a_k}}}\right).$$

Case 2. If $t_{k'}$ and $u_{k'}$ have opposite signs, then we consider $f^{(a_{\hat{k}})}(\alpha)$ where $\hat{k}$ is the next largest i after k' such that either t_i or u_i is non-zero. If there is no such $\hat{k}$ then t_i and u_i are only non-zero when $i = k'$. We will consider this case later.

$f^{(a_{\hat{k}})}(\alpha)$ is equal to

$$\frac{10^n a_{\hat{k}}!t_{\hat{k}}}{d_{\hat{k}}^2}+\frac{10^m a_{\hat{k}}!u_{\hat{k}}}{e_{\hat{k}}^2}+\frac{10^n a_{k'}\cdots(a_{k'}-a_{\hat{k}}+1)\alpha^{a_{k'}-a_{\hat{k}}}t_{k'}}{d_{k'}^2}$$
$$+\frac{10^m a_{k'}\cdots(a_{k'}-a_{\hat{k}}+1)\alpha^{a_{k'}-a_{\hat{k}}}u_{k'}}{e_{k'}^2}$$

where one (but not both) of $t_{\hat{k}}$ and $u_{\hat{k}}$ may be zero. Thus $f^{(a_{\hat{k}})}(\alpha)$ may be zero when

$$\alpha = \left(\frac{-\frac{10^n a_{\hat{k}}!t_{\hat{k}}}{d_{\hat{k}}^2}-\frac{10^m a_{\hat{k}}!u_{\hat{k}}}{e_{\hat{k}}^2}}{\frac{10^n a_{k'}\cdots(a_{k'}-a_{\hat{k}}+1)t_{k'}}{d_{k'}^2}+\frac{10^m a_{k'}\cdots(a_{k'}-a_{\hat{k}}+1)u_{k'}}{e_{k'}^2}}\right)^{\frac{1}{a_{k'}-a_{\hat{k}}}}.$$

If this happens, we let $\hat{\alpha}$ denote this value of α and define

$$\mathcal{J} = \left(\hat{\alpha}-\frac{1}{10^{\sigma n}},\hat{\alpha}+\frac{1}{10^{\sigma n}}\right)$$

where σ is to be chosen later. Then if $\alpha \in \mathcal{J}$ we have

$$\left|\int_{\alpha\in\mathcal{J}} e(f(\alpha))d\alpha\right| < \frac{2}{10^{\sigma n}}.$$

If instead $\alpha \notin \mathcal{J}$, we have

$$\left(\hat{\alpha} + \frac{1}{10^{\sigma n}}\right)^{a_{k'}-a_{\hat{k}}} > \hat{\alpha}^{a_{k'}-a_{\hat{k}}} + \left(\frac{1}{10^{\sigma n}}\right)^{a_{k'}-a_{\hat{k}}}$$

and therefore

$$|f^{(a_{\hat{k}})}(\alpha)| > \left|\left(\tfrac{1}{10^{\sigma n}}\right)^{a_{k'}-a_{\hat{k}}} \left(\tfrac{10^n a_{k'}\cdots(a_{k'}-a_{\hat{k}}+1)t_{k'}}{d_{k'}^2} + \tfrac{10^m a_{k'}\cdots(a_{k'}-a_{\hat{k}}+1)u_{k'}}{e_{k'}^2}\right)\right| > 0.$$

Therefore by Lemma 1.1,

$$\left|\int_{\alpha\notin\mathcal{J}} e(f(\alpha))d\alpha\right| \ll \frac{(10^{\sigma n})^{\frac{a_{k'}-a_{\hat{k}}}{a_{\hat{k}}}}}{\left(\frac{10^n a_{k'}\cdots(a_{k'}-a_{\hat{k}}+1)t_{k'}}{d_{k'}^2} + \frac{10^m a_{k'}\cdots(a_{k'}-a_{\hat{k}}+1)u_{k'}}{e_{k'}^2}\right)^{1/a_{\hat{k}}}}$$

$$< \frac{(10^{\sigma n})^{\frac{a_{k'}-a_{\hat{k}}}{a_{\hat{k}}}}}{\left(\frac{10^n a_{k'}\cdots(a_{k'}-a_{\hat{k}}+1)}{2L+1}\right)^{1/a_{\hat{k}}}},$$

assuming $n < m$. This error dominates the one we had for $\alpha \in \mathcal{J}$ thus we can ignore the latter error.

The expression (3.6) is now

$$\ll \sum_{k'=1}^{k}\sum_{\hat{k}=1}^{k'-1} 10^{\sigma n\frac{(a_{k'}-a_{\hat{k}})}{a_{\hat{k}}}} \log^{4/a_{\hat{k}}} N \log^{8k} N 10^{-n/a_{\hat{k}}}$$

$$= O(10^{\sigma n a_k} \log^{8k+(4/a_k)} N 10^{-n/a_k}).$$

Therefore the main error term in this case is

$$O\left((\log N)^{8k+(4/a_k)} \sum_{n,m=1}^{N} 10^{\sigma n a_k - (n/a_k)}\right).$$

Therefore this error is small enough if we choose σ to be less than

$$\frac{1}{a_k^2}.$$

If instead there is no such $a_{\hat{k}}$, then the only non-zero terms in $f(\alpha)$ are the two with $i = k'$. We have

$$|f^{a_{k'}}(\alpha)| = \left| \frac{10^n a_{k'}! t_{k'}}{d_{k'}^2} + \frac{10^m a_{k'}! u_{k'}}{e_{k'}^2} \right|.$$

The magnitude of the first of these terms is

$$> \frac{10^m 10^{N^c} a_{k'}!}{\log^2 N}$$

whereas the second has magnitude

$$< 10^m a_{k'}! \log^2 N.$$

Therefore $f^{a_{k'}}(\alpha)$ is never zero for N sufficiently large, since then we have $10^{N^c} > \log^4 N$. Thus

$$|f^{a_{k'}}(\alpha)| > 10^m a_{k'}! \left(\frac{10^{n^c}}{\log^2 N} - \log^2 N \right) > 0$$

and hence by Lemma 1.1 we have

$$\left| \int_1^2 e(f(\alpha)) d\alpha \right| \ll \frac{(\log N)^{2/a_{k'}}}{10^{m/a_{k'}} (10^{N^c} - \log^4 N)^{1/a_{k'}}}.$$

Therefore (3.6) is

$$\ll (2L+1)^2 (\log N)^{2/a_k} \frac{1}{10^{m/a_k} (10^{N^c} - \log^4 N)^{1/a_k}}$$
$$\ll (\log N)^{8+(2/a_k)} 10^{-m/a_k}.$$

