On the theory of Point Weight Designs

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Abstract

A point-weight incidence structure is a structure of blocks and points where each point is associated with a positive integer weight. A point-weight design is a point-weight incidence structure where the sum of the weights of the points on a block is constant and there exist some condition that specifies the number of blocks that certain sets of points lie on. These structures share many similarities to classical designs. Chapter one provides an introduction to design theory and to some of the existing theory of point-weight designs.

Chapter two develops a new type of point-weight design, termed a row-sum point-weight design, that has some of the matrix properties of classical designs. We examine the combinatorial aspects of these designs and show that a Fisher inequality holds and that this is dependent on certain combinatorial properties of the points of minimal weight. We define these points, and the designs containing them, to be either 'awkward' or 'difficult' depending on these properties.

Chapter three extends the combinatorial analysis of row-sum point-weight designs. We examine structures that are simultaneously row-sum and point-sum point-weight designs, paying particular attention to the question of regularity. We also present several general construction techniques and specific examples of row-sum point-weight designs that are generated using these techniques.

Chapter four concentrates on the properties of the automorphism groups of point-weight designs with particular emphasis on row-sum point-weight designs. We introduce the idea of a structure being "t-homogeneous with respect to its orbital partition" and use this to derive a formula for the number of blocks a set of points lies upon. We also discuss the properties of the orbits of subgroups of the automorphism group.

In chapter five we extend the idea of a dual to point-weight incidence structures and, as an extension of this, develop the idea of an underlying dual. We also examine the properties of square point-weight designs, i.e. point-weight designs that have exactly as many points as blocks. We find that there exists a result of a similar nature to the Bruck-Chowla-Ryser theorem of symmetric designs.

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Notation

$Aut\ \mathcal{S}$	the automorphism group of the structure $\mathcal{S}$.
b	the number of blocks in a (possibly point-weight) incidence structure.
B, B_i	blocks of a (possibly point-weight) incidence structure, i.e. members of $\mathcal{B}$.
$\mathcal{B}$	the set of blocks of a (possibly point-weight) incidence structure.
$fix_V(\alpha)$	the set of elements of V that are mapped to themselves under the action of the group element α.
$g.c.d.(a_1, \ldots, a_n)$	the greatest common denominator of $a_1, \ldots, a_n$.
G, H	a group.
$H_p(M)$	the Hasse symbol of the matrix M with respect to the prime p.
I	the incidence set of a (possibly point-weight) incidence structure. This is a subset of $V \times \mathcal{B}$ that specifies which points lie on which blocks.
I	the identity matrix.
J	the matrix in which every entry is 1.
k	the sum of the weights of the points on any block in a point-weight design, or the number of points on any block in a (non-point-weight) design.
$l.c.m.(a_1, \ldots, a_n)$	the lowest common multiple of $a_1, \ldots, a_n$.
n, n_x	the order of the structure or of the point x.
$\mathcal{S}$, $\mathcal{T}$	a (possibly point-weight) incidence structure, usually $(V, \mathcal{B}, I, w)$.
$\mathcal{S}_x$	the derived structure of $\mathcal{S}$ at the point x.
$\mathcal{S}^x$	the point-residue of the structure $\mathcal{S}$ at x.
$Stab\, V$	the subgroup of elements in a group that map V onto itself.
r, r_x	the number of blocks with which a single point (or, more specifically, the point x) is incident.

t	the size of the set of points for which the design condition specifies the number of blocks with which this set must be incident. Normally the set will have either t points or the sum of the weights of the points in the set will add up to t.
$t-(v,k,\lambda)$	a classical design.
$t-(v,k,\lambda;W)$	a point-sum point-weight design.
u	the number of points of a point-weight incidence structure.
$\mathcal{U}$	the underlying incidence structure $(V,\mathcal{B},I)$ of a point-weight incidence structure $(V,\mathcal{B},I,w)$.
v	the sum of the weights of the points of a point-weight incidence structure or the number of points in a (non-point-weight) incidence structure.
V	the set of points of a (possibly point-weight) incidence structure.
$(V,\mathcal{B},I)$	an incidence structure.
$(V,\mathcal{B},I,w)$	a point-weight incidence structure.
w	the weight function of a point-weight incidence structure. This maps every point in V to its weight.
W	the weight set of a point-weight incidence structure, $Im(w)$.
x,y,z	points in a (possibly point-weight) incidence structure, i.e. members of V.
$\mathbb{Z}^+$	the set of all integers greater than zero, i.e. $\{1,2,3,\ldots\}$.
$\iota(S)$	the number of blocks that contain the set of points S.
λ	a constant associated with the design condition. Generally a set S that is, in some sense, of size t must be incident with a number of blocks dependent only upon λ and the weights of the points of S.
$\pi_t-(v,k,\lambda;W)$	a row-sum point-weight design.
$\sigma(S)$	the sum of the weights all the points in S.
$\sigma_t-(v,k,\lambda;W)$	a weight-sum point-weight design.
$\langle H\rangle$	the group generated by the set of group elements H.

Chapter 1

Introduction

Design theory has been studied for over one hundred years. Loosely it is concerned with the different ways a set of points may be linked together into sets called blocks such that some collection of points lies an identifiable number of blocks. Central to this is the concept of a design. A design is an incidence structure containing v points such that every block contains k points and every t points lie on exactly λ blocks. We will refer to these designs as 'classical designs' in order to differentiate them from point-weight designs.

It has been shown that there is a secret sharing scheme associated with every classical design and that these secret sharing schemes share certain properties with classical designs, specifically that the number of people who know any particular secret is constant and each person knows the same number of secrets. This was interpreted as showing that every person in the scheme was of equal importance, a condition which is not always desirable in a secret sharing scheme. As a response to this the concept of a point-weight design was proposed at Royal Holloway in the mid-1990s.

A point-weight design is a point-weight incidence structure that contains u points of total weight v, every block has total weight k and there exists some condition which specifies how many blocks certain sets of points may lie on. We also insist that a point-weight design with points that are all of the same weight must be isomorphic to a classical design. These structures share the same style of definition as classical designs but need not have a constant number of points on any block or the same number of blocks incident with any point. This definitions were introduced in the PhD theses of Richard Horne [9] and Tracey Powlesland [13]. They each introduced a specific design condition: that the number of blocks that are incident with a set of points S is constant when, in the case of Horne, S contains t points or, in the case of Powlesland, S contains points whose total weight is t.

This thesis introduces a third design condition: that a set S of t points should lie on a non-constant but calculable number of blocks. In particular we concentrate on a design condition that gives a closed form for the matrix MM^T where M

is the incidence matrix of the structure. This is a standard property of classical designs and hence we may analyse their structure using some previously unusable techniques of classical design theory.

In this chapter we introduce incidence structures and classical designs, the vast majority of the results we present in this chapter can be found in Hughes and Piper [12] or Beth, Jungnickel and Lenz [3]. A more generalized treatment can be found in Dembowski [7]. We also introduce point-weight incidence structures and two classes of point-weight designs: point-sum point-weight designs, as developed in [9], and weight-sum point-weight designs, as developed in [13].

In the second and third chapters we present a new kind of point-weight design, termed a row-sum point-weight design, and derive some of the combinatorial properties of that structure. We also begin to cite some examples and compare it to the existing types of point-weight design.

The fourth chapter deals with the relationship between point-weight designs and their automorphism or generating groups. We examine the common technique for constructing designs by picking a set of base blocks and permuting them within the point set. We also examine the properties of the orbits of a row-sum point-weight design under the action of a subgroup of the automorphism group.

The fifth chapter is concerned with square point-weight incidence structures and designs. We define the notion of the dual of a point-weight incidence structure and that of the underlying dual of a point-weight incidence structure and show that, with one exception, the dual or underlying dual of a point-weight design can not be a point-weight design. We then present non-existence results for square point-weight designs similar to those of classical symmetric designs.

1.1 Incidence structures

Definition 1.1.1 (Incidence structures) *An incidence structure $\mathcal{S}$ is a triple $(V, \mathcal{B}, I)$ where V and $\mathcal{B}$ are disjoint sets and $I \subseteq V \times \mathcal{B}$ presents a binary relation between them.*

We will call the set V the point set and elements of V points. We will call the set $\mathcal{B}$ the block set and elements of $\mathcal{B}$ blocks.

In an incidence structure $\mathcal{S}$, let x be a point (i.e. $x \in V$) and B be a block (i.e. $B \in \mathcal{B}$). If $(x, B) \in I$ then we say 'x is incident with B', 'x lies on B', 'x is a point of B', 'x is contained in B' and 'B contains x'. We call (x, B) a *flag*. We denote the number of points of $\mathcal{S}$ by v and the number of blocks of $\mathcal{S}$ by b.

It is possible that either V or $\mathcal{B}$ is empty however we shall assume that this does not occur in this thesis. Obviously a block may be associated with the points that it contains however it is possible for two blocks to contain the same points. If this occurs then we say the structure has *repeated blocks*. If a structure has no repeated blocks then each block is uniquely determined by the points it contains.

Definition 1.1.2 (Point-weight incidence structures) *A point-weight incidence structure is a quadruple $(V, \mathcal{B}, I, w)$ where $(V, \mathcal{B}, I)$ is an incidence structure and $w : V \to \mathbb{Z}^+$ is a function that assigns to each point of V a positive integer. We call w the weight function of $\mathcal{S}$.*

If x is a point of a point-weight incidence structure $\mathcal{S}$ then we say that 'x has weight $w(x)$' or that 'the weight of x is $w(x)$'.

Definition 1.1.3 *The weight-set W of a point-weight incidence structure $\mathcal{S} = (V, \mathcal{B}, I, w)$ is defined to be the image of the function w.*

Essentially a point-weight incidence structure is an incidence structure along with a weight-function w. Therefore, given a point-weight incidence structure, it is natural to consider the incidence structure from which it arose: the underlying incidence structure.

Definition 1.1.4 *If $\mathcal{S}$ is a point-weight incidence structure $(V, \mathcal{B}, I, w)$ then the underlying incidence structure of $\mathcal{S}$ is the incidence structure $\mathcal{U} = (V, \mathcal{B}, I)$.*

These definitions are very general so it is important to tell if two structures are basically the same but with some different labelling.

Definition 1.1.5 *Two incidence structures $\mathcal{S}_1 = (V_1, \mathcal{B}_1, I_1)$ and $\mathcal{S}_2 = (V_2, \mathcal{B}_2, I_2)$ are isomorphic if there exists bijections $\theta_V : V_1 \to V_2$ and $\theta_{\mathcal{B}} : \mathcal{B}_1 \to \mathcal{B}_2$ such that $(x, B) \in I_1$ if and only if $(\theta_V(x), \theta_{\mathcal{B}}(B)) \in I_2$. The pair $\theta = (\theta_V, \theta_{\mathcal{B}})$ is called an isomorphism.*

Hence any incidence structure with no repeated blocks is isomorphic to an incidence structure of the form $(X, Y, \in)$ where $Y \subseteq \mathcal{P}(X)$, $\mathcal{P}(X)$ denotes the set of subsets of the set X and $\in$ denotes the binary set relation of "belonging to". Since this thesis is concerned only with structures with no repeated blocks we will therefore generally assume that any incidence structure we refer to is of this form.

Definition 1.1.6 *Two point-weight incidence structures $\mathcal{S}_1 = (V_1, \mathcal{B}_1, I_1, w_1)$ and $\mathcal{S}_2 = (V_2, \mathcal{B}_2, I_2, w_2)$ are isomorphic if $\mathcal{S}_1 = (V_1, \mathcal{B}_1, I_1)$ is isomorphic to $\mathcal{S}_2 = (V_2, \mathcal{B}_2, I_2)$ and, for all $x \in V_1$, we have $w_2(\theta_V(x)) = w(x)$.*

However this definition is a little two general as it is possible that too point-weight incidence structures are basically the same but have different weight assignments.

Definition 1.1.7 *Two point-weight incidence structures $\mathcal{S}_1 = (V_1, \mathcal{B}_1, I_1, w_1)$ and $\mathcal{S}_2 = (V_2, \mathcal{B}_2, I_2, w_2)$ are equivalent if $\mathcal{S}_1 = (V_1, \mathcal{B}_1, I_1)$ is isomorphic to $\mathcal{S}_2 = (V_2, \mathcal{B}_2, I_2)$ and there exists a $\mu \in \mathbb{Q}$ such that, for all $x \in V_1$, we have $w_2(\theta_V(x)) = \mu \cdot w(x)$.*

Hence any point-weight incidence structure is equivalent to a point-weight incidence structure with $g.c.d.(W) = 1$. Lastly we introduce some useful notation.

Definition 1.1.8 *Let $\mathcal{S}$ be the point-weight incidence structure $(V, \mathcal{B}, I, w)$ and let $S \subseteq V$ then:*

$$\begin{aligned}\sigma(S) &= \sum_{x \in S} w(x) \\ \iota(S) &= |\{B \in \mathcal{B} : S \subseteq B\}|\end{aligned}$$

1.2 Classical designs

Definition 1.2.1 *A $t-(v,k,\lambda)$ design is an incidence structure $\mathcal{S}$ with v points that satisfies two conditions:*

1. *Every block contains exactly k points.*
2. *Every set of $t > 0$ points is contained in exactly $\lambda > 0$ blocks.*

We will refer to condition 1 as the *constant block size condition* and condition 2 as the *design condition*. A structure that satisfies the design condition is called a t-structure.

We have the following results for $t-(v,k,\lambda_t)$ structures.

Result 1.2.2 *If $\mathcal{S}$ is $t-(v,k,\lambda_t)$ design and $0 \leq s < t$ then $\mathcal{S}$ is a $s-(v,k,\lambda_s)$ design where λ_s satisfies:*

$$\lambda_s \binom{k-s}{t-s} = \lambda_t \binom{v-s}{t-s}.$$

Proof Let S be any set of s points and consider the pairs (S', B) where S' is a set of t points, B is a block and $S \subseteq S' \subseteq B$.

Assume that S lies on λ_s blocks then in each block there are $\binom{k-s}{t-s}$ ways to choose a set S' that lies on that block. Conversely, there are $\binom{v-s}{t-s}$ ways to pick a set S' that contains S and S' lies on exactly λ_t blocks. Hence

$$\lambda_s \binom{k-s}{t-s} = \lambda_t \binom{v-s}{t-s}$$

and so any set S of size s lies on the same number of blocks.

■

So in any classical design any single point is incident with the same number of blocks. We conventionally denote this number to be r.

Result 1.2.3 *If $\mathcal{S}$ is a $t-(v,k,\lambda_t)$ then:*

1. $bk = vr$ *and*

2. *if* $t \geq 2$ *then* $r(k-1) = \lambda_2(v-1)$ *where* $\mathcal{S}$ *is a* $2-(v,k,\lambda_2)$ *design.*

Proof

1. Let us attempt to count the number of flags (x, B) in $\mathcal{S}$. There exists v points and each point lies on r blocks, hence there exists vr flags but there also exists b blocks and each block contains k points, hence there exists bk flags. So $bk = vr$.

2. Pick a point x and count the flags (y, B) such that x and y are contained B. On one hand there are $v-1$ other points and there exists λ_2 blocks containing both those points, but on the other hand there are r blocks through x and $k-1$ other points on each block. So $r(k-1) = \lambda_2(v-1)$.

■

Definition 1.2.4 *An incidence structure is square if $v = b$ (i.e. $|V|$=$|\mathcal{B}|$). A square classical design is called a symmetric design.*

1.3 Related structures

If we start with an incidence structure there are several ways to generate useful new yet related incidence structures.

Definition 1.3.1 (Dual structures) *Suppose that $\mathcal{S}$ is a $(V, \mathcal{B}, I)$ incidence structure. We obtain the dual structure $\mathcal{S}' = (V', \mathcal{B}', I')$ by 'exchanging' the points and blocks, yet retaining incidence. So:*

$$\begin{aligned} V' &= \mathcal{B} \\ \mathcal{B}' &= V \\ I' &= \{(B, x) : B \in V',\ x \in \mathcal{B}' \text{ and } (x, B) \in I\}. \end{aligned}$$

Dual structures are of particular interest to design theory due to the following result:

Result 1.3.2 *If $\mathcal{S}$ is a $2-(v,k,\lambda)$ design then $\mathcal{S}'$, the dual structure of $\mathcal{S}$, is a 2-design if and only if $\mathcal{S}$ is symmetric.*

The proof of this result can be found in most books on design theory, including [3] and [12]. Two other important structures that can be constructed from any incidence structure are:

Definition 1.3.3 (Derived structure) *Suppose $\mathcal{S} = (V, \mathcal{B}, I)$ is an incidence structure and that x is a point of $\mathcal{S}$. The derived structure of $\mathcal{S}$ at x, denoted $\mathcal{S}_x = (V_x, \mathcal{B}_x, I_x)$, is obtained by removing the point x, all blocks not incident with x and all points no longer incident with any block. So:*

$$\begin{aligned}
\mathcal{B}_x &= \{B \in \mathcal{B} : (x, B) \in I\} \\
V_x &= V \setminus (\{x\} \cup \{y \in V : \text{there exists no block containing } x \text{ and } y\}) \\
I_x &= I \cap (V_x \times \mathcal{B}_x).
\end{aligned}$$

Definition 1.3.4 (Point-residue) *Suppose $\mathcal{S}$ is a $(V, \mathcal{B}, I)$ incidence structure and x is a point of $\mathcal{S}$. The point-residue of $\mathcal{S}$ at x, denoted $\mathcal{S}^x = (V^x, \mathcal{B}^x, I^x)$, is obtained by removing the point x, all blocks that are incident with x and then all points that are no longer incident with any block. So:*

$$\begin{aligned}
\mathcal{B}^x &= \{B \in \mathcal{B} : (x, B) \notin I\} \\
V^x &= V \setminus (\{x\} \cup \{y \in V : (y, B) \in I \text{ implies } (x, B) \in I\}) \\
I^x &= I \cap (V^x \times \mathcal{B}^x)
\end{aligned}$$

Both of these concepts can be naturally extended from acting on a single point to a set of points. For example the derived structure of $\mathcal{S}$ at a set of points S, $\mathcal{S}_S$, is the structure obtained by removing the points of S, the blocks that do not contain S and all points that no longer lie on any blocks.

Result 1.3.5 *If $\mathcal{S}$ is a $t-(v, k, \lambda_t)$ design with $t \geq 2$ and x is a point of $\mathcal{S}$ then:*

1. *$\mathcal{S}_x$ is a $(t-1)-(v-1, k-1, \lambda_t)$ design.*
2. *$\mathcal{S}^x$ is a $(t-1)-(v-1, k, \lambda_{t-1}-\lambda_t)$ design providing $v > k$, where λ_{t-1} is the number of blocks a set of $t-1$ points lies on in $\mathcal{S}$.*

Proof

1. As $t \geq 2$, we have that for any two points there exists at least one block that is incident with both of them. So $\mathcal{S}_x$ contains $v-1$ points and each block contains $k-1$ points. Consider a set S of $t-1$ points of $\mathcal{S}_x$. The blocks of $\mathcal{S}_x$ that contain S are precisely those blocks of $\mathcal{S}$ that contain $S \cup \{x\}$, hence there exists λ blocks that contain S in $\mathcal{S}_x$ and $\mathcal{S}_x$ is a $(t-1)-(v-1, k-1, \lambda)$ design.

2. It is obvious that each block of $\mathcal{S}^x$ contains k points. Suppose that a point y only lies on blocks that also contain x, then y lies on λ_2 blocks, where λ_2 is the constant number of blocks that contain any two points. However, by (1.2.2), this means that every point is contained in exactly λ_2 blocks and so $v = k$. Therefore there exists $v-1$ points in $\mathcal{S}^x$.

Now consider a set of $t-1$ points of $\mathcal{S}^x$, S say. The blocks of $\mathcal{S}^x$ that contain S are precisely those blocks of $\mathcal{S}$ that contain S but do not contain x, hence there are $\lambda_{t-1}-\lambda_t$ blocks that contain S in $\mathcal{S}^x$ and $\mathcal{S}^x$ is a $(t-1)-(v-1,k,\lambda_{t-1}-\lambda_t)$ design.

■

It is worth noting that the definitions of derived structures and point-residues can be applied naturally to point-weight incidence structures too. If $\mathcal{S}$ is a $(V,\mathcal{B},I,w)$ and $\mathcal{S}'=(V',\mathcal{B}',I')$ is a sub-structure of $(V,\mathcal{B},I)$ then we may induce a weight function w' on $\mathcal{S}'$ by $w'(x)=w(x)$. Therefore we may define the derived structure (respectively point-residue) of a point-weight incidence structure as the derived structure (respectively point-residue) of the underlying incidence structure along with the induced weight function.

1.4 Incidence matrices

Two of the most fundamental tools used in analysing incidence structures are the incidence matrix of the structure and the automorphism group, and both of these concepts can be extended to point-weight incidence structures. In the section we will deal with incidence matrices and examine automorphism groups in section 1.5.