Now suppose that $k' \neq k''$. If $k' > k''$ then we consider the k'-th derivative. We have

$$f^{(a_{k'})}(\alpha) = \frac{10^n a_{k'}! t_{k'}}{d_{k'}^2}.$$

Thus this case proceeds as for the earlier single sum part of the proof. We obtain that (3.6) is

$$\ll \sum_{k'=1}^{k} \frac{d_{k'}^{2/a_{k'}}}{(10^n)^{1/a_{k'}}} \sum_{t_i,u_i=-L}^{L} \frac{1}{|t_{k'}|^{1/a_{k'}}}$$

$$\ll \frac{d_{k'}^{2/a_k}}{(10^n)^{1/a_k}}(2L+1)^{2k}.$$

This is as in the earlier proof except we have $(2L+1)^{2k}$ replacing $(2L+1)^k$. Hence, by following the same argument through and using $L = \log^4 N$, we obtain the error term

$$O\left((\log N)^{8k} \sum_{n,m=1}^{N} 10^{-\frac{n}{a_k}}\right).$$

Arising from the intersection

$$\int_1^2 \sum_{n=1}^{N} \chi(\mathcal{I}_n)' \sum_{m=1}^{N} \chi(\mathcal{I}_m)' d\alpha,$$

we also have the main term

$$\sum_{n,m=1}^{N} \left(\frac{6}{\pi^2}\right)^{2k}$$

and the error term obtained when the main term and main error terms from the single sum are multiplied. This latter is

$$O\left(\frac{N \log^{4k} N}{10^{\frac{N}{a_k}}}\right).$$

We also have the error from the sum over n and m with $n - N^c < m < n$. This is $O(N^{1+c})$. Therefore again by Lemma 1.3 as in the previous proof, we have that

$$\sum_{n=1}^{N} \chi(\mathcal{I}_n)' = \left(\frac{6}{\pi^2}\right)^k N + O\left(N^{\frac{2+c}{3}} (\log N)^{\frac{1}{2}+\gamma}\right)$$

for almost all α. Since we have already shown that for all α,

$$\sum_{n=1}^{N} \chi(\mathcal{I}_n) - \sum_{n=1}^{N} \chi(\mathcal{I}_n)' = O(N(\log N)^{-1/2}),$$

it follows that

$$\sum_{n=1}^{N} \chi(\mathcal{I}_n) = \left(\frac{6}{\pi^2}\right)^k N + O(N(\log N)^{-1/2})$$

for almost all α, which completes the proof of the theorem.

Chapter 4

Integer Parts of Sequences 2: Sums of Two Squares

We now consider numbers $n = a^2 + b^2$ which are sums of two squares. As for square-free numbers there is also a function which gives the number of sums of two squares in an interval, although this time the function is more complex. We will use the following well-known formula:

$$\sum_{\substack{n=1 \\ n=a^2+b^2}}^{N} 1 = \sum_{n=2}^{N} \frac{K}{(\log n)^{1/2}} + O\left(\frac{N}{\log N}\right) \tag{4.1}$$

where K is the constant:

$$2^{-1/2} \prod_{p \equiv 3 \bmod 4} \left(1 - \frac{1}{p^2}\right)^{-1/2},$$

p denoting a prime (here and throughout this chapter).

The problem is more difficult here since sums of two squares are not evenly distributed but become more sparse as n increases. Thus we prove a result for almost all (α, β) where α and β are independent. In another sense though, the problem is easier. For the main term in the double sum over m and n, it is only necessary that we consider $n < m/10$ (or some such fraction of m) whereas in chapter three we needed $n < m - m^c$ i.e n and m much closer together. We prove:

Theorem 4.1. *For almost all (α, β), the number of solutions for $n \leq N$ to $[10^n\alpha]$ and $[10^n\beta]$ simultaneously equal to a sum of two squares is*

$$\frac{K^2}{\log 10} \log N + o(\log N)$$

as $N \to \infty$, where K is the constant in the theorem above.

Proof. Without loss of generality we need only consider $(\alpha, \beta) \in [1, 2)^2$. Let

$$\mathcal{I}_n = \left(\bigcup_{\substack{r \text{ s.s.} \\ 10^n \leq r < 2.10^n}} \left[\frac{r}{10^n}, \frac{r+1}{10^n}\right), \bigcup_{\substack{s \text{ s.s.} \\ 10^n \leq s < 2.10^n}} \left[\frac{s}{10^n}, \frac{s+1}{10^n}\right) \right)$$

where we use s.s. as an abbreviation of a sum of two squares. Then $(\alpha, \beta) \in \mathcal{I}_n$ precisely when $[10^n\alpha]$ and $[10^n\beta]$ are sums of two squares for $1 < \alpha, \beta < 2$. Therefore the number of solutions to $[10^n\alpha]$ and $[10^n\beta]$ simultaneously sums of two squares, for $n \leq N$, when $1 < \alpha < 2$, is

$$\sum_{n=1}^{N} \chi(\mathcal{I}_n) \qquad \text{where} \qquad \chi(\mathcal{I}_n) = \begin{cases} 1 & \text{if } (\alpha, \beta) \in \mathcal{I}_n \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 1.2 it is sufficient to consider for any $U, V \leq N$:

$$\int_1^2 \int_1^2 \left(\sum_{n=U}^{V} \chi(\mathcal{I}_n) - \frac{K^2}{\log 10} \sum_{n=U}^{V} \frac{1}{n} \right)^2 d\alpha d\beta.$$

This is equal to

$$= \int_1^2 \int_1^2 \left(\sum_{n=U}^{V} \chi(\mathcal{I}_n) \sum_{m=U}^{V} \chi(\mathcal{I}_m) + \left(\frac{K^2}{\log 10} \sum_{m=U}^{V} \frac{1}{m} \right)^2 - 2 \frac{K^2}{\log 10} \sum_{n=U}^{V} \frac{1}{n} \sum_{n=U}^{V} \chi(\mathcal{I}_n) \right) d\alpha d\beta.$$

We start by finding a lower bound for

$$\int_1^2 \int_1^2 \sum_{n=U}^{V} \chi(\mathcal{I}_n) d\alpha d\beta = \sum_{n=U}^{V} \lambda(\mathcal{I}_n).$$

This is equal to

$$\sum_{n=U}^{V} \left(\sum_{\substack{r \text{ s.s.} \\ 10^n \leq r < 2.10^n}} \frac{1}{10^n} \right) \left(\sum_{\substack{s \text{ s.s.} \\ 10^n \leq s < 2.10^n}} \frac{1}{10^n} \right)$$

$$= \sum_{n=U}^{V} \frac{K^2}{10^{2n}} \left(\sum_{r=10^n}^{2.10^n} \frac{1}{(\log r)^{1/2}} \right)^2 + O\left(\sum_{n=U}^{V} \frac{1}{n^{3/2}} \right)$$

by the formula (4.1). This is

$$\geq \sum_{n=U}^{V} \frac{K^2}{10^{2n}} \left(\frac{10^n}{(\log 2 + n \log 10)^{1/2}} \right)^2 = \frac{1}{\log 10} \sum_{n=U}^{V} \frac{K^2}{n} + O\left(\frac{1}{n^2} \right).$$

We also need to find

$$\int_1^2 \int_1^2 \sum_{n=U}^{V} \sum_{m=U}^{V} \chi(\mathcal{I}_n \cap \mathcal{I}_m) d\alpha d\beta$$

which is equal to

$$\sum_{n=U}^{V} \sum_{m=U}^{V} \lambda(\mathcal{I}_n \cap \mathcal{I}_m).$$

The required intervals are therefore

$$\left[\frac{r_1}{10^n}, \frac{r_1 + 1}{10^n} \right) \bigcap \left[\frac{r_2}{10^m}, \frac{r_2 + 1}{10^m} \right)$$

and

$$\left[\frac{s_1}{10^n}, \frac{s_1+1}{10^n}\right) \bigcap \left[\frac{s_2}{10^m}, \frac{s_2+1}{10^m}\right).$$

We assume to start with, that $n < m$. It is sufficient to assume that the length of the intersection of the two intervals,

$$\left[\frac{r_1}{10^n}, \frac{r_1+1}{10^n}\right) \qquad \text{and} \qquad \left[\frac{r_2}{10^m}, \frac{r_2+1}{10^m}\right),$$

is the maximum possible whenever they intersect, i.e of length $1/10^m$. Thus we obtain the conditions

$$\frac{r_1}{10^n} < \frac{r_2+1}{10^m} \qquad \text{and} \qquad \frac{r_2}{10^m} < \frac{r_1+1}{10^n}.$$

These imply that

$$r_1 10^{m-n} - 1 \leq r_2 < r_1 10^{m-n} + 10^{m-n}.$$

Similarly for r replaced by s. Therefore it is necessary to find

$$\frac{1}{10^{2m}} \sum_{*} 1$$

where the sum (*) is over r_1, r_2, s_1, s_2 sums of two squares and $r_1 10^{m-n} - 1 \leq r_2 < r_1 10^{m-n} + 10^{m-n}$ and $s_1 10^{m-n} - 1 \leq s_2 < s_1 10^{m-n} + 10^{m-n}$.

We first consider the case when $m/10 \leq n < m$: Letting $\theta = 10^{n-m}$ we can write the conditions above equivalently as $-\theta \leq \theta r_2 - r_1 < 1$ and $-\theta \leq \theta s_2 - s_1 < 1$ We can now use a modification of Lemma 8.6 of [2] to obtain the number of solutions of

$$|\theta r_2 - r_1| < 2$$

and similarly of $|\theta s_2 - s_1| < 2$ for r_1, r_2, s_1, s_2 all sums of two squares and $10^m \leq r_2, s_2 \leq 2.10^m$. This is more than the number of solutions to our problem but since we only require an upper bound, this is sufficient.

We need to find the number of solutions of

$$\left|\frac{r_2}{q} - r_1\right| < 2$$

where q is the integer $10^{m-n} = 1/\theta$, i.e. the number of r_2 and r_1 such that $r_2 = qr_1 + b$ where $r_2 \in [10^m, 2.10^m]$, for $|b| < 2.10^{m-n}$.

We will need the fact that a sum of two squares has a prime factorisation of the form

$$2^k \prod u_i^2 \prod t_j$$

where each $u_i \equiv 3 \bmod 4$ and each $t_j \equiv 1 \bmod 4$, the u_i's and t_j's not necessarily distinct (see [14], page 104). We consider first the case when $b = 0$:

If $b = 0$ then $r_2 = r_1 10^{m-n}$ and therefore if r_1 is a sum of two squares then r_2 is also a sum of two squares, since the prime factorisation of r_2 is that of r_1 with the additional factors 2^{m-n} and 5^{m-n}, and $5 \equiv 1 \bmod 4$. Therefore the number of solutions for r_1 and r_2 sums of two squares is equal to the number of sums of two squares in the interval $[10^n, 2.10^n)$. This is of order

$$K \sum_{t=10^n}^{2.10^n} \frac{1}{(\log t)^{1/2}} \ll \frac{10^n}{n^{\frac{1}{2}}}.$$

Now instead assume that $b \neq 0$. For each b, the solutions have the form

$$r_1 = x, r_2 = qx + b.$$

It is well known that if $\mathcal{B}$ is the set of primes congruent to 3 mod 4, then there exists a constant C such that

$$\sum_{\substack{p<y \\ p\in\mathcal{B}}} \frac{1}{p} \geq \frac{1}{2} \log\log y - C. \tag{4.2}$$

We let $r_2 = r_2' u^2$ and $r_1 = r_1' v^2$, where u is the product of the primes $u_i \equiv 3 \bmod 4$ in the prime factorisation of r_2 and likewise v is the product of the primes $v_i \equiv 3 \bmod 4$ in the prime factorisation of r_1.

We now let $(u, v) = g$ and define $u' = u/g$ and $v' = v/g$ so that $(u', v') = 1$. Therefore $r_2 = r_2' u'^2 g^2$ and $r_1 = r_1' v'^2 g^2$ so we need

$$r_2' = \frac{10^{m-n}}{u'^2 g^2} x + \frac{b}{u'^2 g^2}$$

and

$$r_1' = \frac{x}{v'^2 g^2}.$$

We note that since $g^2 | r_2$ and $g^2 | r_1 = x$, we have $g^2 | b$.