Definition 1.4.1 (Incidence matrix) *Suppose $\mathcal{S}=(V,\mathcal{B},I)$ is an incidence structure with u points and b blocks, and*

$$\begin{array}{rcl} V &=& \{x_i : 1 \le i \le u\} \\ \mathcal{B} &=& \{B_j : 1 \le j \le b\}. \end{array}$$

Then $M=[m_{i,j}]$ is an incidence matrix of $\mathcal{S}$ if

$$m_{i,j}=\begin{cases} 1 \text{ if } x_i \text{ is incident with } B_j \\ 0 \text{ otherwise.} \end{cases}$$

It is obvious that different labellings of the points and blocks of $\mathcal{S}$ will lead to different incidence matrices M and M', however M' will be the same as M with suitable column and row permutations applied to it. It is therefore unimportant which particular incidence matrix we use for a particular incidence structure.

Result 1.4.2 *Let $\mathcal{S}$ be a $2-(v,k,\lambda)$ design with points $x_1,\ldots,x_v$ and assume that each point is incident with r blocks. If M is an incidence matrix for $\mathcal{S}$ then*

$$M\cdot M^T=(r-\lambda)I+\lambda J \tag{1.1}$$

where J is the matrix whose every entry is 1.

Proof The (i,j) entry of $M \cdot M^T$ is equal to the number of blocks incident with x_i and x_j. Obviously if $i \neq j$ then this is λ and if $i = j$ then this is $r = (r-\lambda)+\lambda$.

■

The incidence matrix is a compact way of supplying all the necessary information about an incidence structure. A matrix of the above form for a point-weight incidence structure would not give all the information required to reconstruct the point-weight incidence structure as it would not give the weights of the points. We call that matrix the *incidence matrix for the underlying incidence structure* or the *underlying incidence matrix*.

Definition 1.4.3 (Point-weight incidence matrix) *Suppose* $\mathcal{S} = (V, \mathcal{B}, I, w)$ *is a point-weight incidence structure with* u *points and* b *blocks, and*

$$\begin{aligned} V &= \{x_i : 1 \leq i \leq u\} \\ \mathcal{B} &= \{B_j : 1 \leq j \leq b\}. \end{aligned}$$

Then $M = [m_{i,j}]$ *is an incidence matrix of* $\mathcal{S}$ *if*

$$m_{i,j} = \begin{cases} w(x_i) & \text{if } x_i \text{ is incident with } B_j \\ 0 & \text{otherwise.} \end{cases}$$

We now present a result of linear algebra which is used to calculate the determinants of the incidence matrices of specific classical and point-weight designs. It should be noted that all the poles in the following calculation are removable, so we assume that any poles that occur are removed.

Result 1.4.4 *If* M *is a* $u \times u$ *matrix of the form*

$$\begin{aligned} M &= diag(\alpha_1 - \lambda, \ldots, \alpha_u - \lambda) + \lambda J \\ &= \begin{pmatrix} \alpha_1 & \lambda & \ldots & \lambda \\ \lambda & \alpha_2 & \ldots & \lambda \\ \vdots & \vdots & \ddots & \vdots \\ \lambda & \lambda & \ldots & \alpha_u \end{pmatrix} \end{aligned}$$

then

$$det(M) = \Big(1 + \lambda \sum_{j=1}^{u} \frac{1}{\alpha_j - \lambda}\Big) \prod_{j=1}^{u} (\alpha_j - \lambda). \tag{1.2}$$

Proof We reduce the matrix M to upper triangular form using row and column operations. Note that if there exists values of j such that $\alpha_j = \lambda$ then this leads to a removable pole in (1.2). Assume that $\alpha_j = \lambda$ for $1 \leq j \leq m$.

If $m \geq 2$ then there obviously exist two rows that are exactly the same and so $det(M) = 0$. However in this case (1.2) is also equal to 0. So assume that $m = 0$ or $m = 1$.

Subtract the first row from each of the other rows and then subtract from first column a factor of $\frac{\lambda-\alpha_1}{\alpha_i-\lambda}$ of the ith column. Now we have a matrix that is of form:

$$\begin{pmatrix} x & \lambda & \lambda & \dots & \lambda \\ 0 & \alpha_2 - \lambda & 0 & \dots & 0 \\ 0 & 0 & \alpha_3 - \lambda & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \alpha_u - \lambda \end{pmatrix}$$

where

$$\begin{aligned} x &= \alpha_1 - \sum_{j=2}^{u} \lambda \frac{\lambda - \alpha_1}{\alpha_j - \lambda} \\ &= \alpha_1 + \lambda \sum_{j=2}^{u} \frac{\alpha_1 - \lambda}{\alpha_j - \lambda} \\ &= \alpha_1 - \lambda + \lambda \sum_{j=1}^{u} \frac{\alpha_1 - \lambda}{\alpha_j - \lambda} \\ &= (\alpha_1 - \lambda)\Big(1 + \lambda \sum_{j=1}^{u} \frac{1}{\alpha_j - \lambda}\Big). \end{aligned}$$

Hence

$$det(M) = \Big(1 + \lambda \sum_{j=1}^{u} \frac{1}{\alpha_j - \lambda}\Big) \prod_{j=1}^{u} (\alpha_j - \lambda).$$

■

Corollary 1.4.5 *If $\mathcal{S}$ is a $2-(v,k,\lambda)$ design then*

$$det(M \cdot M^T) = rk(r-\lambda)^{v-1}$$

Proof We know from (1.1) that $M \cdot M^T$ is of the correct form to use the previous result. Hence

$$\begin{aligned} det(M \cdot M^T) &= \Big(1 + \lambda \sum_{j=1}^{v} \frac{1}{r-\lambda}\Big) \prod_{j=1}^{v} (r-\lambda) \\ &= \Big(\frac{r-\lambda+\lambda v}{r-\lambda}\Big)(r-\lambda)^v \\ &= (r-\lambda+\lambda v)(r-\lambda)^{v-1}. \end{aligned}$$

So it only remains to show that $r - \lambda + \lambda v = rk$. But this is shown in 1.2.3, hence we have the result.

■

Corollary 1.4.6 (Fisher's Inequality) *If $\mathcal{S}$ is a $2-(v,k,\lambda)$ design with b blocks and $v > k$ then $b \geq v$.*

Proof We start by showing that in $\mathcal{S}$ we have that $r - \lambda > 0$. We know by (1.2.2) that r is constant and equal to $\lambda\frac{v-1}{k-1}$ but $v > k$ so $\frac{v-1}{k-1} > 1$. Hence $r > \lambda$.

Now if M is an incidence matrix for $\mathcal{S}$ we have that

$$det(M \cdot M^T) = rk(r-\lambda)^{v-1}.$$

All of the factors on the RHS are greater than 0. So $det(M \cdot M^T) \neq 0$, which means that $rank(M \cdot M^T) = rank(M) = v$. But M has b columns so $rank(M) \leq b$, hence $v \leq b$.

■

1.5 Automorphism groups

As promised, we now define an automorphism group.

Definition 1.5.1 (Automorphism group) *Suppose $\mathcal{S}$ is an incidence structure $(V, \mathcal{B}, I)$ and let Aut $\mathcal{S}$ be the set of all functions $\theta : V \cup \mathcal{B} \to V \cup \mathcal{B}$ such that:*

1. *$\theta(V) = V$ and $\theta(\mathcal{B}) = \mathcal{B}$,*
2. *if x is a point and B is a block then $(x, B) \in I$ if and only if $(\theta(x), \theta(B)) \in I$.*

The set Aut $\mathcal{S}$ is called the automorphism group of $\mathcal{S}$.

It is elementary to show that *Aut* $\mathcal{S}$ is in fact a group.

We may regard the automorphism group of a (possibly point-weight) incidence structure as the action of the automorphism group on the set of points and blocks. We will, in section 4.3, use the general theory of groups acting on sets to derive a theoretic result which we then apply to automorphism groups acting on point-weight incidence structures. It is worthwhile, therefore, to take a moment to formally define actions and G-spaces.

Definition 1.5.2 *Let G be a group and let Ω be a set. An action of G on Ω is a map $\mu : \Omega \times G \to \Omega$ satisfying:*

1. *$\mu(\mu(\omega, g), h) = \mu(\omega, gh)$ for all $\omega \in \Omega$ and $g, h \in G$,*
2. *$\mu(\omega, 1) = \omega$ for any $\omega \in \Omega$, where 1 is the identity element of G.*

The set Ω together with this action is called a G-space.

For further information on G-spaces the reader is referred to [5]. We will generally write ω^g for $\mu(\omega, g)$.

It is easy to see that if $\mathcal{S}$ is a (possible point-weight) incidence structure with a point set V and a block set $\mathcal{B}$ then $Aut\ \mathcal{S}$ acts on $V \cup \mathcal{B}$. This action can be decomposed into the action of $Aut\ \mathcal{S}$ on V and the action of $Aut\ \mathcal{S}$ on $\mathcal{B}$ as the action of $Aut\ \mathcal{S}$ will never map a point onto a block or vice versa. We give a few more definitions that will be needed.

Definition 1.5.3 *The orbits of the G-space Ω are the equivalence classes of an equivalence relation $\sim$ defined by $\alpha \sim \beta$ if and only if $\alpha, \beta \in \Omega$ and there exists an element $g \in G$ such that $\alpha^g = \beta$.*

Definition 1.5.4 *A G-space Ω is n-transitive if for any two vectors of n distinct elements of Ω, $(\alpha_1, \ldots, \alpha_n)$ and $(\beta_1, \ldots, \beta_n)$ say, there exists an element $g \in G$ such that $\alpha_i^g = \beta_i$ for all $1 \leq i \leq n$.*

A G-space Ω is n-homogeneous if for any two sets of n distinct elements of Ω, S_1 and S_2 say, there exists an element $g \in G$ such that $S^g = T$.

These two notions are related by a theorem of Livingstone and Wagner [11].

Result 1.5.5 *If Ω is a t-homogeneous G-space and $2 < 2t \leq |\Omega|$ then G acts $(t-1)$-transitively on Ω.*

These concepts can be applied specifically to (point-weight) incidence structures. We call $Aut\ \mathcal{S}$ transitive on points (respectively blocks) if for any two points $x, y \in V$ (respectively blocks $B_1, B_2 \in \mathcal{B}$) there exists an automorphism θ such that $\theta(x) = y$ (respectively $\theta(B_1) = B_2$). We call an automorphism group n-transitive on points if for any two vectors of n distinct points of $\mathcal{S}$, $(x_1, \ldots, x_n), (y_1, \ldots y_n) \in V^n$, there exists an automorphism θ such that $\theta(x_i) = y_i$ for all $1 \leq i \leq n$. We call an automorphism group n-homogeneous on points if for any two sets of n points, $X, Y \subseteq V$ there exists an automorphism θ such that $\theta(X) = Y$.

1.6 Tactical decompositions

The main result of this section, Block's Lemma, was first proposed in [4] but a proof of it is more readily available in [3]. It is, in fact, a result in matrix theory that can be applied to incidence matrices.

Definition 1.6.1 (Tactical decomposition) *Let M be a $m \times n$ matrix over $\mathbb{R}$ and let $\{R_1, \ldots, R_s\}$ be a partition of the set of row indices $\{1, 2, \ldots, m\}$. Similarly let $\{C_1, \ldots, C_t\}$ be a partition of the set of column indices $\{1, 2, \ldots, n\}$.*

If the $|R_i| \times |C_j|$ submatrix, M_{ij} of M, given by taking the entries of M in the rows indexed by R_i and the columns indexed by C_j has constant row sums $r_{i,j}$ and constant column sums $c_{i,j}$, for every $1 \le i \le s$ and $1 \le j \le t$, then this partition is called a tactical decomposition of M.

Define the column sum matrix C to be the matrix whose (i,j)th entry is $c_{i,j}$ and the row sum matrix R to be the matrix whose (i,j)th entry is $r_{i,j}$.

Result 1.6.2 (Block's Lemma) *Suppose M is a $m \times n$ matrix over $\mathbb{R}$ and $(M_{i,j})$ is a tactical decomposition of M with $1 \le i \le s$ and $1 \le j \le t$. If C and R are the column and row sum matrices defined above then:*

$$\begin{array}{rcccl} t & \le & rank(R) + n - rank(M) & \le & s + n - rank(M) \\ s & \le & rank(C) + m - rank(M) & \le & t + m - rank(M). \end{array}$$

The proof of this result is mostly linear algebra and therefore unenlightening from a design theory point of view, hence it has been omitted.

One application of tactical decompositions is in square designs where the existence of certain square classical designs is dependent on the existence of solutions to certain diophantine equations in the complete field of p-adic numbers. We start by considering the p-adic numbers. The following definitions and result also hold, unless explicitly noted, for the "infinite prime" too, i.e. for $\mathbb{Q}_\infty = \mathbb{R}$.

Definition 1.6.3 *If α and β are non-zero p-adic numbers then we define the Hilbert symbol to be:*

$$(\alpha, \beta)_p = \begin{cases} 1 & \text{if there exists a solution to } \alpha x^2 + \beta y^2 = 1 \text{ in } \mathbb{Q}_p \\ -1 & \text{otherwise} \end{cases}$$

where $\mathbb{Q}_p$ is the complete field of p-adic numbers.

Result 1.6.4 *For any prime p and p-adic numbers $\alpha, \beta, \lambda, \mu$ the following properties hold:*

1. $(\alpha, \beta)_p = (\beta, \alpha)_p$,
2. $(\alpha\lambda^2, \beta\mu^2)_p = (\alpha, \beta)_p$,
3. $(\alpha, -\alpha) = 1$,
4. *If p is a finite prime and co-prime to $2\alpha\beta$ then $(\alpha, \beta)_p = 1$.*
5. $(\alpha, \beta\lambda)_p = (\alpha, \beta)_p(\alpha, \lambda)_p$,
6. $(\alpha, \alpha)_p = (\alpha, -1)_p$,
7. $(\alpha\lambda, \beta\lambda)_p = (\alpha, \beta)_p(\lambda, -\alpha\beta)_p$.

The basics of p-adic numbers can be found in [2] and the above results can be found in [10]. We may now use the congruence theory developed by Hasse and Minkowski.

Definition 1.6.5 *If $n \times n$ is a symmetric integral matrix and p is any prime then we define the Hasse Symbol to be*

$$H_p(M) = (-1, -D(n))_p \prod_{i=1}^{n-1} (D(i), -D(i+1))_p$$

where $D(i)$ is the determinant of the upper left $i \times i$ submatrix of M.

Definition 1.6.6 *Two square integer matrices M and N are congruent if there exists a non-singular matrix C over $\mathbb{Q}$ such that $CMC^T = N$.*

Result 1.6.7 *If the symmetric integral matrices L and M are congruent then $H_p(L) = H_p(M)$ for all primes p.*

The converse to this theorem is also true, that if L and M are integral symmetric matrices and $H_p(L) = H_p(M)$ for all primes p (including the infinite prime) then L is congruent to M. However we will not use the result.

The non-existence results derived from this theory can be very powerful and they can even be used to derive the Bruck-Chowla-Ryser theorem. This is demonstrated in [7] but requires the following theorem (which can be found in [17]).

Result 1.6.8 *The equation*

$$aX^2 + bY^2 + cZ^2 = 0$$

where $abc \neq 0$ has a non-trivial solution in $\mathbb{Q}$ (and hence a non-trivial solution in $\mathbb{Z}$) if and only if it has a non-trivial solution in $\mathbb{R}$ and $\mathbb{Q}_p$ for all primes p

1.7 Point-weight designs

A point-weight design extends the idea of a classical design to a point-weight incidence structure. It would seem natural that any such extension would have to have the following three properties: it must have a statement analogous to the constant block size condition, it must have a statement analogous to the design condition and if all the weights of the points were equal then it must be a classical design. We present two such extensions: point-sum point-weight designs (initially developed in [9]) and weight-sum point-weight designs (initially developed in [13]).

Definition 1.7.1 (Point-sum point-weight designs) *The point-weight incidence structure $(V, \mathcal{B}, I, w)$ is a $t-(v, k, \lambda; W)$ point-sum point-weight design if:*

1. *the sum of the weights of all the points is v (i.e. $\sigma(V) = v$).*

2. *the sum of the weights of all the points on any block is k (i.e. $\sigma(B) = k$ for all $B \in \mathcal{B}$).*

3. *any set of $t > 0$ points is contained in exactly $\lambda > 0$ blocks (i.e. if $S \subseteq V$ and $|S| = t$ then $\iota(S) = \lambda$).*

4. *W is the weight-set of the point-weight incidence structure.*

We recall, from Horne [9] and Powlesland [13], a few of the more relevant combinatorial results.

Result 1.7.2 *If $\mathcal{S}$ is a $1-(v, k, \lambda; W)$ point-weight design then $bk = v\lambda$.*

Proof We count weighted flags, i.e. consider the sum:

$$\sum_{(x,B)\in I} w(x).$$

On one hand there exists λ blocks through each point of $\mathcal{S}$ so the weight of each point will be counted λ times i.e. $\sum_{(x,B)\in I} w(x) = \sum_{x\in V} \lambda w(x) = \lambda v$ but on the other hand each block contains points whose weights add to k hence $\sum_{(x,B)\in I} w(x) = bk$. Hence $bk = v\lambda$.

■

We may define all the related structures on a point-sum point-weight design as before, in particular:

Result 1.7.3 *If $\mathcal{S}$ is a $t-(v, k, \lambda; W)$ point-weight design with $t \geq 2$ and x is any point of $\mathcal{S}$ then S_x is a $(t-1)-(v-w(x), k-w(x), \lambda; W')$ design and either $W' = W$ or $W' = W \setminus \{w(x)\}$.*

Proof Since $t > 1$, we have that for any two distinct points there exists at least one block that is incident with both of them. So the sum of the weights of the points of $\mathcal{S}_x$ is $v - w(x)$, similarly the sum of the weights of the blocks of $\mathcal{S}_x$ is $k - w(x)$.

Consider any set S of $t-1$ points of $\mathcal{S}_x$. The number of blocks of $\mathcal{S}_x$ that contain S is equal to the number of blocks of $\mathcal{S}$ that contain S and x, which is λ.

Lastly since $\mathcal{S}_x$ contains all the points of $\mathcal{S}$ except for x, it must contain a point of each of the weights of $\mathcal{S}$ except for possibly $w(x)$, hence $W' = W$ or $W' = W \setminus \{w(x)\}$.

■

In [9] it was shown that in a $2-(v,k,\lambda;W)$ the number of blocks containing a point was dependent only on λ, v, k and the weight of that point. Consequently all points of the same weight were incident with the same number of points. This was extended to the $t=3$ case by [15] and using his techniques we have extended it to the case when $t \geq 2$.

Result 1.7.4 *If $\mathcal{S}$ is a $2-(v,k,\lambda;W)$ point-weight design with more than 2 points then any two points of equal weight are incident with the same number of blocks.*

Proof For any point $x \in V$, the number of blocks containing x is equal to the number of blocks in $\mathcal{S}_x$. Now $\mathcal{S}_x$ is a $1-(v-w(x), k-w(x), \lambda; W')$ point-weight design and so if r_x is the number of blocks contain the point x we have

$$r_x = \lambda \frac{v-w(x)}{k-w(x)} \tag{1.3}$$

by (1.7.2). Hence any two points of the same weight are incident with the same number of blocks.

■

Result 1.7.5 *If $\mathcal{S}$ is a $3-(v,k,\lambda;W)$ point-weight design with more than 3 points then any two points of equal weight are incident with the same number of blocks.*

Proof For any point $x \in V$, the number of blocks containing x is equal to the number of blocks in $\mathcal{S}_x$. We note that $\mathcal{S}_x$ is a $2-(v-w(x), k-w(x), \lambda; W')$ point-weight design. In $\mathcal{S}_x$ we attempt to evaluate the sum:

$$\sum_{(y,z,B)\in J} w(y)w(z)$$

where $J = \{(y,z,B) \in V_x \times V_x \times \mathcal{B}_x : y \neq z \text{ and } y,z \in B\}$.

We may attempt to evaluate this in two ways: firstly we may sum over all distinct points of $\mathcal{S}_x$ and then over all blocks that contain those two points or, secondly, we may sum over all blocks and all pairs of points that are contained in that block. Firstly:

$$\begin{aligned}
\sum_{(y,z,B)\in J} w(y)w(z) &= \sum_{y\in V_x} \sum_{z\in V_x\setminus\{y\}} \sum_{B\ni y,z} w(y)w(z) \\
&= \sum_{y\in V_x} \sum_{z\in V_x\setminus\{y\}} \lambda w(y)w(z) \\
&= \lambda \sum_{y\in V_x} \sum_{z\in V_x\setminus\{y\}} w(y)w(z)
\end{aligned}$$

Consider the incidence structure $\mathcal{S}^*$ given by changing each point x of weight $w(x)$ into $w(x)$ points of weight 1 and preserving incidence in the natural way. So:

$$\begin{aligned} V^* &= \{x_i : x \in V \text{ and } 1 \le i \le w(x)\} && (1.4)\\ \mathcal{B}^* &= \mathcal{B} && (1.5)\\ I^* &= \{(x_i, B) : (x, B) \in I\} && (1.6) \end{aligned}$$

For any two points $y, z \in V_x$ there are $w(y)w(z)$ ways of linking a point y_i with a point z_j in ${\mathcal{S}_x}^*$. So if we fix a block $B \in \mathcal{B}_x$ then there are

$$\sum_{y \in B} \sum_{z \in B \setminus \{y\}} w(y)w(z)$$

ways of choosing two points of that block in ${\mathcal{S}_x}^*$ that are obtained from different points of $\mathcal{S}_x$. This is the same as picking any two points of B in ${\mathcal{S}_x}^*$ excluding those points which are obtained from the same point. Hence:

$$\sum_{y \in B} \sum_{z \in B \setminus \{x\}} w(y)w(z) = \binom{k}{2} - \sum_{y \in B} \binom{w(y)}{2}.$$

So:

$$\begin{aligned} \sum_{(y,z,B) \in J} w(y)w(z) &= \sum_{B \in \mathcal{B}_x} \sum_{y \in B} \sum_{z \in B \setminus \{x\}} w(y)w(z)\\ &= \sum_{B \in \mathcal{B}_x} \left(\binom{k}{2} - \sum_{y \in B} \binom{w(y)}{2}\right)\\ &= r_x \binom{k}{2} - \sum_{B \in \mathcal{B}_x} \sum_{y \in B} \binom{w(y)}{2}\\ &= r_x \binom{k}{2} - \sum_{y \in V_x} r_{x,y} \binom{w(y)}{2} \end{aligned}$$

where r_x is the number of blocks in S_x (i.e. the number of blocks incident with x in $\mathcal{S}$) and $r_{x,y}$ is the number of blocks incident with y in $\mathcal{S}_x$ (which is calculable by (1.7.4) as $\mathcal{S}_x$ is a $2 - (v - w(x), k - w(x), \lambda; W')$ point-weight design).