By the first equation we have that $x \equiv -b\overline{10^{m-n}} \bmod u'^2 g^2$. Therefore $r_1' g^2 \equiv -\overline{v'^2} b \overline{10^{m-n}} \bmod u'^2 g^2 = -\overline{v'^2} b \overline{10^{m-n}} + c u'^2 g^2$ for some integer c. We thus have

$$r_1' = \frac{-\overline{v'^2} b \overline{10^{m-n}}}{g^2} + c u'^2. \tag{4.3}$$

Therefore $x = -v'^2 \overline{v'^2} b \overline{10^{m-n}} + c u'^2 v'^2 g^2$ and thus we also obtain

$$r_2' = \frac{b - 10^{m-n} v'^2 \overline{v'^2} b \overline{10^{m-n}}}{u'^2 g^2} + 10^{m-n} v'^2 c. \tag{4.4}$$

Since r_2' is an integer we have that $u'^2 | F = (b - 10^{m-n} v'^2 \overline{v'^2} b \overline{10^{m-n}})$.

By Theorem 2.3 of [15] and (4.2), the number of solutions in c of (4.3) and (4.4) is

$$\ll \prod_{\substack{p|E \\ p \equiv 3 \bmod 4}} \left(1 - \frac{1}{p}\right)^{\rho(p)-2} \frac{y}{\log y}$$

where

$$\begin{aligned} E &= u'^2 v'^2 10^{m-n} \left(\frac{-10^{m-n} v'^2 \overline{v'^2} b \overline{10^{m-n}} + b + 10^{m-n} v'^2 \overline{v'^2} b \overline{10^{m-n}}}{g^2} \right) \\ &= \frac{10^{m-n} u'^2 v'^2 b}{g^2} \end{aligned}$$

and $\rho(p)$ is the number of solutions in c to

$$10^{m-n} v'^2 u'^2 c^2 - \frac{2.10^{m-n} v'^2 \overline{v'^2} b \overline{10^{m-n}} c}{g^2} + \frac{bc}{g^2}$$

$$+\frac{\overline{v'^2}^2 b^2 \overline{10^{m-n}}^2 10^{m-n} v'^2}{u'^2 g^4} - \frac{\overline{v'^2} b^2 \overline{10^{m-n}}}{u'^2 g^4} \equiv 0 \bmod p$$

and y, which needs to be > 1, is the range of c. Since the range for c is $\frac{10^n}{u'^2 v'^2 g^2} \neq 1$, this condition holds and therefore the number of solutions is

$$\ll \prod_{\substack{p|E \\ p \equiv 3 \bmod 4}} \left(1 - \frac{1}{p}\right)^{\rho(p)-2} \frac{10^n}{u'^2 v'^2 g^2 (\log 10^n - \log u'^2 - \log v'^2 - \log g^2)}.$$

If $p|E$ and $p \equiv 3 \bmod 4$ then we have that p does not divide 10^{m-n} since $2, 5 \not\equiv 3 \bmod 4$ and thus must divide b or u' or v'. Suppose that $p|b$, but $p \not| u'$ or v'. Then we have that $\rho(p)$ is equal to the number of solutions of

$$10^{m-n} v'^2 u'^2 c^2 \equiv 0 \bmod p$$

which is 1. So it follows that $\rho(p) = 1$ for all p in the product. Thus the product over $p|b, p \not| u', p \not| v', p \equiv 3 \bmod 4$ for a particular b is

$$\leq \prod_{\substack{p|b \\ p \equiv 3 \bmod 4}} \left(1 - \frac{1}{p}\right)^{-1}.$$

This is

$$\leq \prod_{p|b} \left(1 - \frac{1}{p}\right)^{-1} = \prod_{p|b} \frac{p}{p-1} = \prod_{p|b} \frac{p}{\phi(p)} = \frac{b}{\phi(b)}.$$

Suppose now that $p|u'$ or $p|v'$ but $p \not| b$. Then $\rho(p) = 1$ in these cases too and we get a contribution of

$$\frac{u'v'}{\phi(u')\phi(v')}.$$

If $p|v'$ and $p|b$ then $\rho(p) = p$ since all integers c satisfy the required equation. This gives a contribution of

$$\prod_{\substack{p|v', p|b \\ p \equiv 3 \bmod 4}} \left(1 - \frac{1}{p}\right)^{p-2} \leq 1.$$

This is smaller than the extra contribution we have counted by summing over all $p|b$ instead of $p|b, p\nmid u', p\nmid v'$ and can therefore be ignored.

Finally if $p|u'$ and $p|b$ then again we have $\rho(p) = p$, since $u'^2|F$ and so we can also ignore this contribution.

We need to sum over the possible values of b, u', v', and g. We obtain

$$\sum_{\substack{|b|<2.10^{m-n}/g^2 \\ (b,10)=1}} \frac{b}{\phi(b)} \sum_{u',\, v',\, g} \left(\frac{u'v'}{\phi(u')\phi(v')} \right) .$$

$$\left(\frac{10^n}{u'^2 v'^2 g^2 (\log 10^n - \log u'^2 - \log v'^2 - \log g^2)} \right) . \quad (4.5)$$

When $u'v' \leq n$ then

$$\sum_{u',v'} \left(\frac{u'v'}{\phi(u')\phi(v')} \right) \frac{1}{u'^2 v'^2 g^2} = O(1)$$

and

$$\frac{10^n}{(\log 10^n - \log u'^2 - \log v'^2 - \log g^2)} \ll \frac{10^n}{\log 10^n} .$$

When $u'v' > n$ then

$$\log 10^n - \log u'^2 - \log v'^2 - \log g^2 \geq \log 2,$$

since c is an integer > 1, and also

$$\sum_{u',v'} \left(\frac{u'v'}{\phi(u')\phi(v')} \right) \frac{10^n}{u'^2 v'^2} \ll \frac{10^n}{n} .$$

Therefore the sum (4.5) is

$$\ll \frac{\phi(10)}{10} 10^{m-n} \frac{10^n}{\log 10^n} \ll \frac{10^m}{n}$$

by Lemma 7.4 of [2].

Since $n < m$ we can neglect the result for the case $b = 0$ since we have

$$\frac{10^n}{n^{1/2}} = o\left(\frac{10^m}{n} \right) .$$

The same result is obtained upon replacing r by s and so our upper bound for the number of solutions is

$$O\left(\frac{10^{2m}}{n^2}\right).$$

We finally need to divide by 10^{2m} and sum over n and m as follows:

$$\sum_{m=U}^{V}\sum_{n=m/10}^{m}\frac{1}{n^2}\leq\sum_{m=U}^{V}\frac{90}{m}.$$

The case $n = m$ gives a contribution of $\sum_{n=U}^{V}\lambda(\mathcal{I}_n)$ which is

$$O\left(\sum_{n=U}^{V}\frac{1}{N}\right).$$

We now consider the case when $n < m/10$. We use the analogue for sums of two squares of Huxley's theorem for primes in short intervals [16] which states that if $x^{(7/12)+\varepsilon} \leq y < x$ for some $\varepsilon > 0$, then the number of sums of two squares in the interval $(x, x + y)$ is

$$\frac{Ky}{(\log x)^{1/2}}\left(1+O\left(\frac{1}{(\log x)^{1/2}}\right)\right)$$

as $x \to \infty$.

The condition $n < m/10$ implies that $10^{9m/10} \leq 10^{m-n} < 10^m$ and so the condition required for the theorem is satisfied for $x = 10^m$ and $y = 10^{m-n}$. We therefore obtain that the number of solutions for r_2 equal to a sum of two squares for $r_1 10^{m-n} \leq r_2 < r_1 10^{m-n} + 10^{m-n}$ is

$$\frac{K10^{m-n}}{(\log 10^m)^{1/2}}\left(1+O\left(\frac{1}{(\log 10^m)^{1/2}}\right)\right)$$

as $m \to \infty$.

We also need the number of solutions to r_1 equal to a sum of two squares with $10^n \leq r_1 \leq 2.10^n$. This is

$$\frac{K10^{n}}{(\log 10^n)^{1/2}}\left(1+O\left(\frac{1}{(\log 10^n)^{1/2}}\right)\right)$$

and so the number of solutions for r_1 and r_2 sums of two squares is

$$\frac{K^2 10^m}{(\log 10)(mn)^{1/2}} + O\left(\frac{10^m}{nm^{1/2}}\right),$$

the other two error terms, which are

$$O\left(\frac{10^m}{mn^{1/2}}\right) \qquad \text{and} \qquad O\left(\frac{10^m}{nm}\right),$$

being smaller.

We similarly get the same result for s_1 and s_2 and so the total is

$$\left(\frac{K^2}{\log 10}\right)^2 \frac{10^{2m}}{mn} + O\left(\frac{10^{2m}}{mn^{3/2}}\right).$$

Finally all we need to do is divide by 10^{2m} and sum over n and m. We have:

$$\left(\frac{K^2}{\log 10}\right)^2 \sum_{m=U}^{V} \sum_{\substack{n=U \\ n<m/10}}^{V} \frac{1}{mn}$$

$$= \left(\frac{K^2}{\log 10}\right)^2 \sum_{m=U}^{V} \sum_{n=U}^{V} \frac{1}{mn} + O\left(\left(\frac{K^2}{\log 10}\right)^2 \sum_{m=U}^{V} \sum_{\substack{n=U \\ m/10<n<m}}^{V} \frac{1}{mn}\right).$$

This error is at most

$$O\left(\sum_{m=U}^{V} \frac{1}{m} \sum_{n=m/10}^{m} \frac{1}{n}\right) = O\left(\sum_{m=U}^{V} \frac{1}{m}(\log m - \log m + \log 10)\right)$$

$$= O\left(\sum_{m=U}^{V} \frac{1}{m}\right).$$

Also we have

$$\sum_{m=U}^{V} \sum_{n=U}^{m/10} \frac{1}{mn^{3/2}} = O\left(\sum_{m=U}^{V} \frac{1}{m}\right).$$

Therefore we have

$$\int_1^2\int_1^2\left(\sum_{n=U}^{V}\chi(\mathcal{I}_n)\sum_{m=U}^{V}\chi(\mathcal{I}_m)+\left(\frac{K^2}{\log 10}\sum_{n=U}^{V}\frac{1}{n}\right)^2\right.$$
$$\left.-2\frac{K^2}{\log 10}\sum_{n=U}^{V}\frac{1}{n}\sum_{n=U}^{V}\chi(\mathcal{I}_n)\right)d\alpha d\beta=O\left(\sum_{m=U}^{V}\frac{1}{m}\right).$$

We now use a variation of Lemma 1.2 given in [2], Lemma 1.5. Since we have

$$\int_1^2\int_1^2\left(\sum_{n=u}^{v}(F(n,\alpha)-x_n)\right)^2 d\alpha d\beta$$
$$=\int_1^2\int_1^2\left(\sum_{n=U}^{V}\left(\chi(\mathcal{I}_n)-\frac{K^2}{\log 10}.\frac{1}{n}\right)\right)^2 d\alpha d\beta<A\sum_{n=u}^{v}\frac{1}{n}$$

for a constant A and

$$\sum_{n=1}^{N}\frac{1}{n}$$

diverges as $N\to\infty$, we obtain the result that

$$\sum_{n=1}^{N}\chi(\mathcal{I}_n)=\frac{K^2}{\log 10}\sum_{n=1}^{N}\frac{1}{n}+O\left((\log N)^{1/2}\left(\log(\log N+2))^{3/2+\varepsilon}+\frac{K^2}{\log 10}\right)\right)$$

for any $\varepsilon>0$ for almost all (α,β). Hence the theorem follows.

Chapter 5

Integer Parts of Sequences 3: Primes

We now investigate integer parts of sequences that take prime values. Making the integer parts of sequences simultaneously prime is more difficult than making them simultaneously sums of two squares. The results that can be proved are therefore weaker. The aim here will be to consider the set of (α, β) in $[0, 1)^2$ for which $[10^n\alpha]$ and $[10^n\beta]$ are simultaneously prime infinitely often. It can be proved that this set has Hausdorff dimension equal to 2. The problem of finding the Hausdorff dimension of the set of α such that $[10^n\alpha]$ and $[10^n\alpha^2]$ are simultaneously prime infinitely often, seems to be much more difficult.

We give a definition of Hausdorff dimension below; first we need to define the diameter d of a ball $A \in \mathbb{R}^n$:

$$d(A) = \max_{x,y \in A} |x - y|$$

Definition Let δ and S be such that

1. For any $\varepsilon > 0$, there exists a covering (ξ_i) of the set S with $d(\xi_i) < \varepsilon$ such that

$$\sum_{i=1}^{\infty}(d(\xi_i))^{\gamma} < 1 \qquad \text{for all } \gamma > \delta$$

and

2. There exists $\varepsilon > 0$ such that for all coverings (ξ_i) of S such that $d(\xi_i) < \varepsilon$ we have

$$\sum_{i=1}^{\infty}(d(\xi_i))^{\gamma} \geq 1$$

for all $\gamma < \delta$.

Then the set S has Hausdorff dimension δ.

Let S be the set of $(\alpha, \beta) \in [0,1)^2$ with $[10^n\alpha]$ and $[10^n\beta]$ simultaneously prime infinitely often. This set has measure zero by Theorem 3 of [12]. Even so it turns out that it has full Hausdorff dimension and is dense in $[0,1)^2$.