Therefore:

$$r_x = \frac{1}{\binom{k}{2}} \left\{ \lambda \sum_{y \in V_x} \sum_{z \in V_x \setminus \{y\}} w(y)w(z) + \sum_{y \in V_x} \lambda \frac{v - w(x) - w(y)}{k - w(x) - w(y)} \binom{w(y)}{2} \right\}$$

and this expression is the same for any two points x and x' of the same weight because there exists a bijection between the points of V_x and $V_{x'}$ such that corresponding points have the same weight.

■

Theorem 1.7.6 *Suppose $\mathcal{S}$ is a $(V, \mathcal{B}, I, w)$ point-weight incidence structure with the following properties:*

1. *the sum of the weights of the points on any block is a constant, k,*
2. *there exists an integer t, with $1 \leq t < u$, such that any two sets of t points with equally many points of each weight are incident with the same number of blocks.*

Then the following holds: if S_1, S_2 are sets of points with $1 \leq |S_1| = |S_2| \leq t$ and both sets contain equal numbers of points of each weight then $\iota(S_1) = \iota(S_2)$. In particular, any two points of the same weight are incident with the same number of blocks.

Proof We will begin with examining the properties of the derived structure. Take a set of points $T \subseteq V$ with $1 \leq |T| < t$ and consider $\mathcal{S}_T$. Let $\iota_T(S)$ be the number of blocks of $\mathcal{S}_T$ that contain the set $S \subseteq V_T$. Note that $\mathcal{S}_T$ has the following properties:

1. the sum of the weights of the points on any block is $k - \sigma(T)$,
2. if S_1 and S_2 are sets of $t - |T|$ points with equally many points of each weight then $\iota_T(S_1) = \iota(S_1 \cup T) = \iota(S_2 \cup T) = \iota_T(S_2)$,
3. $t - |T| < t - 1 = |V_T|$.

We use induction on the value of t.

Suppose that $\mathcal{S}$ is a point-weight incidence structure that satisfies the conditions of the theorem with $t = 2$ (the case $t = 1$ is trivial). Pick any $x \in V$, the number of blocks of $\mathcal{S}$ that are incident with x, r_x say, is equal to the number of blocks of $\mathcal{S}_x$. Consider the following sum in $\mathcal{S}_x$:

$$\begin{aligned}
\sum_{(y,B) \in I_x} w(y) &= \sum_{B \in \mathcal{B}_x} \sum_{y \in B} w(y) \\
&= \sum_{B \in \mathcal{B}_x} (k - w(x)) \\
&= r_x (k - w(x))
\end{aligned}$$

but also:

$$\begin{aligned}
\sum_{(y,B) \in I_x} w(y) &= \sum_{y \in V_x} \sum_{B \ni y} w(y) \\
&= \sum_{y \in V_x} \iota_T(y) w(y)
\end{aligned}$$

Hence:

$$r_x = \frac{1}{k - w(x)} \sum_{y \in V_x} \iota_T(y) w(y)$$

which is independent of which particular point of weight $w(x)$ was initially chosen. Also if S_1 and S_2 are sets of points of size two and both sets contain equal numbers of points of the same weight then $\iota(S_1) = \iota(S_2)$ by definition.

Let $t \geq 3$ and assume the following induction hypothesis. If $\mathcal{S}$ is a point-weight incidence structure for which there exists an integer $1 \leq s \leq t-1$ such that any two sets of s points, with exactly the same number of points of each weight, are incident with the same number of blocks then any two sets of points S_1 and S_2 with $1 \leq |S_1| = |S_2| \leq s$ and equally many points of each weight has $\iota(S_1) = \iota(S_2)$.

Now consider a point-weight incidence structure $\mathcal{S}$ that satisfies all of the properties listed in the statement of the theorem and pick a set of points $T \subseteq V$ such that $|T| = t - s$. Again we note that the number of blocks that contains T is the same as the number of blocks in the point-weight design $\mathcal{S}_T$ and $\mathcal{S}_T$ is a point-weight incidence structure of the form given in the induction hypothesis above. Let the number of blocks of $\mathcal{S}_T$ be denoted r_T and consider $\mathcal{S}_T$.

Let $I' = \{(S, B) : S \subseteq V_T,\ B \in \mathcal{B}_T,\ |S| = s \text{ and } S \subseteq B\}$, be the set of all ordered pairs of (S, B) where S is a set of s points and B is a block that contains S. We will attempt to evaluate the sum:

$$\sum_{(S,B) \in I'} \Big(\prod_{z \in S} w(z) \Big).$$

Now, if we fix S then that set of points is contained in exactly $\iota_T(S) = \iota(S \cup T)$ blocks of $\mathcal{B}_x$, hence:

$$\begin{aligned}
\sum_{(S,B) \in I'} \Big(\prod_{z \in S} w(z) \Big) &= \sum_{S \subseteq V_T : |S| = s} \sum_{B \supseteq S} \Big(\prod_{z \in S} w(z) \Big) \\
&= \sum_{S \subseteq V_T : |S| = s} \iota_T(S) \Big(\prod_{z \in S} w(z) \Big). \qquad (1.7)
\end{aligned}$$

There exists a bijection, $\theta : V_T \to V_{T'}$, between the points of V_T and $V_{T'}$ such that $w(x) = w(\theta(x))$ provided T and T' contain the same number of points of each weight. Hence there exists a bijection between the subsets of V_T and $V_{T'}$ of size s such that corresponding sets contain equal number of points of each weight, which means that the expression (1.7) is the same for V_T and $V_{T'}$.

Now consider the incidence structure $\mathcal{S}^*$ given by changing each point x of $\mathcal{S}$ into $w(x)$ points and preserving incidence in the natural way. So:

$$\begin{aligned}
V^* &= \{x_i : x \in V \text{ and } 1 \leq i \leq w(x)\} \\
\mathcal{B}^* &= \mathcal{B} \\
I^* &= \{(x_i, B) : (x, B) \in I\}.
\end{aligned}$$

Suppose S is a set of s points of $\mathcal{S}_T$. There are $\prod_{x\in S} w(x)$ ways of choosing a set S^* of s elements of $\mathcal{S}_T{}^*$ such that each element of S^* was obtained from a distinct point of S.

There exists $r_T\binom{k}{s}$ ways of picking the pair (S^*, B) where S^* is a set of s points of $\mathcal{S}_T{}^*$ and B is a block that contains S^*. However this includes the sets S^* whose elements are obtained from a set of points S of $\mathcal{S}_x$ of size less than s. We shall attempt to calculate how many of these "bad" pairs exist.

Let

$$J_i = \{(j_1, \ldots, j_i) \in \mathbb{Z}^i : \sum_{n=1}^{i} j_n = s \text{ and } j_n > 0 \text{ for all } 1 \leq n \leq i\}.$$

Suppose S is a set of $i < s$ points of S_T and label these points $z^{(1)}, \ldots, z^{(i)}$. The number of ways of choosing an ordered pair (S^*, B) where S^* is a set of s points of $\mathcal{S}_T{}^*$ obtained from the set S and B is a block that contains S is

$$p(S) = \iota(S) \sum_{(j_1,\ldots,j_i)\in J_i} \prod_{h=1}^{i} \binom{w(z^{(h)})}{j_h}.$$

Note that this expression is the same if we are working in $S_{T'}$ and T' has the same number of points of each weight as T.

If we sum this over all possible sets of points S of size less than s and all possible blocks B that contain S then we have:

$$\sum_{(S,B)\in I'} \Big(\prod_{z\in S} w(z)\Big) \;=\; r_T\binom{k}{s} - \sum_{i=1}^{s-1} \Big\{ \sum_{S\subseteq V_T:|S|=i} p(S) \Big\}. \tag{1.8}$$

Equating this expression with the expression derived in (1.7) shows that r_T is the same as $r_{T'}$ whenever T and T' contain the same number of points of each weight. Hence all sets of points T with the same number of points of each weight are incident with the same number of blocks.

■

Corollary 1.7.7 *If $\mathcal{S}$ is a $t-(v,k,\lambda;W)$ point-weight design with u points and $u > t \geq 2$ then any two points of equal weight are incident with the same number of blocks.*

Proof We can apply (1.7.6) as $\iota(S) = \lambda$ for any set S of t points.

■

The other major results on point-sum point-weight designs that are of interest are:

Result 1.7.8 *If $\mathcal{S}$ is a $t-(v,k,\lambda;W)$ point-weight design with $v>k$, $t>1$ and $|W|\geq 2$ then the underlying structure of $\mathcal{S}$ is not a classical design.*

Result 1.7.9 *If $\mathcal{S}$ is a $t-(v,k,\lambda;W)$ point-weight design with $v>k$, $t>1$ and $\mathcal{U}$ is the underlying incidence structure of $\mathcal{S}$ then Aut $\mathcal{S}$ = Aut $\mathcal{U}$.*

Result 1.7.10 *Suppose $\mathcal{S}$ is a $t-(v,k,\lambda;W)$ point-weight design with u points. Let $\mathcal{U}$ be the underlying incidence structure of $\mathcal{S}$ and let M be an incidence matrix of $\mathcal{U}$. Then every point-weight design with underlying incidence structure $\mathcal{U}$ is equivalent to $\mathcal{S}$ if and only if $rank(M)=u$.*

Proofs of all of these results can be found in [9]. Lastly we give a concrete example of a point-sum point-weight design that will be useful later:

Result 1.7.11 *If $n\geq 3$ then the degenerate projective plane with $n+1$ points is the underlying structure of a $2-(2n-1,n,1;\{1,n-1\})$ point-weight design.*

Proof The degenerate projective plane with $n+1$ points is the incidence structure $(V,\mathcal{B},I)$ given by:

$$\begin{aligned}
V &= \{x_1,\ldots,x_n,x_{n+1}\}\\
\mathcal{B} &= \{B_1,\ldots B_n,B_{n+1}\}\\
I &= \{(x_{n+1},B_i):1\leq i\leq n\}\cup\{(x_i,B_i):1\leq i\leq n\}\cup\{(x_i,B_{n+1}):1\leq i\leq n\}.
\end{aligned}$$

We extend this to a point-weight incidence structure by adding a weight function w given by

$$w(x_i)=\begin{cases} 1 & \text{if } 1\leq i\leq n\\ n-1 & \text{if } i=n+1.\end{cases}$$

Now the sum of the weights is obviously $2n-1$ and the sum of the weights of the points on any block is n. If we take two points of $\{x_1,\ldots x_n\}$ then both of these points are incident with B_{n+1} only but if we take one point of $x_i\in\{x_1,\ldots x_n\}$ and the point x_{n+1} then these two points are incident with the block B_i only.

Hence the structure is a $2-(2n-1,n,1;\{1,n-1\})$ point-weight design.

■

It is worth noting that no practical uses of point-sum point-weight designs were given in [9] and so we will exhibit a theoretical situation in which a point-sum point-weight design might be practically useful. Suppose that we wish to test several different courses of treatment of a certain medical condition. Suppose further that that medical condition required the patient to take k units of a certain drug per day but that there were many different pills that each contained a different amounts of that drug and only a certain number of each tablet could be taken in one day. Lastly suppose that we want to conduct the test in an unbiased way, i.e. that any 2 pills are only in λ treatment courses. An efficient way of organising the treatments would be construct a $2-(v, k, \lambda; W)$ point-weight design with the points corresponding to the possible pills, the weights of the points being the amount of drug the corresponding pill contained and then the blocks would be the different treatment courses.

The other type of point-weight design that has been developed are weight-sum point-weight designs.

Definition 1.7.12 (Weight-sum point-weight design) *A point-weight incidence structure is a $\sigma_t - (v, k, \lambda; W)$ weight-sum point-weight design if:*

1. *the sum of the weights of all the points is v (i.e. $\sigma(V) = v$).*
2. *the sum of the weights of all the points on any block is k (i.e. $\sigma(B) = k$ for all $B \in \mathcal{B}$).*
3. *any set of points whose total weight is $t > 0$ is contained in exactly $\lambda > 0$ blocks (i.e. if $S \subseteq V$ and $\sigma(S) = t$ then $\iota(S) = \lambda$).*
4. *W is the weight-set of the point-weight incidence structure.*

Weight-sum point-weight designs are really only mentioned for completeness as the properties of weight-sum point-weight designs are vastly different to those of point-sum point-weight designs or row-sum point-weight designs (which will be developed in chapter 2). They are, however, of more practical use and make a contribution to the theory of secret sharing schemes.

Chapter 2

Row-sum point-weight designs

In this chapter we introduce a third type of point-weight design which we call *row-sum point-weight designs* that are entirely consistent with the conditions laid down for a point-weight design and exhibit some intriguing combinatorial and algebraic properties. We begin by examining the combinatorial properties of such structures.

2.1 Basic definitions

The motivation for the definition of row-sum point-weight designs comes from an algebraic property of classical designs. Many of the properties of classical designs are derived from the fact if M is an incidence matrix for a classical design then $MM^T = \lambda J + (r - \lambda)I$ where r is the number of blocks each point is incident with. A row-sum point-weight design is a type of point-weight design that has a closed expression for MM^T (where M is the design's incidence matrix).

Definition 2.1.1 (Row-sum point-weight designs) *A point-weight incidence structure $\mathcal{S} = (V, \mathcal{B}, I, w)$ is called a $\pi_t - (v, k, \lambda; W)$* row-sum point-weight design *if*

1. *the sum of the weights of all the points is v.*

2. *the sum of the weights of the points lying on any one block is k.*

3. *if S is a set of $t > 0$ points then*

$$\iota(S) \prod_{x \in S} w(x) = \lambda$$

where $\lambda > 0$.

4. *the image of w is W.*

This will commonly be referred to as a $\pi_t - (v, k, \lambda; W)$.

Lemma 2.1.2 *If $\mathcal{S}$ is a $\pi_2-(v,k,\lambda;W)$ with u points and an incidence matrix M then*

$$MM^T = diag(w(x_1)^2 r_{x_1}, w(x_2)^2 r_{x_2}, ..., w(x_u)^2 r_{x_u}) + \lambda(J-I) \tag{2.1}$$

where r_{x_i} is the number of blocks with which the point x_i is incident (for $1 \leq i \leq u$).

Hence we have found a definition for point-weight designs that is consistent with our motivation i.e. whenever $t=2$ we have a closed expression for MM^T where M is the incidence matrix for the design. Note that if $\mathcal{S}$ is a $\pi_t-(v,k,\lambda;W)$ where $|W|=1$ then every point has the same weight w and so every t points lie on exactly $\frac{\lambda}{w^t}$ points. Therefore the underlying structure of $\mathcal{S}$ is a $t-(\frac{v}{w},\frac{k}{w},\frac{\lambda}{w^t})$ block design. Thus we have shown that, similar to point-sum and weight-sum point-weight designs, row-sum point-weight designs with weight sets of size one reduce to classical designs.

Let us start by considering row-sum point-weight designs that are equivalent.

Lemma 2.1.3 *If $\mathcal{T}$ is a point-weight incidence structure that is equivalent to a $\pi_t-(v,k,\lambda;W)$ point-weight design $\mathcal{S}$ with scale factor μ, then $\mathcal{T}$ is a $\pi_t-(\mu v,\mu k,\mu^t\lambda;\mu W)$ row-sum point-weight design.*

In particular if $\mathcal{S}$ is a $\pi_2-(v,k,\lambda;W)$ with weight function w then $\mathcal{T}$ is a $\pi_2-(v',k',\lambda';W')$ with weight function $w' \equiv \mu w$. If the point $x \in V_{\mathcal{S}}$ is mapped to the point $y \in V_{\mathcal{T}}$ under the equivalence map then

$$w'(y)^2 r_y - \lambda' = \mu^2(w(x)^2 r_x - \lambda)$$

where r_z is the number of blocks with which the point z is incident. In particular $w(x)^2 r_x - \lambda$ and $w(y)^2 r_y - \lambda'$ have the same sign.

Proof Suppose that $\mathcal{S}=(V,\mathcal{B},I,w)$ and $\mathcal{T}=(V',\mathcal{B}',I',w')$. Since the two structures are equivalent there exists incidence preserving bijections $\theta_V : V \to V'$ and $\theta_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}'$ such that $w'(\theta_V(x)) = \mu w(x)$. Hence

$$\begin{aligned}\sum_{y\in V'} w'(y) &= \sum_{x\in V} w'(\theta_V(x)) \\ &= \sum_{x\in V} \mu w(x) \\ &= \mu \sum_{x\in V} w(x) \\ &= \mu v\end{aligned}$$

and for any block $B' \in \mathcal{B}'$ there exists a block $B \in \mathcal{B}$ such that $B' = \theta_{\mathcal{B}}(B)$ and

$$\sum_{y\in\theta_{\mathcal{B}}(B)} w'(y) = \sum_{x\in B} w'(\theta_V(x))$$

$$\begin{aligned}
&= \sum_{x \in B} \mu w(x) \\
&= \mu \sum_{x \in B} w(x) \\
&= \mu k.
\end{aligned}$$

Lastly suppose that S' is a set of t points of $\mathcal{T}$. Then there exists a set S of t points of $\mathcal{S}$ such that $S' = \theta_V(S)$. So,

$$\begin{aligned}
\iota(S') &= \iota(S) \\
&= \frac{\lambda}{\prod_{x \in S} w(x)} \\
&= \frac{\mu^t \lambda}{\prod_{x \in S} \mu w(x)} \\
&= \frac{\mu^t \lambda}{\prod_{x \in S} w'(\theta_V(x))} \\
&= \frac{\mu^t \lambda}{\prod_{y \in S'} w'(y)}.
\end{aligned}$$

Therefore $\mathcal{T}$ is a $\pi_t - (\mu v, \mu k, \mu^t \lambda; \mu W)$ and in particular if $t = 2$ then $\mathcal{T}$ is a $\pi_2 - (\mu v, \mu k, \mu^2 \lambda; \mu W)$ point-weight design. Suppose $t = 2$ and pick any $y \in V'$. There exists a point $x \in V$ such that $\theta_V(x) = y$ and so,

$$\begin{aligned}
w'(y)^2 r_y - \lambda' &= (\mu w(x))^2 r_x - \mu^2 \lambda \\
&= \mu^2 (w(x)^2 r_x - \lambda).
\end{aligned}$$

The purpose of this part of the lemma will become clear when we investigate awkward and difficult designs in (2.4).

■

This means that whenever we consider a $\pi_t - (v, k, \lambda; W)$ we may assume that the weight set W has a greatest common divisor of 1, as there will always exist an equivalent design for which this is true. Lastly we give the most obvious lemma about the combinatorial properties of a row-sum point-weight design.

Lemma 2.1.4 *If $\mathcal{S} = (V, \mathcal{B}, I, w)$ is a $\pi_1 - (v, k, \lambda; W)$ with u points and b blocks then $u\lambda = bk$.*

Proof Consider the weighted flag sum

$$\sum_{(x,B) \in V \times \mathcal{B} : x \in B} w(x).$$

By evaluating this sum in two different ways we have that

$$\begin{aligned}\sum_{x\in V}\sum_{B\ni x} w(x) &= \sum_{B\in\mathcal{B}}\sum_{x\in B} w(x)\\ \sum_{x\in V}\frac{\lambda}{w(x)}w(x) &= \sum_{B\in\mathcal{B}} k\\ u\lambda &= bk. \end{aligned} \tag{2.2}$$

■

2.2 Related structures

In this section we investigate which of the point-weight structures we defined in (1.3) are also row-sum point-weight designs. We start by considering derived structures, see (1.3.3) and (1.7.3).

Lemma 2.2.1 *If S is a set of $s < t$ points of a $\pi_t - (v, k, \lambda; W)$ point-weight design $\mathcal{S}$ with $t \geq 2$ then $\mathcal{S}_S$ is a $\pi_{t-s} - (v - \sigma(S), k - \sigma(S), \frac{\lambda}{\prod_{x\in S} w(x)}; W')$ point-weight design.*

Proof Since $t \geq 2$ we have that any two points must lie on at least one block. So $\mathcal{S}_S$ contains all the points of $\mathcal{S}$ except those that lie in S. Hence the sum of the weights of all the points is $v - \sigma(S)$ and the sum of the weights of the points lying on any one block is $k - \sigma(S)$. Now let T be any set of $t - s$ points of $V \setminus S$. We have that

$$\iota(S \cup T)\prod_{x\in S} w(x)\prod_{y\in T} w(y) = \lambda.$$

However $\iota(T)$ in $\mathcal{S}_S$ is equal to $\iota(S \cup T)$ in $\mathcal{S}$. Hence in $\mathcal{S}_S$ we have for any set of $t - s$ points T:

$$\iota(T)\prod_{y\in T} w(y) = \frac{\lambda}{\prod_{x\in S} w(x)}$$

which implies the result. Note that $W \setminus w(S) \subseteq W' \subseteq W$.

■

Therefore we immediately get the following corollary.

Corollary 2.2.2 *If $\mathcal{S}$ is a $\pi_t - (v, k, \lambda; W)$ point-weight design with $t \geq 2$ and x is a point of $\mathcal{S}$ then $\mathcal{S}_x$ is a $\pi_{t-1} - (v - w(x), k - w(x), \frac{\lambda}{w(x)}; W')$ point-weight design.*

The concept of a derived structure at a point or set of points is a very powerful tool and we will be using it a great deal. To begin with it allows us to prove the following lemma:

Lemma 2.2.3 *If $\mathcal{S}$ is a $\pi_t - (v, k, \lambda; W)$ point-weight design with u points, S is a set of $t-1$ points of $\mathcal{S}$ and $t \geq 2$ then*

$$\iota(S) = \frac{\lambda(u-t+1)}{(k-\sigma(S))\prod_{x\in S} w(x)}$$

Proof From (2.2.1) we know that the derived structure of $\mathcal{S}$ at S is a $\pi_1 - (v - \sigma(S), k - \sigma(S), \frac{\lambda}{\prod_{x\in S} w(x)}; W')$ point-weight design with $u - t + 1$ points and so we may apply (2.1.4). We also note that the number of blocks of $\mathcal{S}_S$ is equal to the number of blocks that are incident with S in $\mathcal{S}$. Therefore:

$$(u-t+1)\frac{\lambda}{\prod_{x\in S} w(x)} \quad = \quad \iota_{\mathcal{S}}(S)(k-\sigma(S))$$

and so,

$$\iota_{\mathcal{S}}(S) \quad = \quad \frac{(u-t+1)\lambda}{(k-\sigma(S))\prod_{x\in S} w(x)}.$$

■

This in turn allows us to deduce that:

Corollary 2.2.4 *If $\mathcal{S}$ is a point-weight incidence structure that is simultaneously a $\pi_t - (v, k, \lambda; W)$ and a $\pi_{t-1} - (v, k, \lambda'; W)$ point-weight design, where $t \geq 2$, then $|W| = 1$.*

Proof Suppose the point-weight incidence structure $\mathcal{S}$ is a $\pi_t - (v, k, \lambda; W)$ and a $\pi_{t-1} - (v, k, \lambda'; W)$ point-weight design with $|W| \geq 2$. Let S be a set of $t-1$ points. We have, if we use (2.2.3), two ways of calculating $\iota(S)$:

$$\iota(S) \quad = \quad \frac{\lambda'}{\prod_{x\in S} w(x)}$$
$$\iota(S) \quad = \quad \frac{(u-t+1)\lambda}{(k-\sigma(S))\prod_{x\in S} w(x)}$$

which means that

$$\frac{\lambda'}{\lambda} \quad = \quad \frac{u-t+1}{k-\sigma(S)}.$$

However this is a contradiction as the RHS of this equation is not constant if $|W| \geq 2$ as there always exists at least two sets of $t-1$ points S_1, S_2 such that $\sigma(S_1) \neq \sigma(S_2)$. Hence we must have that $|W| = 1$.

■

Another structure we considered in (1.3) was the point-residue structure, see (1.3.4).

Lemma 2.2.5 *If $\mathcal{S}$ is a $\pi_t - (v, k, \lambda; W)$ point-weight design with $t \geq 2$ and x is a point of $\mathcal{S}$ then the point residue of $\mathcal{S}$ at x, $\mathcal{S}^x$, is a $\pi_{t-1} - (v', k', \lambda'; W')$ if and only if $|W'| = 1$.*

Proof Let S be a set of $t-1$ points of $\mathcal{S}^x$. There exists a corresponding set of points in $\mathcal{S}$. Hence

$$\begin{aligned}
\iota_{\mathcal{S}^x}(S) &= \iota_{\mathcal{S}}(S) - \iota_{\mathcal{S}}(S \cup \{x\}) \\
&= \frac{\lambda(u-t+1)}{(k-\sigma(S))\prod_{y\in S} w(y)} - \frac{\lambda}{w(x)\prod_{y\in S} w(y)} \\
&= \frac{\lambda}{\prod_{y\in S} w(y)}\Big(\frac{u-t+1}{k-\sigma(S)} - \frac{1}{w(x)}\Big).
\end{aligned}$$

Therefore if $\mathcal{S}^x$ is a $\pi_{t-1} - (v', k', \lambda'; W')$ point-weight design then

$$\lambda' = \lambda\Big(\frac{u-t+1}{k-\sigma(S)} - \frac{1}{w(x)}\Big)$$

and this expression is constant. However this expression is constant if and only if $|W'| = 1$ for the same reasons given in the previous lemma.

■

A discussion of the last type of related structure defined in (1.3), the dual structure, is contained in Chapter 5. We will end this section by giving a theorem about row-sum point-weight designs that share the same underlying structure.

Theorem 2.2.6 *If $\mathcal{S}$ is a $\pi_t - (v, k, \lambda; W)$ point-weight design with more than t points that has the same underlying structure as $\mathcal{T}$, which is also a $\pi_t - (v', k', \lambda'; W')$ point-weight design then $\mathcal{S}$ is equivalent to $\mathcal{T}$.*

Proof Let S be any set of $t-1$ points and x and y be points not in S. Consequently we have that:

$$\begin{aligned}
\iota(S \cup \{x\})w(x)\prod_{z\in S} w(z) &= \lambda \\
\iota(S \cup \{x\})w'(x)\prod_{z\in S} w'(z) &= \lambda' \\
\iota(S \cup \{y\})w(y)\prod_{z\in S} w(z) &= \lambda \\
\iota(S \cup \{y\})w'(y)\prod_{z\in S} w'(z) &= \lambda'
\end{aligned}$$

So,

$$\frac{w(x)}{w'(x)}\prod_{z\in S}\frac{w(z)}{w'(z)}=\frac{\lambda}{\lambda'}=\frac{w(y)}{w'(y)}\prod_{z\in S}\frac{w(z)}{w'(z)}$$

which implies that the weight of every point of $\mathcal{S}$ not in S is a multiple of the weight of the equivalent point of $\mathcal{T}$. However we can always pick different sets S until we have shown that every point is a multiple of the weight of the equivalent point. Hence $\mathcal{S}$ is equivalent to $\mathcal{T}$.

$\blacksquare$

This tells us that the way we can assign weights to a structure and make it a π_t row-sum point-weight design is essentially unique. This fact will come in very useful later.

2.3 A combinatorial analysis of the parameters

We have already, in (2.1.4), started a simple combinatorial analysis of row-sum point-weight designs and we will now expand upon this. We will primarily be concerned here with $\pi_t-(v,k,\lambda;W)$ designs where $t\geq 2$.

In (2.1.4) we derived an expression for the number of blocks in a $\pi_1-(v,k,\lambda;W)$ point-weight design. The proof revolves around the fact that we can find an explicit formula for the number of blocks which are incident with a given point. The question of how many blocks a given point lies upon in a $\pi_t-(v,k,\lambda;W)$ design has essentially already been proven in the introductory chapter where we have shown that any two points of equal weight are incident with the same number of blocks in a $t-(v,k,\lambda;W)$ point-weight design. This technique can also be used in $\pi_t-(v,k,\lambda;W)$ point-weight designs.

Theorem 2.3.1 *If $\mathcal{S}$ is a $\pi_t-(v,k,\lambda;W)$ point-weight design with more than t points then any two sets of at most t points that have the same number of points of each weight are incident with the same number of blocks. In particular, the number of blocks incident with a point is equal for all points of the same weight.*

Proof If $\mathcal{S}$ is a $\pi_1-(v,k,\lambda;W)$ design then the result is trivial as each point x is incident with $\frac{\lambda}{w(x)}$ blocks and if $t>1$ then we may apply (1.7.6) with

$$\iota(S)=\frac{\lambda}{\prod_{x\in S}w(x)}$$

where S is any set of t points.

$\blacksquare$

The converse (that two points that lie on the same number of blocks must have the same weight) however is not generally true and we will return to this question in (3.1). However, even if this converse is not true, the theorem itself can be quite a lot of use. The following results use this fact and follow the lines suggested by [1].

Lemma 2.3.2 *Suppose $\mathcal{S}$ is a $\pi_t - (v, k, \lambda; W)$ or a $t - (v, k, \lambda; W)$ point-weight design with b blocks, where $W = \{w_1, \ldots w_n\}$. Suppose further that there exists u_i points of weight w_i and each of those points is incident with r_i blocks. Then*

$$bk = \sum_{i=1}^{n} u_i w_i r_i.$$

Proof In a manner similar to (2.1.4) we will evaluate a sum in two ways. Suppose $\mathcal{S}$ is a $(V, \mathcal{B}, I, w)$ point-weight incidence structure. Then

$$\begin{aligned} \sum_{(x,B)\in I} w(x) &= \sum_{x\in V} \sum_{B\ni x} w(x) \\ &= \sum_{x\in V} r_x w(x) \\ &= \sum_{i=1}^{n} u_i w_i r_i \end{aligned}$$

but also

$$\begin{aligned} \sum_{(x,B)\in I} w(x) &= \sum_{B\in\mathcal{B}} \sum_{x\in B} w(x) \\ &= \sum_{B\in\mathcal{B}} k \\ &= bk. \end{aligned}$$

Hence the result.

■

The result shown in [1] is that $bk = v_1r_1 + \ldots + v_nr_n$ for classical designs and if we set $v_i = w_iu_i$ in a point-weight design, i.e. that v_i is the sum of the weights of the points with weight w_i, which is not an unreasonable thing to do, then the results are identical. The next result in this section is also essentially due to Adhikary, [1].

Lemma 2.3.3 *If $\mathcal{S} = (V, \mathcal{B}, I, w)$ is a $\pi_2 - (v, k, \lambda; W)$ point-weight design then*

$$r_x + r_y \leq b + \frac{\lambda}{w(x)w(y)}$$

where r_z is the number of blocks with which the point z is incident.

Proof Consider the complementary structure $\mathcal{S}^*$ of $\mathcal{S}$ given by

$$\begin{aligned} V^* &= V \\ \mathcal{B}^* &= \mathcal{B} \\ I^* &= \{(x, B) \in V^* \times \mathcal{B}^* : (x, B) \notin I\} \end{aligned}$$

Now any two points x and y must be incident with a non-negative number of blocks in $\mathcal{S}^*$, i.e. $\iota^*(\{x, y\}) \geq 0$. The exact number of blocks of x and y are incident with in $\mathcal{S}^*$ is

$$\begin{aligned} \iota^*(\{x, y\}) &= b - r_x - r_y + \iota(\{x, y\}) \\ &= b - r_x - r_y + \frac{\lambda}{w_x w(y)} \end{aligned}$$

and so

$$b + \frac{\lambda}{w(x)w(y)} \geq r_x + r_y$$

■

We also have an obvious restriction on the value of λ.

Lemma 2.3.4 *If S is a set of $s \leq t$ points of a $\pi_t - (v, k, \lambda; W)$ point-weight design then $\prod_{x \in S} w(x)$ divides λ.*

Corollary 2.3.5 *If $\mathcal{S}$ is a $\pi_t - (v, k, \lambda; W)$ point-weight design where $W = \{w_1, \ldots, w_n\}$ and each of point of weight w_i is incident with u_i blocks then*

$$\lambda \geq l.c.m.\{w_1^{min\{t,u_1\}}, \ldots, w_n^{min\{t,u_n\}}\}.$$

Proof If S is a set of points containing $min\{t, u_i\}$ points of weight w_i, we know that $\prod_{x \in S} w(x) = w_i^{min\{t,u_i\}}$ divides λ for all $1 \leq i \leq n$. Hence the lowest common multiple of these numbers divides λ.

■

Lastly we note that:

Lemma 2.3.6 *If $\mathcal{S}$ is a $\pi_t - (v, k, \lambda; W)$ and S is a set of $t-1$ points of $\mathcal{S}$ then*

$$\iota(S) \geq \frac{\lambda}{w(x) \prod_{y \in S} w(y)} \quad \textit{for all } x \in V \setminus S \tag{2.3}$$

Proof For any $x \in V \setminus S$ let $T = S \cup \{x\}$ then:

$$\iota(S) \geq \iota(T) = \frac{\lambda}{w(x) \prod_{y \in S} w(y)}$$

■

2.4 Incidence matrices

Now we return to the incidence matrix. The combination of (2.1.2) and (1.4.4) gives us the following result:

Lemma 2.4.1 *If $\mathcal{S}$ is a $\pi_2 - (v, k, \lambda; W)$ point-weight design with incidence matrix M then:*

$$\begin{aligned} det(MM^T) &= (1 + \lambda \sum_{x \in V} \frac{1}{w(x)^2 r_x - \lambda}) \prod_{x \in V} (w(x)^2 r_x - \lambda) \\ &= \prod_{x \in V} (w(x)^2 r_x - \lambda) + \lambda \sum_{x \in V} \prod_{y \in V \setminus \{x\}} (w(y)^2 r_y - \lambda). \end{aligned} \tag{2.4}$$

This allows us to sub-divide row-sum point-weight designs depending on the values of $w(x)^2 r_x - \lambda$:

Definition 2.4.2 *A point $x \in V$ is called* awkward *if $w(x)^2 r_x - \lambda = 0$ and a design is called* awkward *if it contains an awkward point. A point $x \in V$ is called* difficult *if $w(x)^2 r_x - \lambda < 0$ and a design is called* difficult *if it contains a difficult point.*

Definition 2.4.3 *Suppose $\mathcal{S}$ is a $\pi_2 - (v, k, \lambda; W)$ point-weight design with incidence matrix M. $\mathcal{S}$ is said to be* nice *if $det(MM^T) \neq 0$.*

Lemma 2.4.4 *If a $\pi_2 - (v, k, \lambda; W)$ point-weight design is neither awkward or difficult then it is nice.*

Proof Since the design contains neither awkward or difficult points we have that $w(x)^2 r_x - \lambda > 0$ for all $x \in V$, hence every term in the calculation of the determinant is positive and so the determinant is non-zero.

■

Theorem 2.4.5 *If x is an awkward point in a $\pi_2 - (v, k, \lambda; W)$ point-weight design $\mathcal{S}$, then $w(x) \leq w(y)$ for all $y \in V$ and if z is a point of the same weight as x then x and z are incident with exactly the same blocks, hence z is awkward too.*

Proof We have already shown that if $|W|= 1$ then $\mathcal{S}$ is equivalent to a structure in which every point is of weight 1 and the underlying structure is of a classical design. We have already shown, in the proof of (1.4.6), that if $r_x = \lambda$ in a classical design then $k = v$, i.e. there exists only one block that contains all the points. In that case the theorem is trivially satisfied. So we may assume that $|W|\geq 2$.

Suppose $y \neq x$ is a point of $\mathcal{S}$. The number of blocks incident with both x and y must be less than or equal to the number of blocks incident with x alone, i.e.

$$\tfrac{\lambda}{w(x)w(y)} \leq r_x = \tfrac{\lambda}{w(x)^2}.$$

Hence $w(x) \leq w(y)$.

Now suppose z is another point of minimal weight. Every block containing x and z, of which there are $\frac{\lambda}{w(x)w(z)} = \frac{\lambda}{w(x)^2}$, also contains x. However there are only $\frac{\lambda}{w(x)^2}$ blocks that contain x so every block containing x also contains z.

Suppose there exists a block $B \in \mathcal{B}$ that contains z but does not contain x. Pick a point $z' \neq z$ that lies on B (such a point exists because $w(z) < w(x) + w(z) \leq k$ but the sum of the weights of the points in B is equal to k) then the number of blocks containing z and z' is greater the number of blocks containing x and z' because every block that x also contains z but there exists at least one block that contains z and z' but not x. So:

$$\frac{\lambda}{w(z)w(z')} > \frac{\lambda}{w(x)w(z')}$$

which is a contradiction as $w(x) = w(z)$. So there can exist no block that contains z but does not contain x, i.e. x and z must be incident with exactly the same blocks. In particular this means that $r_x = r_z$ and therefore $w(x)^2 r_x = w(z)^2 r_z = \lambda$, which implies that z is awkward too.

■

If a point lies on exactly the same blocks as another point then we can get a kind of converse to the above theorem.

Lemma 2.4.6 *If x and y are points of the same weight of a $\pi_2 - (v, k, \lambda; W)$ point-weight design and x lies on every block that contains y then x and y are awkward.*

Proof There exist $\frac{\lambda}{w(x)w(y)}$ blocks that contain both x and y, and since every block containing y contains x, y must lie on $\frac{\lambda}{w(x)w(y)} = \frac{\lambda}{w(y)^2}$ blocks. Hence y is awkward, but then so is x as they are both of the same (necessarily) minimal weight.

■

Corollary 2.4.7 *If x and y are two points of a $\pi_2 - (v, k, \lambda; W)$ point-weight design such that $w(x) = w(y)$ and this weight is not minimal then there exists a block B that contains x but does not contain y.*

Theorem 2.4.8 *If x is a difficult point of a $\pi_2 - (v, k, \lambda; W)$ point-weight design then $w(x) < w(y)$ for all $y \in V \setminus \{x\}$.*

Proof Since x is difficult we have that the number of points x is incident with, r_x, is less than $\frac{\lambda}{w(x)^2}$. So for any $y \in V \setminus \{x\}$ we have that:

$$\frac{\lambda}{w(x)w(y)} \leq r_x < \frac{\lambda}{w(x)^2}$$

Hence $w(x) < w(y)$.

■

Corollary 2.4.9 *If a $\pi_2 - (v, k, \lambda; W)$ point-weight design is difficult then it has a single difficult point of minimal weight.*

Corollary 2.4.10 *A $\pi_2 - (v, k, \lambda; W)$ design cannot be both difficult and awkward.*

Proof If $\mathcal{S}$ is a difficult design, then it has a single difficult point of minimal weight. Since that point is not awkward the design cannot be awkward, as all awkward points are of minimal weight too.

■

Lemma 2.4.11 *If $\mathcal{S}$ is an awkward $\pi_2 - (v, k, \lambda; W)$ point-weight design with incidence matrix M then $det(MM^T) = 0$ if and only if $\mathcal{S}$ contains more than one awkward point.*

Proof We have already shown (2.4.1) that the following is an expression for the determinant of MM^T:

$$det(MM^T) = \prod_{x \in V}(w(x)^2 r_x - \lambda) + \lambda \sum_{x \in V} \prod_{y \in V \setminus \{x\}} (w(y)^2 r_y - \lambda).$$

So if $\mathcal{S}$ has any awkward points then the first part of this expression will always be zero, however the latter part of this expression will be positive unless $\mathcal{S}$ has *more than one* awkward point. Therefore $\mathcal{S}$ is nice if and only if it contains a single awkward point.

■

We may now present an analogy to Fisher's Inequality, (1.4.6).

Lemma 2.4.12 (Fisher's Inequality) *Suppose $\mathcal{S}$ is a $\pi_2 - (v, k, \lambda; W)$ point-weight design with u points and b blocks.*

1. *If $\mathcal{S}$ is nice then $b \geq u$.*

2. *If $\mathcal{S}$ is difficult then $b \geq u - 1$.*

3. *If $\mathcal{S}$ is awkward with m awkward points then $b \geq u - m + 1$.*

Proof Let M be an incidence matrix for $\mathcal{S}$. We know that $rank(MM^T) = rank(M)$ and that $rank(M) \leq b$ as M has b columns, so we have to find lower bounds for $rank(MM^T)$. We will deal with each case in turn.

1. If $\mathcal{S}$ is nice then $det(MM^T) \neq 0$ and so $rank(MM^T) = u$. Hence $u \leq b$.

2. If $\mathcal{S}$ is difficult then it contains a single difficult point. If $\mathcal{S}$ is difficult and nice then $b \geq u \geq u - 1$. Suppose $\mathcal{S}$ is not nice then we may apply elementary row operations to MM^T as in (1.4.4) to reduce it to the form

$$\left[\begin{array}{c|cccc} 0 & \lambda & \lambda & \dots & \lambda \\ \hline 0 & w(x_2)^2 r_{x_2} - \lambda & 0 & \dots & 0 \\ 0 & 0 & w(x_3)^2 r_{x_3} - \lambda & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & w(x_u)^2 r_{x_u} - \lambda \end{array}\right].$$

   From this we can clearly see that $rank(MM^T) = u - 1$. Hence $b \geq u - 1$.