Theorem 5.1. *The set S described above has Hausdorff dimension 2.*

Proof. We will show that $\exists\,\varepsilon > 0$ such that for any $\delta > 0$ and for any collection of intervals (ξ_i) satisfying both

$$d(\xi_i) < \varepsilon \qquad \text{for all}\, i, \text{ and} \qquad \sum_{i=1}^{\infty} d(\xi_i)^{2-\delta} < 1,$$

the collection of intervals (ξ_i) does not cover the set S. This will imply S has Hausdorff dimension ≥ 2 and therefore Hausdorff dimension 2. We will do this by constructing a sequence of nested compact sets $\mathcal{I}_I \supseteq \mathcal{I}_{I+1} \supseteq \mathcal{I}_{I+2} \cdots$, such that

$$\bigcap_{i=I}^{\infty} \mathcal{I}_i \subset S$$

but $\bigcap_{i=I}^{\infty} \mathcal{I}_i$ (which is non-empty) is not covered by the collection of intervals (ξ_i).

We define

$$K_i = \frac{10^{2.3^i}}{4^{i^2+1}(\log 10)^{2i}} \qquad \text{and} \qquad \varepsilon_i = 10^{-i^2/\delta}$$

and choose I large enough so that both

$$K_{i+1} < \frac{10^{2.3^{i+1}}}{2.4^{i^2+1}3^{2i+2}(\log 10)^{2i+2}} - \frac{4.10^{2.3^{i+1}}}{10^{i^2}3^{2i}(\log 10)^2} - 10^{2(i+1)^2/\delta} \tag{5.1}$$

and

$$\left(\frac{3}{4}.\frac{10^{2.3^i}}{3^{i+1}\log 10}(1+o(1))\right)^2 \geq \frac{1}{2}.\frac{10^{4.3^i}}{3^{2i+2}(\log 10)^2} \tag{5.2}$$

hold for all $i \geq I$.

The $\mathcal{I}_i$s will each consist of M_i boxes of the form

$$\left[\frac{p_1}{10^{3^i}}, \frac{p_1+\frac{3}{4}}{10^{3^i}}\right] \times \left[\frac{p_2}{10^{3^i}}, \frac{p_2+\frac{3}{4}}{10^{3^i}}\right] \qquad (*)$$

where p_1 and p_2 are primes. We will show for all $i \geq I$ that $M_i > K_i$ and that none of the boxes in $\mathcal{I}_i$ intersect with any ξ_j with $\varepsilon_i \leq d(\xi_j) < \varepsilon_{i-1}$. This will prove the result.

We use induction. For the case $i = I$ we let $\mathcal{I}_I$ be the union of all boxes of the form (*) for $i = I$ and $10^{3^i} < p_1, p_2 < 2.10^{3^I}$. By the prime number theorem [14], page 226, we have

$$M_I = (\pi(2.10^{3^I}) - \pi(10^{3^I}))^2 \sim \left(\frac{2.10^{3^I}}{\log 2 + 3^I \log 10} - \frac{10^{3^I}}{3^I \log 10}\right)^2$$
$$> K_I.$$

Also, if we let $\varepsilon = \varepsilon_I$ then no box in $\mathcal{I}_I$ intersects with any ξ_j with $\varepsilon_I \leq d(\xi_j) < \varepsilon_{I-1}$ since $d(\xi_j) < \varepsilon_I$ for all j. This establishes the case $i = I$.

For the inductive step we suppose we have $\mathcal{I}_i$. We have by Huxley's theorem [16] that the number of primes in the interval

$$\left[10^{3^{i+1}}, 10^{3^{i+1}} + \frac{3}{4}10^{3^{i+1}2/3}\right]$$

is

$$\frac{\frac{3}{4}10^{2.3^i}}{\log 10^{3^{i+1}}}(1+o(1))$$

and therefore we can find

$$\geq \frac{1}{2}.\frac{10^{4.3^i}}{(\log 10)^2 3^{2i+2}}$$

boxes of the form

$$\left[\frac{p_1}{10^{3^{i+1}}}, \frac{p_1+\frac{3}{4}}{10^{3^{i+1}}}\right] \times \left[\frac{p_2}{10^{3^{i+1}}}, \frac{p_2+\frac{3}{4}}{10^{3^{i+1}}}\right]$$

contained entirely within each box of $\mathcal{I}_i$ by (5.2). Thus the number of boxes in total is

$$> \frac{10^{2.3^i}}{4^{i^2+1}(\log 10)^{2i}}.\frac{1}{2}.\frac{10^{4.3^i}}{(\log 10)^2 3^{2i+2}} = \frac{10^{2.3^{i+1}}}{2.4^{i^2+1}3^{2i+2}(\log 10)^{2i+2}}$$

by the inductive hypothesis.

The number of these boxes which intersect a set ξ_j of diameter d is

$$< \left(\frac{d10^{3^{i+1}}}{3^i \log 10} + 1\right)^2 < \left(\frac{2d10^{3^{i+1}}}{3^i \log 10}\right)^2 + 1$$

since the number of primes in an interval of length $10^{3^{i+1}}d$ is

$$< \frac{3.10^{3^{i+1}}d}{\log(10^{3^{i+1}}d)}$$

by the Brun-Titchmarsh inequality [15], Theorem 3.7, and we may need to count an extra prime outside the interval which nevertheless gives boxes that intersect the set ξ_j.

Since we are assuming that $\sum_{i=1}^{\infty} d(\xi_i)^{2-\delta} < 1$, we have

$$\sum_{\substack{\xi_j \\ \varepsilon_{i+1} \leq d < \varepsilon_i}} d^2 < \sum_{d<\varepsilon_i} d^2 < \varepsilon_i^\delta \sum_{d<\varepsilon_i} d^{2-\delta} < \varepsilon_i^\delta = 10^{-i^2}.$$

and also

$$\sum_{\substack{\xi_j \\ d > \varepsilon_{i+1}}} 1 < \varepsilon_{i+1}^{\delta-2} \sum_{j=1}^{\infty} \xi_j^{2-\delta} < \varepsilon_{i+1}^{\delta-2} < 10^{2(i+1)^2/\delta}.$$

Hence the number of boxes which do not intersect a set ξ_j with $\varepsilon_{i+1} \leq d(\xi_j) < \varepsilon_i$ is

$$> \frac{10^{2.3^{i+1}}}{2.4^{i^2+1}3^{2i+2}(\log 10)^{2i+2}} - \frac{4.10^{2.3^{i+1}}}{3^{2i}10^{i^2}(\log 10)^2} - 10^{2(i+1)^2/\delta} > K_{i+1}$$

by (5.1). This proves the theorem.