3. If $\mathcal{S}$ is awkward with 1 awkward point then $\mathcal{S}$ is nice and so we have already shown that $b \geq u$. Suppose that $\mathcal{S}$ is awkward with $m \geq 2$ awkward points

and, without loss of generality, assume that MM^T is of the form

$$\left[\begin{array}{ccc|cccc}
\lambda & \dots & \lambda & \lambda & \dots & \dots & \lambda \\
\vdots & & \vdots & \vdots & & & \vdots \\
\lambda & \dots & \lambda & \lambda & \dots & \dots & \lambda \\
\hline
\lambda & \dots & \lambda & w(x_{m+1})^2 r_{x_{m+1}} & \lambda & \dots & \lambda \\
\lambda & \dots & \lambda & \lambda & \ddots & \ddots & \vdots \\
\vdots & & \vdots & \vdots & \ddots & \ddots & \lambda \\
\lambda & \dots & \lambda & \lambda & \dots & \lambda & w(x_u)^2 r_{x_u}
\end{array}\right].$$

Thus, by subtracting the m^{th} row from every other row, we may reduce this matrix to

$$\left[\begin{array}{ccc|cccc}
0 & \dots & 0 & 0 & \dots & \dots & 0 \\
\vdots & & \vdots & \vdots & & & \vdots \\
0 & \dots & 0 & 0 & \dots & \dots & 0 \\
\lambda & \dots & \lambda & \lambda & \dots & \dots & \lambda \\
\hline
0 & \dots & 0 & w(x_{m+1})^2 r_{x_{m+1}} - \lambda & 0 & \dots & 0 \\
\vdots & & \vdots & 0 & \ddots & \ddots & \vdots \\
\vdots & & \vdots & \vdots & \ddots & \ddots & 0 \\
0 & \dots & 0 & 0 & \dots & 0 & w(x_u)^2 r_{x_u} - \lambda
\end{array}\right]$$

which clearly has rank $u - m + 1$. Hence $b \geq u - m + 1$.

■

Lastly we define some terminology that will be of use to us later. Note that it is entirely consistent with the standard terminology of classical designs if $\mathcal{S}$ has the underlying structure of a classical design and every point is of weight 1.

Definition 2.4.13 *If $\mathcal{S}$ is a $\pi_2-(v, k, \lambda; W)$ point-weight design and x is a point of $\mathcal{S}$ then the order of x is defined to be $n_x = w(x)^2 r_x - \lambda$.*

2.5 Conclusion

In this chapter we have introduced the concept row-sum point-weight designs, which will form the central area of study of this thesis and begun to examine them. We have found that there is a uniqueness to row-sum point-weight designs both in the sense of equivalence, that any design is equivalent to one with a co-prime weight set, and in sense of (2.2.6), that any two $\pi_t-(v, k, \lambda; W)$ point-weight design with the same underlying incidence structure are necessarily equivalent.

We have also shown that unlike classical designs, the incidence matrix of a row-sum point-weight design can be singular but that if this is the case then there must exist one or more special point with specific properties.

Furthermore we have also introduced two very powerful devices that will be extensively used in the remaining chapters. We found the derived structure of a row-sum point-weight design is also a row-sum point-weight design and found its explicit parameters (2.2.1). We also found that every point of equal weight is incident with the same number of blocks (2.3.1).

There are a couple of areas which are conspicuous by their absence. The first is that no examples of row-sum point-weight designs have been given in this chapter and this will be remedied in the next chapter where a whole section will be given over to the construction of row-sum point-weight designs. It was felt that, besides settling the question of existence of point-weight designs, there was no advantage in including specific examples here. Secondly we have not investigated point-weight incidence structures that admit to being both $\pi_t - (v, k, \lambda; W)$ and either $t - (v, k, \lambda'; W)$ or $\sigma_t - (v, k, \lambda'; W)$ point weight designs. We will also look at some aspects of this problem in the next chapter.

Chapter 3

Further combinatorial analysis

In this chapter we will continue our combinatorial examination of row-sum point-weight designs, specifically we will examine the two areas mentioned in the conclusion of the last chapter. That is to say we exhibit some constructions and construction techniques that will settle the question of existence for nice, awkward and difficult row-sum point-weight designs, and we will examine the situation where a point-weight incidence structure is both a $\pi_t - (v, k, \lambda; W)$ design and a $s - (v, k, \lambda'; W')$ design.

3.1 Regularity

We have shown in (2.3.1) that any two points of equal weight are incident with the same number of blocks. We will now examine the converse: if two points are incident with the same number of blocks do they have the same weight? This is obviously true for classical designs (where we tend to think of all the points as having the same weight) and has been shown, in [9], to be true for point-sum point-weight designs when $t = 2$. This questions is linked to the problem of regularity.

Definition 3.1.1 *A point-weight incidence structure $\mathcal{S} = (V, \mathcal{B}, I, w)$ is regular if every point is incident with the same number of blocks.*

So if $\mathcal{S}$ is a regular $\pi_t - (v, k, \lambda; W)$ point-weight design then it is also a $1 - (v, k, r; W)$ point-weight design. The focus of this section will mostly be on $\pi_2 - (v, k, \lambda; W)$ point-weight designs and so we begin by calculating the number of blocks a point is incident with in such a design.

Lemma 3.1.2 *If $\mathcal{S}$ is a $\pi_2 - (v, k, \lambda; W)$ point-weight design with u points and x is a point of $\mathcal{S}$ then x is incident with r_x blocks where*

$$r_x = \frac{\lambda(u-1)}{(k - w(x))w(x)}.$$

$$\left[\begin{array}{cccc|cccc} 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{array}\right]$$

Figure 3.1: A regular $\pi_2 - (6, 3, 2; \{1, 2\})$ point-weight design

Proof We use (2.2.3) where the set S contains the single point x.

■

It is tempting to conjecture that there are no regular $\pi_t - (v, k, \lambda; W)$ point-weight designs but Figure 3.1 would provide a contradiction. In fact any design constructed using the method given in (3.2.1) is regular. We will now examine the conditions that a design must satisfy in order to have two points of different weight that are incident with the same number of blocks.

Lemma 3.1.3 *Suppose $\mathcal{S}$ is a $\pi_2 - (v, k, \lambda; W)$ point-weight design with $u > 2$ points. If there exists two points x and y, with $w(x) < w(y)$, that are incident with the same number of blocks then:*

1. $k = w(y) + w(x)$,
2. *there exists a single point of weight* $w(y)$,
3. $\lambda = w(x)w(y)$,
4. *if z is a point of whose weight is not $w(x)$ or $w(y)$ then $w(z)$ divides g.c.d.$(w(x), w(y))$,*
5. *if there exists more than one point of weight $w(x)$ then $w(x) \mid w(y)$.*

Proof We have that

$$\begin{aligned} r_x &= r_y. \\ \text{So } \frac{(u-1)\lambda}{(k-w(x))w(x)} &= \frac{(u-1)\lambda}{(k-w(y))w(y)} \\ \text{and } (k-w(x))w(x) &= (k-w(y))w(y) \\ \text{and } (w(x)-w(y))(k-w(x)-w(y)) &= 0 \\ \text{and so } k &= w(x)+w(y) \text{ since } w(x) \neq w(y). \end{aligned}$$

Now if there were more than one point of weight $w(y)$ then those two points would both have to lie on at least one block (because $\mathcal{S}$ is a $\pi_2 - (v, k, \lambda; W)$

point-weight design) and the weight of that block would therefore be at least $2w(y) > w(y) + w(x) = k$. Hence there cannot exist two points of weight $w(y)$.

As $w(y) + w(x) = k$ and we have excluded the possibility of repeated blocks we must have that the only block that contains both x and y is $\{x, y\}$. But then we have that $\iota(\{x, y\}) = \frac{\lambda}{w(x)w(y)} = 1$ and so $\lambda = w(x)w(y)$.

Now suppose that we have a point z whose weight is neither $w(x)$ or $w(y)$. By (2.3.4) we know that $w(x)w(z)$ divides $\lambda = w(x)w(y)$, hence $w(z) \mid w(y)$ but similarly $w(z) \mid w(x)$ and so $w(z) \mid g.c.d.(w(x), w(y))$.

By a similar argument, if there exists a point $x' \neq x$ with weight $w(x)$ then by (2.3.4) we have that $w(x)^2$ divides $\lambda = w(x)w(y)$ and so $w(x) \mid w(y)$. In particular this means that if $|W|= 2$ and $u \geq 3$ then there must exist a point of weight 1 in $\mathcal{S}$, as W is a co-prime set.

■

Corollary 3.1.4 *If $\mathcal{S}$ is a regular $\pi_2-(v, k, \lambda; W)$ with $u > 2$ points then $|W|\leq 2$.*

Proof We show above that if y is incident with the same number of blocks as x then $w(y) = w(x)$ or $w(y) = k - w(x)$. So if all points are incident with the same number of blocks as x then there can be at most two different weights.

■

So any regular $\pi_2-(v, k, \lambda; W)$ design with $|W|> 1$ has a very specific structure and set of parameters.

Lemma 3.1.5 *Suppose $\mathcal{S}$ is a regular $\pi_2 - (v, k, \lambda; W)$ point-weight design with $u > 2$ points, $|W|> 1$ and each point of $\mathcal{S}$ is incident with r blocks. Then $\mathcal{S}$ is a $\pi_2 - (\lambda + r, \lambda + 1, \lambda; \{1, \lambda\})$ point-weight design with $r + 1$ points with an incidence matrix of the form*

$$\left[\begin{array}{ccc|ccc} \lambda & \dots & \lambda & 0 & \dots & 0 \\ \hline & & & & & \\ & I_r & & & A & \\ & & & & & \end{array}\right]$$

where A is incidence matrix of a $2 - (r, \lambda + 1, \lambda)$ design. Furthermore k divides $r(r - 1)$.

Proof The above corollary shows that $\mathcal{S}$ has $|W|= 2$ and that there exists a single point of highest weight. Let this point be y and have weight $w(y)$. Let $\{x_1, \dots, x_{u-1}\}$ be the remaining points each of which is of the same weight. Since there exist at least three points in $\mathcal{S}$, we have that there exists more than one point of weight $w(x_1)$ hence, by (3.1.3), we have that $w(x_1)$ divides $w(y)$ and so $w(x_i) = 1$ for all $1 \leq i \leq u - 1$ by the co-primality of W.

Therefore $\lambda = w(x_1)w(y) = w(y)$, $k = w(y) + w(x_1) = \lambda + 1$ and $W = \{1, \lambda\}$.

Now y is connected to each of the $u-1$ points x_i by a single block that contains y and x_i only, and there exists no blocks that are incident with y except the ones of this form. Since y is incident with r blocks there must exist r points $x_1, \ldots x_{u-1}$. Hence $r = u-1$ and the blocks that contain y must have an incidence matrix of the form

$$\left[\begin{array}{c} \lambda \ \ldots \ \lambda \\ \hline \\ I_r \\ \\ \end{array}\right]$$

This means there exist r points of weight 1 in $\mathcal{S}$ and one point of weight λ, and so $v = r + \lambda$. $\mathcal{S}$ must be a $\pi_2 - (r+\lambda, \lambda+1, \lambda; \{1, \lambda\})$ and must have an incidence matrix of the form

$$\left[\begin{array}{c|c} \lambda \ \ldots \ \lambda & 0 \ \ldots \ 0 \\ \hline & \\ I_r & A \\ & \end{array}\right]$$

The matrix A is the incidence matrix of $\mathcal{S}^y$. There exist r points of $\mathcal{S}^y$ and each of these points has weight 1. Consequently every block of $\mathcal{S}^y$ contains k points and any two points are contained in λ blocks. Thus the underlying structure of $\mathcal{S}^y$ is that of a $2-(r, \lambda+1, \lambda)$ design and since A is a 0, 1-matrix, it is the incidence matrix of a $2-(r, \lambda+1, \lambda)$ design.

Lastly we note that A also defines a $1-(r, k, r-1)$ design as each point x_i of $\mathcal{S}$ is incident with r blocks and precisely one of these is incident with y. So the number of blocks of $\mathcal{S}^y$ is

$$b = \frac{r(r-1)}{k}$$

by (1.2.2) and this is an integer. Hence k divides $r(r-1)$.

■

One might be tempted to think that the family of constructions shown in (3.2.1) are the only regular $\pi_2 - (v, k, \lambda; W)$ point-weight designs but this is not true. Fig 3.2 gives another example of a regular $\pi_2 - (v, k, \lambda; W)$ point-weight design that is not constructable using (3.2.1). This is not the only possible regular point-weight design as the next lemma shows.

Lemma 3.1.6 *If A is the incidence matrix of a $2-(u, \lambda+1, \lambda)$ design with $\lambda > 1$ then the matrix*

$$M = \left[\begin{array}{c|c} \lambda \ \ldots \ \lambda & 0 \ \ldots \ 0 \\ \hline & \\ I_u & A \\ & \end{array}\right]$$

defines a regular $\pi_2 - (v, k, \lambda; W)$ point-weight design with $|W| \geq 2$.

$$\left[\begin{array}{cccccc|cccccccccc}
2 & 2 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1
\end{array}\right]$$

Figure 3.2: A regular $\pi_2 - (8, 3, 2; \{1, 2\}$ point-weight design

Proof Let $\mathcal{S}$ be the point weight incidence structure define by M and note that this contains points of two different weights. It is obvious that $\mathcal{S}$ has a constant blocks size $k = \lambda + 1$ so we have to show that $\mathcal{S}$ satisfies the design condition and that this design is regular.

If we take two points x_1 and x_2 of weight one then these blocks are incident with $\lambda = \frac{\lambda}{w(x_1)w(x_2)}$ blocks by the definition of A. If we take a point x of weight one and the point y of weight λ then these two points are incident with only $1 = \frac{\lambda}{w(x)w(y)}$ blocks. Hence $\mathcal{S}$ is a $\pi_2 - (u + \lambda, \lambda + 1, \lambda; W)$ point-weight design.

The point y is clearly incident with u blocks. Any point x of weight one is incident with one block that contains y and $u - 1$ blocks that don't contain y, by (1.2.2). Hence x is incident with u blocks and $\mathcal{S}$ is regular.

■

We have now completely characterised regular $\pi_2 - (v, k, \lambda; W)$ designs as point-weight incidence structures with specific structures. We may extend this to any point-weight incidence structure that is both a $\pi_{t+1} - (v, k, \lambda; W)$ and a $t - (v, k, \lambda'; W)$ point-weight design. Figure 3.3 gives a specific example of a point-weight incidence structure that is simultaneously a $\pi_3 - (v, k, \lambda; W)$ and a $2 - (v, k, \lambda'; W)$ point weight design. In fact we can produce a structure that is a $\pi_{t+1} - (v, k, \lambda; W)$ and a $t - (v, k, \lambda'; W)$ point-weight design for any $t \geq 1$.

If we start with parameters $t \geq 2$ and $\lambda \geq 2$ then the point-weight incidence structure defined by

$$\left[\begin{array}{ccc|ccc}
\lambda & \ldots & \lambda & 0 & \ldots & 0 \\
\hline
 & A & & & B &
\end{array}\right]$$

where A is the $(\lambda + t + 1) \times \binom{\lambda+t+1}{t}$ matrix consisting of all column vectors with t 1's and B is equal to $J_{\lambda+t+1} - I_{\lambda+t+1}$, then this structure is both a $\pi_{t+1} - (2\lambda + t + 1, \lambda + t, \lambda; \{1, \lambda\})$ and a $t - (2\lambda + t + 1, \lambda + t, \lambda + 2; \{1, \lambda\})$ point-weight design.

$$\left[\begin{array}{cccccccccc|ccccc}
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 \\
\hline
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0
\end{array}\right]$$

Figure 3.3: A structure that is both a $\pi_3-(6,4,2;\{1,2\})$ and a $2-(6,4,4;\{1,2\})$ point-weight design.

Theorem 3.1.7 *If $\mathcal{S}$ is a point-weight incidence structure that is both a $\pi_{t+1}-(v,k,\lambda;W)$ and a $t-(v,k,\lambda';W)$ point-weight design with $|W|\geq 2$ and contains $u>t+1$ points then $\mathcal{S}$ has an incidence matrix of the form:*

$$\left[\begin{array}{ccc|ccc}
\lambda & \ldots & \lambda & 0 & \ldots & 0 \\
\hline
 & & & & & \\
 & A & & & B & \\
 & & & & &
\end{array}\right]$$

where A is the $(\lambda'+t-1)\times\binom{\lambda'+t-1}{t}$ matrix consisting of all 0,1-column vectors with t 1's and B is the incidence matrix for a $(t+1)-(\lambda'+t-1,\lambda+t,\lambda)$ design. Furthermore all point-weight incidence structures of this type are $\pi_{t+1}-(\lambda+\lambda'+t-1,\lambda+t,\lambda;\{1,\lambda\})$ and $t-(\lambda+\lambda'+t-1,\lambda+t,\lambda';\{1,\lambda\})$ point-weight designs.

Proof We will proceed using induction on the value of t. First note that we have already shown that the above theorem is true for $t=1$ in (3.1.5), as in this case $\lambda'=r$. So, for an induction hypothesis, we assume that if $\mathcal{S}$ is a point-weight incidence structure that is both a $\pi_{s+1}-(v,k,\lambda;W)$ and a $s-(v,k,\lambda';W)$ point-weight design with $|W|\geq 2$, $u>s+1$ points and $1\leq s<t$ then $\mathcal{S}$ is of the form given in the statement of the theorem.

Let $\mathcal{S}$ be a point-weight incidence structure that is simultaneously a $\pi_{t+1}-(v,k,\lambda;W)$ and a $t-(v,k,\lambda';W)$ point-weight design with $|W|\geq 2$, $u>t+1$ points and $t\geq 2$.

Pick a point z of $\mathcal{S}$. $\mathcal{S}_z$ is a $\pi_t-(v-w(z),k-w(z),\frac{\lambda}{w(z)};W')$ and a $(t-1)-(v-w(z),k-w(z),\lambda';W')$ point-weight design with $u-1>t$ points (by 2.2.1 and 1.7.3). We know that if there exists only one point of weight $w(z)$ then $W'=W\setminus\{w(z)\}$ else $W'=W$. We will try to pick z such that there exists more than one point of weight $w(z)$ but this will not be possible if each point of $\mathcal{S}$ is of a different weight.

If every point of $\mathcal{S}$ is of a different weight then $|W|=u$. Now $\mathcal{S}_x$ has, by the induction hypothesis, $|W'|\leq 2$ so $|W|\leq 3$. However, in this case, this is a contradiction as $|W|= u > t+1 \geq 3$ as $t \geq 2$. So we must assume that there exists a point z such that there exists more than one point of that weight and so $|W'|=|W|= 2$.

This means that, by the induction hypothesis, $\mathcal{S}_x$ has an incidence matrix of the form

$$\left[\begin{array}{ccc|ccc} \lambda & \dots & \lambda & 0 & \dots & 0 \\ \hline & & & & & \\ & A' & & & B' & \\ & & & & & \end{array}\right]$$

where A' is the $(\lambda'+t-2) \times \binom{\lambda+t-2}{t}$ matrix consisting of all column vectors with $t-1$ 1's and B' is the incidence matrix for a $t-(\lambda'+t-2, \lambda+t-1, \lambda)$ design. Since $W = W'$ we know that z either has weight λ or weight 1. Let x be the point of $\mathcal{S}$ that has weight λ in $\mathcal{S}_z$, hence $x \neq z$. Similarly let y be a point of $\mathcal{S}$ that has weight 1 in $\mathcal{S}_z$ and $y \neq z$. Note that as $u > t+1 \geq 3$ there must exist more than one point of weight 1 in $\mathcal{S}_z$.

If we consider $\mathcal{S}_y$ then this structure must also be of the above form as it is a $\pi_t - (v-1, k-1, \lambda; W)$ and a $(t-1)-(v-1, k-1, \lambda'; W)$ point-weight design. However $\mathcal{S}_y$ contains a row that corresponds to the point z and it is not the row that corresponds to the point x. Hence $w(z) = 1$. So $\mathcal{S}$ has an incidence matrix of the form

$$\left[\begin{array}{ccc|ccc|ccc|ccc} \lambda & \dots & \lambda & 0 & \dots & 0 & \lambda & \dots & \lambda & 0 & \dots & 0 \\ \hline 1 & \dots & 1 & 1 & \dots & 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ \hline & & & & & & & & & & & \\ & A' & & & B' & & & C' & & & D' & \\ & & & & & & & & & & & \end{array}\right]$$

where the first row corresponds to the point x and the second row corresponds to the point z. $\mathcal{S}_x$ must have an incidence matrix of the form:

$$\left[\begin{array}{ccc|ccc} 1 & \dots & 1 & 0 & \dots & 0 \\ \hline & & & & & \\ & A' & & & C' & \\ & & & & & \end{array}\right]$$

and have an underlying incidence structure that is a $t-(u-1, t, 1)$ design. Hence it must be the trivial design and C' must be a $(u-2) \times \binom{u-2}{t-1}$ matrix that consists of all the remaining 0,1-column vectors of Hamming weight $t-1$.