This proof can clearly be generalised to show that the set of $(\alpha_1, \ldots, \alpha_m) \in [0,1)^m$, such that $[10^n\alpha_1], [10^n\alpha_2], \ldots, [10^n\alpha_m]$ are simultaneously prime infinitely often, has Hausdorff dimension m.

Theorem 5.2. *The set of* (α, β) *with* $[10^n\alpha]$ *and* $[10^n\beta]$ *simultaneously prime infinitely often, is dense in* $\mathbb{R}^2$.

Proof. We consider the interval $[0,1)^2$ again. We need to show that any open set, B, of this interval contains a point of our set S.

B must contain a set of the form $(a,b) \times (c,d)$ where (a,b) and (c,d) are open intervals. Let N_1 be such that the first N_1 digits of a after the decimal point agree with those of b after the decimal point in each of the N_1 places but the $N_1 + 1$th digits do not agree. In a similar way let N_2 be such that the first N_2 digits of c after the decimal point agree with those of d after the decimal point in each of the N_2 places but the $N_2 + 1$th digits do not agree.

We will construct two real numbers α and β which satisfy $[10^n\alpha]$ and $[10^n\beta]$ simultaneously prime infinitely often and $(\alpha, \beta) \in (a, b) \times (c, d)$.

We define the first N_1 digits of α after the decimal point to be the same as those of a. The first N_2 digits of β are similarly defined to be the same as those of c. The $N_1 + 1$st digit of α and the $N_2 + 1$st digit of β can then be chosen so that $(\alpha, \beta) \in (a, b) \times (c, d)$. Now we let $N > \max(N_1 + 1, N_2 + 1)$ and define the rest of the first N digits of α and β to be any digits.

We can define the next $n - N$ digits of α in such a way that $[10^n\alpha]$ is prime for this n as long as n is sufficiently large. This is because we need to find a prime in the interval $[x10^n, x10^n + 10^{n-N}]$, where x is a real number between 0 and 1, and by Huxley's theorem [16] there are

$$O\left(\frac{10^{n-N}}{\log 10^n}\right)$$

primes in this interval if

$$10^{n-N} \geq x^{(7/12)+\varepsilon}10^{(7n/12)+\varepsilon}.$$

This condition holds if we choose $n > 12N/5$. We can define the corresponding $n - N$ digits of β similarly so that $[10^n\beta]$ is prime. Let this particular value of n be n_1. Now we repeat this process, defining a further $n_2 - n_1$ digits of α and of β so that $[10^n\alpha]$ and $[10^n\beta]$ are both prime for $n = n_2$. We can do this if $n_2 > 12n_1/5$. This process can be repeated indefinitely and so this proves the existence of an α and β such that $[10^n\alpha]$ and $[10^n\beta]$ are simultaneously prime infinitely often with $(\alpha, \beta) \in B$. This proves the theorem.

Almost all points on the line $(\alpha, \alpha) \in \mathbb{R}^2$ intersect with the set

$$\mathcal{A} = \{(\alpha, \beta) : [10^n\alpha], [10^n\beta] \text{ simultaneously prime infinitely often } \}$$

by [17]. We will now prove that this is the only line in $\mathbb{R}^2$ with this property; all others intersect $\mathcal{A}$ in a set of measure zero.

Theorem 5.3. *Let the set $\mathcal{A}$ be defined as above. Then $(\alpha, A\alpha + B) \cap \mathcal{A}$ has measure zero unless $A = 1$ and $B = 0$.*

Proof. Without loss of generality assume that $1 \leq \alpha \leq 2$ and $A \geq 0$.

We require $[10^n\alpha]$ and $[10^n(A\alpha + B)]$ simultaneously prime infinitely often. We define the following sequence of sets:

$$A_i = \left[\frac{p_{i1}}{10^i}, \frac{p_{i1}+1}{10^i}\right) \bigcup \cdots \bigcup \left[\frac{p_{ik}}{10^i}, \frac{p_{ik}+1}{10^i}\right), \qquad i \geq 1,$$

where $p_{i1}, \ldots p_{ik}$ are the primes between 10^i and 2.10^i. We also define the sets B_i:

$$\left[\frac{q_{i1}}{10^iA} - \frac{B}{A}, \frac{q_{i1}+1}{10^iA} - \frac{B}{A}\right) \bigcup \cdots \bigcup \left[\frac{q_{ik}}{10^iA} - \frac{B}{A}, \frac{q_{ik}+1}{10^iA} - \frac{B}{A}\right), \qquad i \geq 1,$$

where $q_{i1}, \ldots q_{ik}$ are the primes between $10^i(A + B)$ and $10^i(2A + B)$. We thus need $\beta = A\alpha + B$ and $\alpha \in A_i \cap B_i$ for infinitely many i.

We must find the size of the intersection $A_i \cap B_i$. Consider the intersection of an interval from A_i with one from B_i. The intervals are of the form

$$\left[\frac{p_1}{10^i}, \frac{p_1+1}{10^i}\right) \qquad \text{and} \qquad \left[\frac{p_2}{10^iA} - \frac{B}{A}, \frac{p_2+1}{10^iA} - \frac{B}{A}\right).$$

Case 1. Suppose that

$$\frac{p_1}{10^i} \leq \frac{p_2}{10^iA} - \frac{B}{A}.$$

Then we also need

$$\frac{p_2}{10^iA} - \frac{B}{A} < \frac{p_1+1}{10^i}.$$

Suppose first that $A > 1$. The length of the intersection is thus $\leq 1/10^i A$. We need

$$p_1 \leq \frac{p_2}{A} - \frac{B.10^i}{A} < p_1 + 1,$$

equivalently

$$0 < \frac{p_2}{A} - p_1 - \frac{B.10^i}{A} < 1.$$

We thus need to determine the number of solutions to

$$\left| \frac{p_2}{A} - p_1 - \frac{B.10^i}{A} \right| < 1,$$

where $10^i \leq p_1, p_2 < 2.10^i$ are primes. To do this, we modify the proof of Lemma 8.6 in [2] to take into account the $B.10^i/A$ term:

By theorem 2.1 in [2], there exist $a, q \in \mathbb{Z}$ with $1 \leq a \leq q \leq 10^{3i/4}$ and $(a, q) = 1$ such that

$$\left| \frac{1}{A} - \frac{a}{q} \right| < \frac{1}{q 10^{3i/4}}.$$

Let

$$\lambda = \frac{1}{A} - \frac{a}{q} > 0.$$

We split the range for p_2 into blocks $[H, H + Z]$ where $Z = q10^{i/20}$. Then in this range we have

$$\frac{p_2}{A} - p_1 - \frac{B.10^i}{A} = \frac{p_2 a}{q} - p_1 + H\lambda - \frac{B.10^i}{A} + O(10^{-7i/10}).$$

If we let

$$C = \left[H\lambda - \frac{B.10^i}{A} + \frac{1}{2} \right]$$

then we need an upper bound for the number of solutions of

$$\left| \frac{p_2 a}{q} - p_1 + C \right| < 2.$$

This is equivalent to the number of solutions of

$$p_2 a = (p_1 - C)q + b \qquad \text{where } |b| < 2q.$$

For each b the solutions have the form

$$p_2 = \bar{a}b + xq,\ p_1 = C + xa + b\left(\frac{a\bar{a} - 1}{q}\right)$$

where $a\bar{a} = 1 \bmod q$.

Now we can use Theorem 2.3 in [15] to obtain a bound for the number of solutions in x. This is

$$\ll \frac{Z}{q\log^2 10^i}\frac{aq}{\phi(aq)} \sum_{\substack{|b|<2q \\ (b,q)=1=(qC+b,a)}} \frac{qC+b}{\phi(qC+b)}$$

$$\ll \frac{Z}{q\log^2 10^i}\frac{aq}{\phi(aq)} \sum_{\substack{(n,aq)=1 \\ |n-Cq|<2q}} \frac{n}{\phi(n)} \ll \frac{Z}{i^2}$$

if C is not so large that $\log C$ is of order equal to a power of q, since by Lemma 7.4 of [2],

$$\sum_{\substack{(n,aq)=1 \\ |n-Cq|<2q}} \frac{n}{\phi(n)} = (Cq + 2q - Cq + 2q)\frac{\phi(aq)}{aq}\prod_{p \not| aq}\left(1 + \frac{1}{p(p-1)}\right)$$

$$+O(\tau(aq)\log 2(Cq + 2q)).$$

When we sum over the blocks we obtain $O(10^i/i^2)$ as required.

In the case when C is large relative to q, we make use of the averaging over C. Write

$$C_0 = \left[\lambda 10^i - \frac{B.10^i}{A} + 1/2\right], C_1 = C_0 - 2$$

and

$$C_2 = \left[2\lambda 10^i - \frac{B.10^i}{A}\right] + 3.$$

We can assume that $C_0 \geq q^2$. Then our bound is now

$$\ll \sum_{0 \leq m \leq 10^i/Z} \frac{Z}{qi^2} \frac{aq}{\phi(aq)} \sum_{\substack{|b|<2q \\ (b,q)=1=(qC+b,a) \\ C=[(10^i+mZ)\lambda-(B.10^i/A)+1/2]}} \frac{qC+b}{\phi(qC+b)}$$

$$\ll \frac{Z}{qi^2} \frac{aq}{\phi(aq)} \frac{1}{Z\lambda} \sum_{\substack{(n,aq)=1 \\ qC_1 \leq n \leq qC_2}} \frac{n}{\phi(n)} \ll \frac{10^i}{i^2}$$

since by Lemma 7.4 of [2],

$$\sum_{\substack{(n,aq)=1 \\ qC_1 \leq n \leq qC_2}} \frac{n}{\phi(n)} = \left(2\lambda 10^i - \frac{B.10^i}{A} + 3 - \lambda 10^i + \frac{B.10^i}{A} + 2\right) q \frac{\phi(aq)}{q}$$

$$+O\left(\tau(aq) \log 2 \left(2\lambda 10^i - \frac{B.10^i}{A} + 3\right) q\right).$$

For the case $A < 1$, the range of the intersection is $\leq 1/10^i$ and we need the number of solutions of $0 < p_2 - Ap_1 - B10^i < A$. By the same method this gives the required result.

Suppose now that $A = 1$. In this case we need to find the number of solutions of

$$|p_2 - p_1 - B.10^i| < 1.$$

By Lemma 8.5 of [2] this is at most

$$\frac{K10^i}{i_2} \sum_{|B.10^i - n| < 2} \frac{n}{\phi(n)}$$

provided that $|B|.10^i > 3$. This is equal to

$$\frac{K10^i}{i^2} \frac{630\zeta(3)}{\pi^4} + O(\log(|B|.10^i + 2))$$

by Lemma 2.5 of [2]. For any $B \neq 0$, we can choose i sufficiently large to ensure that $|B|.10^i \geq 3$ and so we obtain the required bound.

There are three other ways the intervals can intersect.

Case 2. We have

$$\frac{p_2}{10^i A} - \frac{B}{A} \leq \frac{p_1}{10^i} < \frac{p_2+1}{10^i A} - \frac{B}{A}.$$

We write this as

$$0 \leq Ap_1 - p_2 + B.10^i < 1 \qquad \text{if} \qquad A < 1$$

and as

$$0 \leq p_1 - \frac{p_2}{A} + \frac{B.10^i}{A} < \frac{1}{A} \qquad \text{if} \qquad A > 1.$$

Case 3. We have

$$\frac{p_2}{10^i A} - \frac{B}{A} \leq \frac{p_1}{10^i} \leq \frac{p_2+1}{10^i A} - \frac{B}{A} - \frac{1}{10^i}$$

which gives

$$0 \leq Ap_1 - p_2 + B.10^i \leq 1 - A$$

where we must have $A < 1$ in this case.

Case 4. We have

$$\frac{p_1}{10^i} \leq \frac{p_2}{10^i A} - \frac{B}{A} \leq \frac{p_1+1}{10^i} - \frac{1}{10^i A}$$

which gives

$$0 \leq \frac{p_2}{A} - p_1 + \frac{B.10^i}{A} \leq 1 - \frac{1}{A}$$

where we must have $A > 1$ in this case.

It is clear that these cases give the same result as before.

We deduce that

$$\lambda(A_i \cap B_i) = O\left(\frac{10^i}{i^2}\frac{1}{A.10^i}\right) = O(1/i^2)$$

and therefore since

$$\sum_{i=1}^{\infty} \lambda(A_i \cap B_i)$$

converges, by the first Borel-Cantelli lemma ([2], page 8), we have that almost all α belong to only finitely many $A_i \cap B_i$. Hence the set of α that belong to infinitely many of the $A_i \cap B_i$ has measure zero. This completes the proof of the theorem.

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