Similarly $\mathcal{S}^x$ has an incidence matrix of the form

$$\left[\begin{array}{ccc|ccc} 1 & \dots & 1 & 0 & \dots & 0 \\ \hline & & & & & \\ & B' & & & D' & \\ & & & & & \end{array}\right]$$

and have the underlying structure of a $(t+1)-(u-1, \lambda+t, \lambda)$ point-weight design. We will therefore have shown that $\mathcal{S}$ has the given structure if we can show that $\lambda' + t - 1 = u - 1$. This is an obvious consequence of the induction hypothesis as the design that is defined by B' in $\mathcal{S}_z$ contains $u-2$ points and is a $(t-1)-(\lambda'+t-2, \lambda+t-1, \lambda)$ design.

Now suppose we have a point-weight incidence structure $\mathcal{S}$ of the form given in the statement of the theorem. Let x be the point of weight λ and let $y_1, \ldots, y_{t+1}$ be points of weight 1. Then

$$\begin{aligned} \iota(\{x, y_1, \ldots, y_t\}) &= 1 = \frac{\lambda}{w(x)w(y_1)\ldots w(y_t)} \\ \iota(\{y_1, \ldots, y_{t+1}\}) &= \lambda = \frac{\lambda}{w(y_1)\ldots w(y_{t+1})} \end{aligned}$$

and so $\mathcal{S}$ is a $\pi_{t+1} - (\lambda+\lambda'+t-1, \lambda+t, \lambda; \{1, \lambda\})$ point-weight design. If we take the points $x, y_1, \ldots, y_{t-1}$ then there are $(\lambda'+t-1)-(t-1)$ ways to add an extra point of weight 1 to that set and each of those sets constitutes one block. Hence

$$\iota(\{x, y_1, \ldots, y_{t-1}\}) = (\lambda'+t-1)-(t-1) = \lambda',$$

whereas

$$\begin{aligned} \iota(\{y_1, \ldots, y_t\}) &= 1 + \lambda \frac{\binom{(\lambda+t-1)-t}{1}}{\binom{(\lambda+t)-t}{1}} \\ &= 1 + (\lambda'-1) \\ &= \lambda' \end{aligned}$$

by noting that there exists one block in A that contains $y_1, \ldots, y_t$ and using (1.2.2) on the $(t+1)-(\lambda'+t-1, \lambda+t, \lambda)$ defined by B.

So $\mathcal{S}$ is a $t-(\lambda+\lambda'+t-1, \lambda+t, \lambda'; \{1, \lambda\})$ point-weight design.

■

The condition that the design $\mathcal{S}=(V, \mathcal{B}, I, w)$ must have $u>t$ is imposed to avoid the trivial design on t points that consists of a single block containing all the points. This structure is obviously a $\pi_t-(v, v, \prod_{x \in V} w(x); W)$ and a $(t-1)-(v, v, 1; W)$ point-weight design for any weight assignment.

3.2 Constructing row-sum point-weight designs

In this chapter we discuss the issues arising from the construction of row-sum point-weight designs. We give three general methods of construction and an example that we will return to in chapter 4.

Lemma 3.2.1 *There exists a nice $\pi_2-(2n-2, n-1, n-2; \{1, n-2\})$ row-sum point-weight design for all $n \geq 4$.*

Proof We start with n points $x_1, ..., x_n$ of weight 1 and one point x_{n+1} of weight $n-2$, so $v=2n-2$. The block set consists of all possible combinations of points that have weight $k=n-1$ i.e. n sets of $n-1$ points of weight 1 and n sets of the form $\{x_i, x_{n+1}\}$ where $i=1, ..., n$.

Now if we take any two points of weight 1 then these points are incident with $n-2$ blocks of the first type and if we take a point of weight 1 and the point x_{n+1} then these two points are incident with just one block. Hence the value $\lambda=n-2$ is correct.

We will now show that this design is nice. By (2.4.5) and (2.4.8) we only need to consider the points of minimal weight and by (2.3.1) we only need to consider one of them. The point x_1 is incident with one block of the form $\{x_1, x_{n+1}\}$ and $n-1$ blocks of the form $\{x_1, \ldots, x_n\} \setminus \{x_j\}$ where $j=2, \ldots, n$. Hence $r_{x_1}=n$ and so $w(x_1)^2 r_{x_1}=n>n-2=\lambda$, which means x_1 is not difficult or awkward and therefore the design is nice (by (2.4.4)).

■

An example of this method of construction is shown in Figure 3.1.

We are also interested in finding specific examples of awkward and difficult designs, hence we will investigate designs that contain a single point of minimal weight or where every point of minimal weight lies on exactly the same blocks.

Lemma 3.2.2 *If $\mathcal{S}$ is a $\pi_2-(v, k, \lambda; W)$ point-weight design with $|W|=2$ and a single point of minimal weight then every block contains that point.*

Proof Let $\mathcal{S}$ be a $\pi_2 - (v, k, \lambda; W)$ point-weight design with $|W|= 2$ and a single point x of minimal weight. $\mathcal{S}$ must have at least two points and the only design with two points is the trivial design consisting of one block. In this case every block trivially contains any point so we may assume that $\mathcal{S}$ has at least three points.

Since $\mathcal{S}$ is a design with $W = \{m, n\}$ then, without loss of generality, we can assume that $m < n$. Hence $\mathcal{S}$ contains a single point of weight m. Suppose there exists a block that doesn't contain the point of minimal weight, then $k = \alpha n = \beta n + m$. Hence n divides m, which is a contradiction as $m < n$.

■

Continuing with the notation of the previous lemma, suppose that x is a point of minimal weight in a $\pi_2 - (v, k, \lambda; W)$ point-weight design and that $\mathcal{S}$ contains α other points. Suppose that each block contains x and β other points and that $W = \{m, n\}$ with $m < n$. It is obvious that $\mathcal{S}_x$ is a $2 - (\alpha, \beta, \frac{\lambda}{n^2})$ and a $1 - (\alpha, \beta, \frac{\lambda}{mn})$ design. Now by the reduction of order formula for classical designs (1.2.2):

$$\begin{aligned} \frac{\lambda}{mn}\binom{\beta-1}{1} &= \frac{\lambda}{n^2}\binom{\alpha-1}{1}. \\ \text{Hence } \frac{\beta-1}{m} &= \frac{\alpha-1}{n}. \end{aligned}$$

From this we can see that m divides $n(\beta - 1)$ and since the $gcd\{n, m\} = 1$ we have that m divides $\beta - 1$. Similarly n divides $\alpha - 1$ and if $\alpha = \gamma n + 1$ then $\beta = \gamma m + 1$.

We may construct a converse to this dissection of difficult point-weight designs with $|W|= 2$ which allows us to form $\pi_2 - (v, k, \lambda; W)$ from a classical design.

Theorem 3.2.3 *If $\mathcal{S}$ is a $2 - (\alpha, \beta, \lambda)$ design with $\alpha > \beta \geq 2$ then we can construct a difficult $\pi_2 - (\alpha(\alpha - 1) + \beta - 1, \alpha\beta - 1, \lambda(\alpha - 1)^2; \{\alpha - 1, \beta - 1\})$ design.*

Proof Suppose $\mathcal{S} = (V, \mathcal{B}, I)$, define a new point-weight structure $(V', \mathcal{B}', I', w')$ by

$$\begin{aligned} V' &= V \cup \{x\}, \\ \mathcal{B}' &= \mathcal{B}, \\ I' &= I \cup \{(x, B) : B \in \mathcal{B}\}, \\ w'(z) &= \begin{cases} \beta - 1 & \text{if } z = x \\ \alpha - 1 & \text{otherwise.} \end{cases} \end{aligned}$$

Hence the sum of the weights of the points is $\alpha(\alpha - 1) + (\beta - 1)$ and the sum of the weights of the points on any one blocks is $\beta(\alpha - 1) + (\beta - 1) = \alpha\beta - 1$.

Take any two points in $V \setminus \{x\}$: these points lie on $\lambda = \frac{\lambda(\alpha-1)^2}{(\alpha-1)^2}$ blocks.

Now take any point $y \in V \setminus \{x\}$ and consider how many blocks contain x and y. This is equal to the number of blocks containing y, which we may calculate by the reduction of order formula for classical designs:

$$\begin{aligned} r_y &= \frac{\lambda\binom{\alpha-1}{1}}{\binom{\beta-1}{1}} \\ &= \lambda\frac{\alpha-1}{\beta-1} \\ &= \frac{\lambda(\alpha-1)^2}{(\alpha-1)(\beta-1)}. \end{aligned}$$

Hence we have constructed a design with the required parameters. It remains to show that x is a difficult point (note that x lies on $\lambda\frac{\binom{\alpha}{2}}{\binom{\beta}{2}}$ blocks):

$$\begin{aligned} w(x)^2 r_x - \lambda(\alpha-1)^2 &= (\beta-1)^2\lambda\frac{\alpha(\alpha-1)}{\beta(\beta-1)} - \lambda(\alpha-1)^2 \\ &= \frac{\lambda(\alpha-1)}{\beta}(\beta-\alpha) \\ &< 0 \text{ as } \beta < \alpha. \end{aligned}$$

$$\left[\begin{array}{cccc} 2 & 2 & 2 & 2 \\ \hline 3 & 3 & 3 & 0 \\ 3 & 3 & 0 & 3 \\ 3 & 0 & 3 & 3 \\ 0 & 3 & 3 & 3 \end{array}\right]$$

Figure 3.4: A $\pi_2 - (14, 11, 18; \{2, 3\})$ derived from a $2 - (4, 3, 2)$.

Hence x is a difficult point and so the design is difficult.

■

Figure 3.4 gives an example of this method of construction.

Notice that the construction as it stands does not guarantee that the weights of the points are co-prime, however we have already shown that it will be equivalent to a design with co-prime weights. In fact we can force the weights to be co-prime by dividing out the common factor of $\alpha - 1$ and $\beta - 1$ when we assign them. We may generalise this construction to use classical designs that have larger values of t.

Theorem 3.2.4 *Suppose $\mathcal{S}$ is a $t - (v, k, \lambda)$ classical design with $t \leq k < v$ then we can construct a $\pi_t - (v_\mathcal{T}, k_\mathcal{T}, \lambda_\mathcal{T}; W)$ point-weight design $\mathcal{T}$ with one special point x such that:*

1. *$|W| = 2$,*
2. *the only point of weight $w(x)$ is x and this weight is minimal,*
3. *the underlying incidence structure of $\mathcal{T}_x$ is $\mathcal{S}$.*

Proof Suppose $\mathcal{S} = (V_\mathcal{S}, \mathcal{B}_\mathcal{S}, I_\mathcal{S})$ and let $\mathcal{T} = (V_\mathcal{T}, \mathcal{B}_\mathcal{T}, I_\mathcal{T}, w_\mathcal{T})$ where:

$$\begin{aligned} V_\mathcal{T} &= V_\mathcal{S} \cup \{x\}, \\ \mathcal{B}_\mathcal{T} &= \mathcal{B}_\mathcal{S}, \\ I_\mathcal{T} &= I_\mathcal{S} \cup \{(x, B) : B \in \mathcal{B}_\mathcal{S}\}. \end{aligned}$$

It only remains to define the weight function. Since $\mathcal{S}$ is a $t - (v, k, \lambda)$ design, it is also a $(t - 1) - (v, k, \lambda')$ design and, since $k < v$, we have that $\lambda' > \lambda$. Let m and n be the unique co-prime integers such that $\frac{n}{m} = \frac{\lambda'}{\lambda}$ and set:

$$w_\mathcal{T}(z) = \begin{cases} m & \text{if } z = x \\ n & \text{otherwise.} \end{cases}$$

Note that $|W|= 2$, that the only point of weight $w(x)$ is x and this point is minimal because $\lambda' > \lambda$. Also note that the underlying incidence structure of $\mathcal{T}_x$ is $\mathcal{S}$. It therefore only remains to show that $\mathcal{T}$ is a $\pi_t - (v_\mathcal{T}, k_\mathcal{T}, \lambda_\mathcal{T}; W)$ point weight design.

Obviously $v_\mathcal{T} = vn + m$ and $k_\mathcal{T} = kn + m$. Set $\lambda_\mathcal{T} = \lambda n^t$. If S is a set of t points of weight n then $\iota(S) = \lambda = \frac{\lambda_\mathcal{T}}{n^t}$ and if S is a set of $t-1$ points of weight n and the point x then $\iota(S) = \lambda' = \frac{\lambda n}{m} = \frac{\lambda_\mathcal{T}}{mn^{t-1}}$. Hence this value of $\lambda_\mathcal{T}$ is correct.

■

At first glance this may seem like a contradiction. If $\mathcal{S}$ was a $3-(v, k, \lambda)$ classical design, then it is also a $2-(v, k, \lambda')$ design and therefore can be used to generate π_3 and π_2 designs. It would seem that this would contradict either (2.2.4), which tell us that if $\mathcal{T}$ is a $\pi_t - (v, k, \lambda; W)$ point-weight design then it cannot be a $\pi_{t-1} - (v, k, \lambda; W)$ point-weight design, or (2.2.6), which tell us that if two $\pi_t - (v, k, \lambda; W)$ point-weight designs have the same underlying incidence structure then they are equivalent. Fortunately neither of these theorems provide a contradiction. In the first case, that of theorem (2.2.4), we are required to have that the $\pi_t - (v, k, \lambda; W)$ and the $\pi_{t-1} - (v, k, \lambda; W)$ point-weight design have the same weight assignment which is not the case with the π_t and π_{t-1} designs generated here. In the second case, that of theorem (2.2.6), we require that both structures are $\pi_t-(v, k, \lambda; W)$ point-weight designs and not one a $\pi_t-(v, k, \lambda; W)$ and one a $\pi_{t-1} - (v, k, \lambda; W)$ design.

As an example consider the classical design on five points with the blocks being all possible sets of four points. This is simultaneously a $3-(5, 4, 2)$, a $2-(5, 4, 3)$ and a $1-(5, 4, 4)$ design and as such it gives rise to both a $\pi_3-(17, 14, 54; \{2, 3\})$ point-weight design with one point of weight 2 and five points of weight 3 and a $\pi_2 - (23, 19, 48; \{3, 4\})$ point-weight design with one point of weight 3 and five points of weight 4.

We now return to the discussion of difficult $\pi_2 - (v, k, \lambda; W)$ point-weight designs that motivated this construction.

Lemma 3.2.5 *Any $\pi_2 - (v, k, \lambda; W)$ with $|W|= 2$ and a single point of minimal weight is difficult.*

Proof If $\mathcal{S}$ is a $\pi_2-(v, k, \lambda; W)$ with $|W|= 2$ and a single point x of minimal weight then by (3.2.2) we have that every block contains that point. This means that the underlying structure of $\mathcal{S}_x$ is a $2-(u, k', \lambda)$ design. We may then augment this classical design in the manner of (3.2.3) to give a difficult $\pi_2-(v^*, k^*, \lambda; W)$ point-weight design. However this design is equivalent to $\mathcal{S}$ because the both have the same underlying incidence structure (see (2.2.6)). Therefore $\mathcal{S}$ is difficult.

■

Corollary 3.2.6 *There exists no nice, awkward $\pi_2 - (v, k, \lambda; W)$ point-weight designs with $|W|= 2$.*

Proof By (2.4.11) we know that if $\mathcal{S}$ is a nice, awkward $\pi_2 - (v,k,\lambda;W)$ point-weight design then it has a single awkward point which must be of minimal weight. However the above corollary gives us that that point must be difficult, which is a contradiction as no point can be both awkward and difficult (see (2.4.10).

■

We will examine the matrix structure of awkward $\pi_2-(v,k,\lambda;W)$ point-weight designs with $|W|=2$. However we will need the following lemma.

Lemma 3.2.7 *There exists no $\pi_2-(v,k,\lambda;W)$ point-weight design $\mathcal{S}$ with $v>k$, $|W|=2$ and a set of points S with $|S|>1$ such that every block contains S.*

Proof We know, by (2.4.6) and the fact that $|S|>1$, that S contains only awkward points. Without loss of generality we may assume that S is the entire set of awkward points, then every point that does not lie in S is of a greater weight than the points of S, by (2.4.5).

Suppose that $V \setminus S$ contains only one point y. Then there exists at least one block that contains S and y, and that block has weight $k=\sigma(S)+w(y)=v$. Hence $k=v$ which is a contradiction. So we know that $V \setminus S$ contains at least two points y_1 and y_2 and that S contains at least two points x_1 and x_2.

Now
$$\frac{\iota(\{y_1,y_2\})}{\iota(\{x_1,y_1\})}=\frac{\lambda w(y_1)w(x_1)}{\lambda w(y_1)w(y_2)}=\frac{w(x_1)}{w(y_2)}$$
and similarly
$$\frac{\iota(\{x_1,y_1\})}{\iota(\{x_1,x_2\})}=\frac{\lambda w(x_1)w(x_2)}{\lambda w(x_1)w(y_1)}=\frac{w(x_2)}{w(y_1)}=\frac{w(x_1)}{w(y_2)}.$$

However $\mathcal{S}_S$ has the same underlying incidence structure as a $2-(\alpha,\beta,\lambda')$ design with b blocks. Hence

$$\begin{array}{rcccl}\iota(\{y_1,y_2\}) &=& \lambda' \,, && \\ \iota(\{x_1,y_1\}) &=& r_{y_1} &=& \lambda'\frac{\alpha-1}{\beta-1}\,, \\ \iota(\{x_1,x_2\}) &=& b &=& \lambda'\frac{\alpha(\alpha-1)}{\beta(\beta-1)}.\end{array}$$

Therefore

$$\begin{aligned}\frac{\beta-1}{\alpha-1} &= \frac{\iota(\{y_1,y_2\})}{\iota(\{x_1,y_1\})}\\ &= \frac{\iota(\{x_1,y_1\})}{\iota(\{x_1,x_2\})}\\ &= \frac{\beta}{\alpha}.\end{aligned}$$

So $k=v$, which is the required contradiction.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 3 & 3 & 0 & 0 & 3 & 0 \\ 3 & 3 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 3 \\ 0 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 3 & 3 & 0 \\ 0 & 0 & 3 & 3 & 0 & 0 & 3 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 3 & 3 & 0 & 0 & 3 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 & 3 & 0 & 0 & 3 & 0 & 3 \end{bmatrix}$$

Figure 3.5: A $\pi_2 - (21, 9, 9; \{1, 3\})$ awkward design.

■

So if $\mathcal{S}$ is an awkward $\pi_2 - (v, k, \lambda; W)$ with $|W|= 2$ then it must have an incidence matrix of the form:

$$\left[\begin{array}{c|c} mJ & 0 \\ \hline nM & nN \end{array}\right]$$

(by (2.4.5) with $m < n$) and must satisfy the following rules:

1. there must exist $\tau > 1$ points of weight m (by (3.2.6)).
2. M is a $1-(\beta, k_1, r)$ design (by considering $\mathcal{S}_X$ where X is the set of awkward points). If $\mathcal{S}$ contains u points then $\beta = u - \tau$ and $\tau m + nk_1 = k$.
3. N consists of blocks of size k_2 where $nk_2 = k$.
4. $\left[\begin{array}{c|c} M & N \end{array}\right]$ is a $2-(\beta, \{k_1, k_2\}, \frac{\lambda}{n^2})$ incidence structure (obtained by considering the incidence structure given by removing all the awkward points).
5. $k_1 < k_2$ as $k = nk_2 = nk_1 + \tau m$.
6. $nmr = \lambda$ where r is defined in 2. This result is obtained by noting that if x and y are points of different weights then $w(x)w(y)\iota(\{x, y\}) = \lambda$.

Figure 3.5 gives an example of an awkward design with $|W|= 2$.

If we were to look for awkward and difficult point-weight designs then it is helpful to know some bounds for the parameters. The following lemmas show some bounds on the block size of such structures.

Lemma 3.2.8 *If $\mathcal{S}$ is an awkward $\pi_2 - (v, k, \lambda; W)$ point-weight design with u points and an awkward point x then $uw(x) = k$.*

Proof Since x is awkward we know that if x is incident with r_x blocks then $w(x)^2 r_x = \lambda$ and we have an expression for r_x given by (3.1.2) so:

$$\begin{aligned}\lambda &= w(x)^2 \frac{\lambda(u-1)}{w(x)(k-w(x))} \\ &= \lambda \frac{w(x)(u-1)}{k-w(x)}\end{aligned}$$

and so

$$k - w(x) = (u-1)w(x)$$

which gives the result.

■

Lemma 3.2.9 *If $\mathcal{S}$ is a difficult $\pi_2-(v,k,\lambda;W)$ point-weight design with u points and a difficult point x then $uw(x) < k$.*

Proof Since x is difficult we know that if x is incident with r_x blocks then $w(x)^2 r_x < \lambda$ and we have an expression for r_x given by (3.1.2) so:

$$\begin{aligned}\lambda &> w(x)^2 \frac{\lambda(u-1)}{w(x)(k-w(x))} \\ &> \lambda \frac{w(x)(u-1)}{k-w(x)}\end{aligned}$$

and so

$$k - w(x) > (u-1)w(x)$$

which gives the result.

■

So we know that if $\mathcal{S}$ is an awkward or difficult $\pi_2 - (v,k,\lambda;W)$ point-weight design then it has a fairly large block size. Furthermore we have exhibited some examples where $|W| = 2$. It is harder to find examples when $|W| > 2$, the following results show the non-existence of such designs when k is very small.

Lemma 3.2.10 *If $\mathcal{S}$ is a $\pi_2 - (v,k,\lambda;W)$ point-weight design with $|W| \geq 3$ and a single point of minimal weight then $k \geq 8$.*

Proof We deal with each possible weight set and value for λ in turn. Obviously the smallest possible weight set is $\{1,2,3\}$ so $k > 4$ as there must exist a block containing a point of weight 2 and 3.

Case 1: $k = 5$

Obviously in this case we must have $W = \{1, 2, 3\}$ as any other weight values would mean that two of the higher weighted points could not exist on the same block. Furthermore there must exist a single point of weight 3 because two points of weight 3 could not exist on the same block either. Every point of weight 2 is linked to the point of weight 3 by a single block containing exactly those two points. Hence $\lambda = 6$, which implies that there exists only one point of weight 2 (or else 4 would have to divide λ).

Consequently there exists no design with $k = 5$ as there can exist no block that contains the point of weight 1 and the point of weight 2.

Case 2: $k = 6$

There are two possibilities for the weight set: $\{1, 2, 3\}$ and $\{1, 2, 4\}$. However in the latter case, since there exists only a single point of weight 1, there can exist no block that contains that point. Hence if there exists a design with $k = 6$ then it must have a weight set of $\{1, 2, 3\}$.

The possible distributions of the weights of points on each block are:

- One point of weight one, one point of weight two and one point of weight 3.
- Three points of weight two.
- Two points of weight three.

So, if there exists two or more points of weight 3 then there exists only one block connecting any two of those points, hence $\lambda = 9$ which is a contradiction as 2 must divide λ by (2.3.4). Hence there exists a single point of weight 3.

So any block containing the single point of weight 1 also contains the single point of weight 3 and a point of weight 2. Hence there exists at most one block connecting the point of weight 1 and any point of weight 2. So $\lambda = 2$ which is a contradiction as 3 divides λ by (2.3.4).

Hence there exists no design with $k = 6$.

Case 3a: $k = 7$ *and* $W = \{1, 2, 4\}$

The possible distributions of the weights of points on each block are:

- One point of weight one, one point of weight two and one point of weight four.
- One point of weight one and three points of weight two.

There can exist at most one point of weight 4 as no block could contain two of them. Since there exists only one block that connects each point of weight 2 with the point of weight 4 we must have $\lambda = 8$.

This means that there exists only 2 blocks that contain the point of weight 1 and the point of weight 4, and so there exists only two points of weight 2. This is a contradiction as two points of weight 2 could never lie on a block without a third point of weight 2 existing.

Case 3b: $k = 7$ and $W = \{1, 2, 3\}$

The possible distributions of the weights of points on each block are:

- One point of weight one and two points of weight three.
- One point of weight one and three points of weight two.
- Two points of weight two and one point of weight three.

Since there must exist blocks connecting points of weight 1 and 3, and blocks connecting points of weight 1 and 2 we must have more than one point of weight 2 and more than one point of weight 3.

Hence 36 divides λ.

In particular this means that there exists at least 4 blocks connecting any two points of weight 3. However these blocks must consist of the two points of weight 3 and the single point of weight 1, which is a contradiction as we may not have two blocks that contain exactly the same points.

Case 3c: $k = 7$ and $W = \{1, 3, 4\}$

There exist no block that could contain the single point of weight 1 and a point of weight 4, which is a contradiction.

Case 3d: $k = 7$ and $W = \{1, 2, 3, 4\}$

The possible distribution of the weights of the points on each block are:

- One point of weight 4 and one point of weight 3.
- One point of weight 4, one point of weight 2 and one point of weight 1.
- Two points of weight 3 and one point of weight 1.
- One point of weight 3 and two points of weight 2.
- Three points of weight two and one point of weight 1.

There exists at most one point of weight 4 as no block could contains two of them. Now there is only one block that could contain the point of weight 4 and any one point of weight 3, hence $\lambda = 12$. This means there is at most one point of weight 3 (as if there exists two points of weight 3 then 9 would have to divide λ by (2.3.4)). Hence there exists no block that contains the point of weight 3 and the point of weight 1.

Case 3e: $k = 7$ and $W = \{1, 2, 5\}$

There can exist no block that contains the single point of weight 1 and the (necessarily) single point of weight 5, which is a contradiction.

■

Lemma 3.2.11 *If $\mathcal{S}$ is a difficult $\pi_2-(v,k,\lambda;W)$ point-weight design with $|W|>2$ then $k \geq 9$.*

Proof Since a difficult design must contain a single point of minimum weight we know from the previous lemma that $k \geq 8$. Hence it only remains to deal with the case $k = 8$. If W contains a point of weight 7 or more than obviously the two highest weighted points could not exist on the same block, which is a contradiction. So we assume that W contains at no point of weight 7 or higher and the sum of the weights of the two highest points is at most 8. The possible weight sets are:

$$\begin{array}{cccc} \{6,2,1\} & \{5,3,2,1\} & \{5,3,2\} & \{5,3,1\} \\ \{5,2,1\} & \{4,3,2,1\} & \{4,3,2\} & \{4,3,1\} \\ \{4,2,1\} & \{3,2,1\} & & \end{array}$$

However we can exclude the following weight sets as there could not exist a block that contains two points of a given weight:

- $W = \{6,2,1\}$ as there cannot exist a block that contains a point of weight 6 and a point of weight 1.
- $W = \{5,3,2\}$ as there cannot exist a block that contains a point of weight 5 and a point of weight 2.
- $W = \{5,3,1\}$ as there cannot exist a block that contains a point of weight 5 and a point of weight 1.
- $W = \{4,3,2\}$ as there cannot exist a block that contains a point of weight 4 and a point of weight 3.
- $W = \{4,2,1\}$ as there cannot exist a block that contains the point of weight 1.

Let us consider the remaining weight sets:

Case 1: $W = \{1,3,4\}$

The possible distributions of the weights of points on each block are:

- One point of weight one, one point of weight three and one point of weight four.
- Two points of weight four.

Hence there can exist only one point of weight 3. For any point of weight 4 there exists precisely one block that contains it and the point of weight 3 (hence $\lambda = 12$), and precisely one block that contains it and the point of weight 1 (hence $\lambda = 4$). This gives the required contradiction.

Case 2a: $W = \{1, 2, 3\}$ *and there exists a single point of weight 3*

The possible distributions of the weights of points on each block are:

- One point of weight one, two points of weight two and one point of weight three.
- Four points of weight two.

Take any point of weight 2, there should be three times as many blocks that contain that point of weight 2 and the point of weight 1 than contain that point of weight 2 and the point of weight 3. However every block that contains that point of weight 2 and the point of weight 1 also contains the point of weight 3, which is the required contradiction.

Case 2b: $W = \{1, 2, 3\}$ *and there exists more than one point of weight 3*

The possible distributions of the weights of points on each block are:

- One point of weight one, two points of weight two and one point of weight three.
- Four points of weight two.
- One point of weight two and two points of weight three.

As there must exist a block that contains the point of weight 1, there exists more than one point of weight 2. Hence 36 divides λ, let $\lambda = 36\alpha$.

Take two points z_1, z_2 of weight 3, a point y of weight 2 and let x be the unique point of weight 1. There exits 18α blocks that contain the points x and y. These are all blocks of the form $\{x, y, y', z\}$ for some point y' of weight 2 and z of weight 3. Of these blocks there can exist at most 6α that contain the point z_1 and 6α that contain the point z_2 because 6α is the number of blocks that contain both y and z_i. Hence there exists at least 3 points of weight 3.

However, as x is a difficult point, the number of blocks on which x lies is less than λ. However x lies on 12α distinct blocks for every point of weight 3, hence x lies on at least $3 \cdot 12\alpha = 36\alpha = \lambda$ blocks. This is the required contradiction.

*C*ase 3: $W = \{1, 2, 5\}$

The possible distributions of the weights of points on each block are:

- One point of weight 1, one point of weight 2 and one point of weight 5.

- Four points of weight 2.

Since there exists only one block connecting the unique block of weight 5 to any point of weight two we have that $\lambda = 10$. Conversely, since there exists only one block connecting the unique block of weight 1 to any point of weight 2 we have that $\lambda = 2$. This is the required contradiction.

Case 4a: $W = \{1, 2, 3, 4\}$ *and there exists more than one point of weight 4*
The possible distributions of the weights of points on each block are:

- Two points of weight 4.
- One point of weight 4, one point of weight 3 and one point of weight 1.
- One point of weight 4 and two points of weight 2.
- Two points of weight 3 and one point of weight 2.
- One point of weight 3, two points of weight 2 and one point of weight 1.
- Four points of weight 2.

Since there exists only one block that can contain two points of weight 4 we know that $\lambda = 16$. However since there exists only one block that can contain any one point of weight 4 and any one point of weight 3 we must have that $\lambda = 12$ which is the required contradiction.

Case 4b: $W = \{1, 2, 3, 4\}$ *and there exists a single point of weight 4*
In this case we must again have that $\lambda = 12$ because there can exist only one block that contains the point of weight 4 and any one point of weight 3. So there must exist 3 blocks that contain the single point of weight 4 and the single point of weight 1, hence there must exist three points of weight 3. Now, by (2.3.4), we have that 9 divides λ which is a contradiction.

Case 5: $W = \{1, 2, 3, 5\}$
The possible distributions of the weights of points on each block are:

- One point of weight 5 and one point of weight 3.
- One point of weight 5, one point of weight 2 and one point of weight 1.
- Two points of weight 3 and one point of weight 2.
- One point of weight 3, two points of weight 2 and one point of weight 1.
- Four points of weight 2.

This means there can exist only one point of weight 5 as there could exist no block that could contain two of them. Since there exists only one block that could contain the single point of weight 5 and any one point of weight 3 we must have that $\lambda = 15$ but, by (2.3.4), we have that 30 divides λ which is a contradiction.

$\blacksquare$

3.3 Comparing row-sum and point-sum point-weight designs

In this section we examine point-weight incidence structures that are simultaneously point-sum and row-sum point-weight designs. In particular, we investigate the values of t and s for which a point-weight incidence structure $\mathcal{S}$ might be both a $\pi_t - (v, k, \lambda; W)$ and a $s - (v, k, \lambda'; W)$ point-weight design.

Lemma 3.3.1 *If $\mathcal{S}$ is a $\pi_t - (v, k, \lambda; W)$ and a $t - (v, k, \lambda'; W)$ point-weight design with more than t points then $|W| = 1$.*

Proof Consider any two points x and y of $\mathcal{S}$ and let S be a set of $t-1$ points in $V \setminus \{x, y\}$. Since $\mathcal{S}$ is a $t - (v, k, \lambda'; W)$ we have that:

$$\iota(S \cup \{x\}) = \iota(S \cup \{y\}) = \lambda'$$

but $\mathcal{S}$ is also a $\pi_t - (v, k, \lambda; W)$, so:

$$\iota(S \cup \{x\})w(x) \prod_{z \in S} w(z) = \iota(S \cup \{y\})w(y) \prod_{z \in S} w(z) = \lambda.$$

Hence $w(x) = w(y)$ and therefore $|W| = 1$.

$\blacksquare$

We now show that it is impossible to have a point-weight incidence structure $\mathcal{S}$ that is both a $\pi_t - (v, k, \lambda; W)$ and a $(t+1) - (v, k, \lambda'; W)$. We start with the simplest cases.

Lemma 3.3.2 *If $\mathcal{S}$ is a $\pi_2 - (v, k, \lambda; W)$ point-weight design and a $3 - (v, k, \lambda'; W)$ then $|W| = 1$.*

Proof Suppose that $\mathcal{S}$ is a $\pi_2 - (v, k, \lambda; W)$ and a $3 - (v, k, \lambda'; W)$ point-weight design and that $|W| > 1$. Take any $x \in V$ then $\mathcal{S}_x$ is:

1. a $\pi_1 - (v - w(x), k - w(x), \frac{\lambda}{w(x)}; W')$,

2. a $2-(v-w(x), k-w(x), \lambda'; W')$,

where $W' = W$ or $W' = W \setminus \{w(x)\}$ if there exists only a single point with weight $w(x)$. Note that $\frac{\lambda}{w(x)}$ is an integer by (2.3.4).

Therefore for all $y \in V \setminus \{x\}$ we have that, in $\mathcal{S}_x$, the point y is incident with the following number of blocks:

1. $r_y = \frac{\lambda}{w(x)}\frac{1}{w(y)}$ and
2. $r_y = \lambda' \frac{(v-w(x))-w(y)}{(k-w(x))-w(y)}$ by (1.7.4).

Hence $w(y)$ satisfies the equation:

$$X^2 + \frac{\lambda' w(x)(w(x)-v)-\lambda}{\lambda' w(x)}X + \frac{\lambda(k-w(x))}{\lambda' w(x)} = 0 \tag{3.1}$$

which is independent of y and $w(y)$. So $w(y)$ can take at most two values.

So now there exists three possibilities for the structure of $\mathcal{S}$:

1. $|W|=3$ and so there exists no other point of weight $w(x)$ in $\mathcal{S}$ and the above equation is of the form $(X-w(y))(X-w(z))=0$ for $y, z \in V \setminus \{x\}$ and $w(y) \neq w(z)$.
2. $|W|=2$ and there exists more than one point of weight $w(x)$ in $\mathcal{S}$. Therefore the above equation is of the form $(X-w(x))(X-w(y))=0$ for some $y \in V$ with $w(y) \neq w(x)$.
3. $|W|=2$ and there exists only one point of weight $w(x)$ in $\mathcal{S}$. This implies that $\mathcal{S}_x$ has the underlying structure of a classical design.

We will consider these possibilities in turn:

Case 1: $|W|=3$

Hence there exists a single point $x \in V$ of weight $w(x)$.

Suppose $W = \{w(x), w(y), w(z)\}$ and consider $\mathcal{S}_y$. The above reasoning holds for $\mathcal{S}_y$ as well as $\mathcal{S}_x$ so from this we may classify $\mathcal{S}$ according to the above possibilities. However since we know that $|W|=3$ we must have that $\mathcal{S}$ is of type 1 and that there exists a single point of weight $w(y)$. Similarly there exists only a single point of weight $w(z)$.

Therefore $V = \{x, y, z\}$ and there exists no $\pi_2-(v,k,\lambda;W)$ point-weight design with this point set and weight set.

Case 2: $|W|=2$ *and there exist more than one point of weight* $w(x)$

Suppose $W = \{w(x), w(y)\}$. If there exists precisely one point of weight $w(y)$ then examining $\mathcal{S}_y$ will lead to Case 3 so we may assume that there exists more than one point of weight $w(y)$ too. Now by examining $\mathcal{S}_y$ and $\mathcal{S}_x$ we find that equation 3.1 will be the same in both cases: $(X - w(x))(X - w(y))$. Hence, from the constant term:

$$\begin{aligned}\frac{\lambda(k-w(x))}{\lambda' w(x)} &= \frac{\lambda(k-w(y))}{\lambda' w(y)} \\ \Leftrightarrow kw(y) - w(x)w(y) &= kw(y) - w(x)w(y) \\ \Leftrightarrow w(x) &= w(y)\end{aligned}$$

which is the required contradiction.

Case 3: $|W| = 2$ and there exists a single point of weight $w(x)$ in $\mathcal{S}$

In this case we have that there exists a single point of weight $w(x)$ in $\mathcal{S}$ and so the underlying structure of $\mathcal{S}_x$ a $2-(\alpha,\beta,\lambda')$ design as all the points of $\mathcal{S}_x$ are of the same weight. We also know that equation 3.1 has a factor of $X - w(y)$ but this is not very useful, instead we will concentrate and the very specific structure $\mathcal{S}$ must have.

Firstly suppose that there exists a block B that does not contain the point x and let y be any other point, then we have that $k = \beta w(y) + w(x) = \gamma w(y)$. Hence $w(y) \mid w(x)$ and so $w(y) = 1$ by the primality of W. Now consider the structure given by $\mathcal{S}_y$, in this case equation 3.1 will be of the form:

$$X^2 + \frac{\lambda'(1-v)-\lambda}{\lambda'}X + \frac{\lambda(k-1)}{\lambda'} = 0$$

which can also be written as

$$(X-1)(X-w(x)) = 0$$

as the points of weight $w(x)$ and 1 must satisfy this equation. So $w(x) = \frac{\lambda(k-1)}{\lambda'}$. Therefore, in $\mathcal{S}$, we have that

$$\iota(\{x,y\}) = \frac{\lambda'}{k-1} < \lambda'$$

as $k > 2$ because $2 < w(x) + w(y) \leq k$. This is a contradiction because any set of three point must lie on exactly λ' blocks, so any set of two points must lie on at least λ' blocks.

We will therefore assume that every block contains the point x. This means that $\mathcal{S}_x$ is a $3-(\alpha,\beta,\lambda')$ design because we haven't removed any blocks from $\mathcal{S}$. However we already know, from the definition of $\mathcal{S}_x$, that any two points also lie on λ' blocks, therefore $\alpha = \beta$ by (1.2.2). This means that there is only one block and this block contains every point.

Let x, y and z be points such that x and y are of different weights. Each pair of points lies on exactly 1 block so

$$\frac{\lambda}{w(x)w(z)} = 1 = \frac{\lambda}{w(y)w(z)}$$

and so $w(x) = w(y)$ which is the required contradiction.

Hence if $\mathcal{S}$ is simultaneously a $\pi_2 - (v, k, \lambda; W)$ and a $3 - (v, k, \lambda'; W)$ point-weight design then $|W| = 1$.

■

We may extend this to all possible values of $t \geq 2$.

Theorem 3.3.3 *If $\mathcal{S}$ is a $\pi_t - (v, k, \lambda; W)$ and a $(t+1) - (v, k, \lambda'; W)$ point-weight design for some $t \geq 2$ then $|W| = 1$.*

Proof We will use induction on the value of t. Obviously we have shown that the theorem is true if $t = 2$ in (3.3.2) so assume that the theorem is true whenever $\mathcal{S}$ is a $\pi_s - (v, k, \lambda; W)$ and a $(s + 1) - (v, k, \lambda'; W)$ point-weight design with $2 \leq s < t$.

Suppose $\mathcal{S}$ is a $\pi_t - (v, k, \lambda; W)$ and a $(t+1) - (v, k, \lambda'; W)$ point-weight design with $|W| \geq 2$. Suppose we pick a point x of $\mathcal{S}$ such that there exists more than one point of weight x then $\mathcal{S}_x$ is a $\pi_{t-1} - (v - w(x), k - w(x), \frac{\lambda}{w(x)}; W)$ and a $t - (v - w(x), k - w(x), \lambda'; W)$ point-weight design with $|W| \geq 2$. However this contradicts the induction hypothesis so we must assume that no such point exists, i.e. that $\mathcal{S}$ consists only of single points of each weight.

Pick any point x of $\mathcal{S}$ and $\mathcal{S}_x$ is a $\pi_t - (v - w(x), k - w(x), \frac{\lambda}{w(x)}; W')$ and a $t - (v - w(x), k - w(x), \lambda'; W')$ point-weight design where $W' = W \setminus \{w(x)\}$. If this is not to contradict the induction hypothesis then we must have $|W'| = 1$ but then $1 = |W'| = |W| - 1 = u - 1$ as each point is of a different weight. Hence $u = 2$ which is a contradiction as $\mathcal{S}$ is a $(t + 1) - (v, k, \lambda'; W)$ design for some $t \geq 2$.

■

I would conjecture that if $\mathcal{S}$ is a $t - (v, k, \lambda'; W)$ point-weight design for some $t \geq 3$ with $|W| \geq 2$ then $\mathcal{S}$ cannot be a $\pi_s - (v, k, \lambda; W)$ point-weight design for any $1 < s \leq t$. This comes down to proving that if $\mathcal{S}$ is a $t - (v, k, \lambda'; W)$ point-weight design with $|W| \geq 2$ then $\mathcal{S}$ is not a $\pi_2 - (v, k, \lambda; W)$ point-weight design. However this conjecture is as yet unproven.

3.4 Comparing row-sum point-weight designs to experimental designs

We have examined row-sum point-weight designs in the context of combinatorial design theory, however design theory is also used in the design and analysis of experiments. A good introduction to the theory of experimental designs can be found in [19]. In the theory of experimental designs a set of plots are subjected to treatments one of v treatments $\tau_1, ..., \tau_v$. Such plots are also partitioned into blocks. The incidence matrix then becomes a display of which treatments are being used on plots that are grouped in different blocks. The use of design theory allows a statistician to estimate certain properties of the treatments and the blocks.

Although in most experiments a classical design (termed a "balanced incomplete block design" or BIBD) is used, the general definition doesn't restrict the choice of treatment or blocks for each plot very much. It is perfectly possible for a treatment to be applied to plots that lie on the same block and hence, for the incidence matrix to contain integer values greater than 1. Informally an experimental design whose incidence matrix contains entries greater than one is called an n-ary design. However the situation is more confusing when one attempts to use a formal definition.

According to Tocher [20] an n-ary design is a design for which the incidence matrix only contains entries from the set $\{0, 1, \ldots, n-1\}$. Presumably we require that there exists an entry of value $n-1$. Some papers, including [18], suggest that every value of $\{0, 1, \ldots, n-1\}$ must appear somewhere in the incidence matrix. However, in this thesis we shall use the definitions given in [8].

Definition 3.4.1 *A block design is an allocation of treatments $\tau_1, \ldots, \tau_v$ onto a finite set of plots such that every plot is associated with exactly one treatment, and an allocation of plots into blocks such that every plot is associated with exactly one block. A block design is said to be proper if every block is associated with the same number of plots and is said to be equireplicate if every treatment is associated with the same number of plots.*

A block design with an incidence matrix M is said to be an n-ary block design if the entries of M constitutes n distinct integers.

A block design is said to be pairwise balanced if $MM^T = D + \lambda J$, where D is a diagonal matrix, λ is a scalar and J is the matrix whose every entry is 1.

Under these definitions it is easy to see that a point-weight incidence structure with weight set W is a $(|W| + 1)$-ary block design with the property that if a treatment τ_i appears on any two blocks then it appears the same number of times on each of those blocks. It is also not difficult to see that a point-weight incidence structure that satisfies the constant block size condition is a proper $(|W| + 1)$-ary block design and that a row-sum point-weight design is a proper, pairwise balanced $(|W| + 1)$-ary block design.

It is also worth noting that the general theory of experimental designs allows two blocks to contain equal numbers of each treatment where as we have specifically excluded the possibility of repeated blocks.

We can also show that a row-sum point-weight design is variance balanced, see [8] and [14].

Definition 3.4.2 *Suppose that M is the incidence matrix of a block design. Let r_i be the number of plots to which the treatment τ_i is applied, $R = diag(r_1, \ldots, r_v)$,*

k_j be the size of the j-th block and $K = diag(k_1, \ldots, k_b)$. A block design is variance balanced if the off diagonal elements of the coefficient matrix $C = R - MK^{-1}M^T$ are equal.

Since any row-sum point-weight design is a proper, pairwise balanced block design we have that, in the notation above,

$$\begin{aligned} C &= R - MK^{-1}M^T \\ &= R - M(\frac{1}{k}I)M^T \\ &= R - \frac{1}{k}MM^T \\ &= R - \frac{1}{k}(D + \lambda J) \\ &= \frac{1}{k}(R - D) - \frac{\lambda}{k}J, \end{aligned}$$

and clearly the off diagonal elements of C are equal. However it is also fairly trivial to see that any proper, variance balanced block design is pairwise balanced. So any proper structure is pairwise balanced if and only if it is variance balanced.

3.5 Conclusion

In this chapter we gave examples and constructions for nice, awkward and difficult row-sum point-weight designs. All of these examples have a weight set of size two and the question of the existence of a row-sum point-weight design with a weight set of size three or greater is still open.

We have also completely characterised the regular $\pi_2 - (v, k, \lambda; W)$ as being those designs with $|W| = 2$ that are constructed in a certain way. Similarly we have also shown that the only point-weight incidence structures that are both $\pi_{t+1} - (v, k, \lambda; W)$ and $t - (v, k, \lambda'; W)$ point-weight designs with $t \geq 2$ also have $|W| = 2$ and are of a similar structure. To the contrary we have shown that there can exist no point-weight incidence structure that is both a $\pi_t - (v, k, \lambda; W)$ and either a $t - (v, k, \lambda'; W)$ or a $(t + 1) - (v, k, \lambda'; W)$. We know from [9] that if $\mathcal{S}$ is a point-weight incidence structure then there can exist at most one value t for which $\mathcal{S}$ is a $t - (v, k, \lambda'; W)$ and we conjectured that if $\mathcal{S}$ is a $t - (v, k, \lambda'; W)$ point-weight design then it is not a $\pi_s - (v, k, \lambda; W)$ for any $1 < s \leq t$.

We also investigated the connection between row-sum point-weight designs and experimental designs. Row-sum point-weight designs are specific examples of proper, pairwise balanced n-ary designs, and as such are variance balanced. It had been hoped that this connection would lead to further results and constructions, however this has not been the case. This is due in part to the aesthetic restriction of repeated blocks.

We have not discussed the relationship between row-sum and weight-sum point-weight designs. It is harder to investigate this relationship as the weight-sum point-weight design might not be given any information about the number of blocks a set of t points lies upon. There are a few examples of structures that are both $\pi_t - (v, k, \lambda; W)$ and $\sigma_s - (v, k, \lambda'; W)$ designs, for example the design given in Figure 3.4 is rather trivially also a $\sigma_5 - (14, 11, 3; \{2, 3\})$ and a $\sigma_{11} - (14, 11, 1; \{2, 3\})$ point-weight design. However the general theory is still an open problem.

Chapter 4

Groups and point-weight designs

In this chapter we examine the relationships between groups and point-weight designs including the automorphism group and the groups that can be used to generate a design from base blocks.

4.1 Automorphism groups

Horne [9] has already defined the automorphism group of a point-weight incidence structure as an automorphism of the underlying structure that preserves the weights of points. Thus any automorphism of a point-weight incidence structure is an automorphism of its underlying incidence structure $\mathcal{U}$, i.e.

$$Aut\ \mathcal{S} \leq Aut\ \mathcal{U}.$$

However it is not certain whether any automorphism of the underlying structure would necessarily preserve the weights of the points (although it does preserve the number of blocks a point is incident with).

It has already been shown that if $\mathcal{S}$ is a point-sum point-weight design then $Aut\ \mathcal{S} = Aut\ \mathcal{U}$ and hence any automorphism of the underlying structure preserves the weights of the points. We will show that the same holds for row-sum point-weight designs.

Lemma 4.1.1 *If σ is an automorphism of a point-weight incidence structure $\mathcal{S}$ then σ is completely defined by its effect on V if and only if $\mathcal{S}$ has no repeated blocks.*

Proof It is well known (see [3]) that for all $\tau \in Aut\,\mathcal{U}$ the effect of τ is completely defined by its effect on V if and only if $\mathcal{U}$ has no repeated blocks. Since $\sigma \in Aut\ \mathcal{S} \leq Aut\ \mathcal{U}$ we have that this applies to σ too as $\sigma \in Aut\ \mathcal{U}$.

■

The following theorem is due to Peter Cameron, [6].

Result 4.1.2 *If $\mathcal{S}$ is a $\pi_t - (v, k, \lambda; W)$ row-sum point-weight designs with more than t points and an underlying incidence structure $\mathcal{U}$ then $Aut\ \mathcal{S} = Aut\ \mathcal{U}$.*

Proof Let $\mathcal{S}$ be a $\pi_t - (v, k, \lambda; W)$ point-weight design with more than t points and let $\mathcal{U}$ be the underlying incidence structure of $\mathcal{S}$. We may assume, without loss of generality, that $W = \{w_1, w_2, \ldots\}$ where $w_1 < w_2 < \ldots$, and then define $U_i = \{x \in V : w(x) = w_i\}$ and $u_i = |U_i|$. Let j be the value for which

$$\sum_{i=1}^{j} u_i \leq t \text{ and } \sum_{i=1}^{j+1} u_i > t.$$

Let $U = \bigcup_{i=1}^{j} U_i$ and $m = |U|$. If S is a set of t points then $\iota(S)$ is maximal if and only if S is of the form $U \cup X$, where X is a set of $t - m$ points of U_{j+1}.

Note that any automorphism of $\mathcal{U}$ preserves the value of ι.

We consider two cases. If $m < t$ then the sets X above are non-empty and their union is U_{j+1}. So if we apply automorphism of the underlying structure to S then in order to preserve the value of $\iota(S)$ we must have that U is mapped to itself and X is mapped to another set $X' \subseteq U_{j+1}$. Hence any automorphism of the underlying structure maps points of weight w_{j+1} to other points of weight w_{j+1}.

Now consider a point $x \in U_i$, where $i > j + 1$, and the set $S = U \cup X \cup \{x\}$, where X is a set of $t - m - 1$ points of U_{j+1}. If we apply any automorphism of $\mathcal{U}$ to S then we have already shown that U is mapped to U, X is mapped to a set $X' \subseteq U_{j+1}$ and so, in order to preserve the value of $\iota(S)$, we must have that x is mapped to another point of the same weight. This means that any automorphism maps points of weight at least w_{j+1} to other points of the same weight.

Lastly if we pick a point $x \in U_i$, where $i < j + 1$, and form the set $S = (U \cup X) \setminus \{x\}$, where X is a set of $t - m + 1$ points of weight greater than w_j. We have already shown that, under the action of any automorphism of the underlying group, every point of X is mapped to a point of the same weight. Again, in order to preserve $\iota(S)$ we must have that x is mapped to a point of the same weight. Hence every point of $\mathcal{S}$ is mapped to a point of the same weight under the action of $Aut\ \mathcal{U}$.

Now, for the second case, suppose that $t = m$. In this case we have that $S = U$ is the unique subset of size t such that $\iota(S)$ is maximal, hence for any $g \in Aut\ \mathcal{U}$ we have $\iota(U^g) = \iota(U)$. Let y be a point of weight w_1 and $U' = U \setminus \{y\}$. Furthermore let x be any point of maximal weight and let $S = U' \cup \{x\}$. If $g \in Aut\ \mathcal{U}$ then certainly $\iota(S) = \iota(S^g)$, so

$$\begin{aligned}
\frac{\lambda w(y)}{w(x) \prod_{z \in U} w(z)} &= \iota(S) \\
&= \iota(S^g) \\
&= \frac{\lambda w(y^g)}{w(x^g) \prod_{z \in U} w(z^g)}
\end{aligned}$$

$$= \frac{\lambda w(y^g)}{w(x^g)\prod_{z\in U} w(z)}.$$

Therefore,

$$\frac{w(x)}{w(x^g)} = \frac{w(y)}{w(y^g)}.$$

Since $w(x)$ is minimal we have that $w(x^g) \geq w(x)$ (i.e. $\frac{w(x)}{w(x^g)} \leq 1$). Similarly, since $w(y)$ is maximal, we have that $w(y^g) \leq w(y)$ and so $\frac{w(y)}{w(y^g)} \geq 1$. This means that $w(x) = w(x^g)$, i.e. points of minimal weight are mapped to points of the same weight by the action of the automorphism group of the underlying structure. We may now repeat this argument with a point x of weight w_2, noting that $w(x) \leq w(x^g)$ as x^g cannot be a point of minimal weight as these are only mapped to points of the same weight under the action of $Aut\ \mathcal{U}$. This means that any point of U is mapped to a point of the same weight under the action of $Aut\ \mathcal{U}$.

We now repeat the argument used in the first case to show that points of weight greater than w_j also map to points of the same weight under the action of the automorphism group of the underlying incidence structure.

■

4.2 Generating designs from groups

In classical design theory it is possible to take a group that acts on a set with certain properties and create a classical design. We will investigate the possibility of using this technique to create point-weighted designs.

According to the definition of $Aut\ \mathcal{S}$ we have that points may only be mapped to points of the same weight. So we may partition the points according to the orbits of $Aut\ \mathcal{S}$ and every point in an orbit of will be of the same weight. It is not necessary for different orbits to have different weights.

Let G be a group that acts on a set V and let $V = V_1 \cup V_2 \cup \ldots \cup V_d$ be the orbit decomposition of V by G. Hence, for all i, $V_i^G = V_i$ and G acts transitively on V_i. Further let $\{\beta_1, \beta_2, \ldots, \beta_e\}$ be a set of *base blocks*, i.e. a set of subsets of V. We will examine the structure create by taking a point set V and a block set $\mathcal{B} = \bigcup_{j=1}^{e} \beta_j^G$.

Firstly, if the above structure is to be a t-point-sum or a t-row-sum point-weight design, the two obvious conditions must hold:

- Any set of t points must lie on some non-zero number of blocks. Hence for all $1 \leq i \leq d$ there exists j with $1 \leq j \leq e$ such that β_j contains at least $min\{|V_i|, t\}$ points of V_i.

- If the structure is to have constant block size then there must exist a weight assignment for the orbits $V_1, \ldots, V_d$ such that each of the base blocks has the same weight.

Obviously if β_j is incident with less than t points then it may be ignored, in the sense that if the base blocks $\{\beta_1, \ldots, \beta_e\}$ generate a t-point-sum or t-row-sum point-weight design then the structure generated from the base blocks $\{\beta_1, \ldots, \beta_e\} \setminus \{\beta_j\}$ will be a point-sum or row-sum point-weight design with the same parameters.

Since the design condition of a point-sum point-weight design does not depend upon the weights of the points we may assign weights to each of the orbits of points in any way in which the base blocks satisfy the constant block size condition. Whether such an assignment of weights exists is an easily checkable matrix condition: if M_β is an incidence matrix for the base blocks then there must exist a positive rational column vector $\underline{x}$ such that $M_\beta^T \underline{x} = \underline{1}$ and the weights of the points (derived from the entries of $\underline{x}$) must be the same for each point in an orbit. However if we attempt to construct a row-sum point-weight design then the design condition depends very heavily upon the weights of the points and this will constrain our ability to chose the weights of the points so that each base block will be of the same size (i.e. the sum of the weights of the points on any base block is the same).

We will focus our attempts on satisfying the design condition.

Definition 4.2.1 *Suppose a group G acts on a set V with orbit decomposition $V_1, \ldots, V_d$ and let S and T be subsets of V. S and T are said to have the same structure if $|S \cap V_i| = |T \cap V_i|$ for all $1 \leq i \leq d$.*

Note that if S and T have the same structure then $|S| = |T|$.

Definition 4.2.2 *Suppose a group G acts on a set V with orbit decomposition $V_1, \ldots, V_d$ and $t \geq 1$. G is said to be t-*homogeneous with respect to its orbital decomposition *if for all subsets S and T of V, where S and T each contain t points and have the same structure, then there exists a $g \in G$ such that $S^g = T$.*

So any group acting on a set V is 1-homogeneous with respect to the orbital decomposition because G acts transitively on the orbits.

Lemma 4.2.3 *Suppose a group G acts on a set V with orbit decomposition $V_1, \ldots, V_d$ and that G acts t-homogeneously on V with respect to its orbital decomposition. Let $\mathcal{B}$ be the set of sets of points of V obtained by applying G to a set of base blocks and let $\mathcal{S}$ be the incidence structure $(V, \mathcal{B}, \in)$. If S is a subset of V containing t points and T is a subset of V that has the same structure as S then $\iota(S) = \iota(T)$ in $\mathcal{S}$.*

Proof Since G is t-homogeneous with respect to its orbital decomposition and S and T have the same structure of t points, there exists an element $g \in G$ such that $S^g = T$.

Suppose $S \subseteq B \in \mathcal{B}$ then $T \subseteq B^g \in \mathcal{B}$ and $B_1^g = B_2^g$ if and only if $B_1 = B_2$.

$$\text{Thus } \iota(S) \leq \iota(T).$$

However the roles of S and T in this argument can be interchanged, so $\iota(S) = \iota(T)$.

■

Theorem 4.2.4 *Suppose a group G acts on a set V with orbit decomposition $V_1, \ldots, V_d$ and that G acts t-homogeneously on V with respect to its orbital decomposition. Furthermore suppose that $\{\beta_1, \ldots, \beta_e\}$ is a set of base blocks of V and that S is a set of t points. Then*

$$\iota(S) \quad = \quad \sum_{j=1}^{e} \frac{|G| \prod_{i=1}^{d} \binom{|\beta_j \cap V_i|}{|S \cap V_i|}}{|Stab\ \beta_j| \prod_{i=1}^{d} \binom{|V_i|}{|S \cap V i|}}. \tag{4.1}$$

Proof We will use a "counting flags" technique: we will count the number of ordered pairs (T, B) where T is a subset of V with the same structure as S and B is a block that contains T.

We begin by picking T and calculating the number of blocks that T lies on. There exists $\prod_{i=1}^{d} \binom{|V_i|}{|S \cap V_i|}$ ways to pick a set T with the same structure as S and, by the above lemma, each of these sets will lie on exactly $\iota(S)$ blocks.

Next we calculate how many sets T of the same structure as S can lie on a given block. We consider the blocks generated by each base block in turn. The number of sets T that can lie on a block generated by base block β_j is $\prod_{i=1}^{d} \binom{|\beta_j \cap V_i|}{|S \cap V_i|}$ and there are $\frac{|G|}{|Stab\ \beta_j|}$ blocks generated by the base block β_j. Hence there are $\sum_{j=1}^{e} \frac{|G|}{|Stab\ \beta_j|} \prod_{i=1}^{d} \binom{|\beta_j \cap V_i|}{|S \cap V_i|}$ ordered pairs.

Equating these two expressions and re-arranging gives that:

$$\iota(S) \quad = \quad \sum_{j=1}^{e} \frac{|G| \prod_{i=1}^{d} \binom{|\beta_j \cap V_i|}{|S \cap V_i|}}{|Stab\ \beta_j| \prod_{i=1}^{d} \binom{|V_i|}{|S \cap V i|}}$$

■

Let us attempt to construct a simple point-sum or row-sum point-weight design using this formula. The case when G acts transitively on V is trivial as every point in the structure must have the same weight and so the structure is a classical design. Hence assume that its orbital decomposition of V is $V = V_1 \cup V_2$ and that there exists only one base block β. In this case the constant block size condition is trivially satisfied as any weight assignment will lead to blocks of the same size.

Lemma 4.2.5 *Suppose G acts t-homogeneously with respect to its orbital decomposition on a set V and that $\beta \subseteq V$ is a base block. If the structure $(V, \beta^G, \in, w)$ for some weight function w is a $t-(v,k,\lambda;W)$ point-weight design with $v > k$ and $t > 1$ then $|W| = 1$.*

Proof Since there is only one base block we must have that all the blocks have the same number of points in them, hence the underlying structure must be a $t-(u,k',\lambda)$ design. However we know from [9] that any $t-(v,k,\lambda;W)$ design with an underlying structure that is a classical design has $|W| = 1$.

■

Hence, in the case where $\mathcal{S}$ has only one base block, we may restrict ourselves to attempting to construct row-sum point-weight designs. However the following lemma restricts the type of row-sum point-weight design that we can construct:

Lemma 4.2.6 *If $\mathcal{S}$ is a nice $\pi_2-(v,k,\lambda;W)$ point-weight design that is constructed from a group G acting on a set of points V with d orbits, and there exists e base blocks which are permuted by G to give the block set $\mathcal{B}$ then $e \geq d$.*

Proof We use Block's lemma (1.6.2). We may divide the rows of M into a partition where two points are in the same partition if and only if they lie in the same point orbit. Similarly we may divide the columns of M into a partition where two blocks are in the same partition if and only if they are derived from the same base block. This forms a tactical decomposition of M with d row partitions and e columns partitions. Hence, if M has u rows, we have:

$$d \;\; \leq \;\; e + u - rank(M)$$

Since M is nice we have that $det(MM^T) \neq 0$ or that $rank(MM^T) = u$ as MM^T is a $u \times u$ matrix. Now, by a simple result of linear algebra, $rank(M) \geq rank(MM^T)$ but since M has u rows, we have that $rank(M) \leq u$ and so $rank(M) = u$. Therefore the above inequality simplifies to $d \leq e$.

■

Corollary 4.2.7 *If $\mathcal{S}$ is a $\pi_2-(v,k,\lambda;W)$ point-weight design and there exists a group G acting on V and a base block β such that V is composed of more than one point orbit and the block set $\mathcal{B} = \beta^G$ then $\mathcal{S}$ is either difficult or awkward with more than one awkward point.*

Proof Since $\mathcal{S}$ has more point orbits then base blocks we must have that $\mathcal{S}$ is not nice. Hence, by (2.4.4) and (2.4.11), it must be either difficult or awkward and have more than one awkward point.

■

Theorem 4.2.8 *If $\mathcal{S}$ is a $\pi_2 - (v, k, \lambda; W)$ point-weight design (with $k < v$) and there exists a group G that acts on V and a base block β such that V is composed of more than one point orbit and the block set $\mathcal{B} = \beta^G$ then $\mathcal{S}$ is a difficult design of the form of (3.2.3).*

Proof Suppose $\mathcal{S}$ is a $\pi_2 - (v, k, \lambda; W)$ point-weight design and there exists a group G that acts on V and a base block β such that V is composed of more than one point orbit and the block set $\mathcal{B} = \beta^G$. Note that β is a block. Let S be the set of awkward or difficult points.

We have that $S \subseteq \beta$ because S is necessarily a point orbit or the union of point of orbits and in the case where S contains more than one awkward point we know by (2.4.5) that every block that contains one point of S must contain all the points of S.

So consider $\mathcal{S}_S$. As we have not removed any blocks this is still a $\pi_2 - (v - \sigma(S), k - \sigma(S), \lambda; W')$ point-weight design that is generated by a single base block $\beta \setminus S$ however none of the points of $\mathcal{S}_S$ are awkward or difficult, so $\mathcal{S}_S$ is nice. This contradicts (4.2.7) unless $\mathcal{S}_S$ is composed of a single point orbit, i.e. $\mathcal{S}_S$ contains points of all the same weight and has the underlying structure of a classical design. In particular this means that $|W|= 2$.

Suppose $|S|> 1$, then $\mathcal{S}$ is a $\pi_2 - (v, k, \lambda; W)$ design with $k < v$ and more than one point that lies on every block, which contradicts (3.2.7). Hence $|S|= 1$ and $\mathcal{S}$ must have the same underlying incidence structure as the difficult design constructed using (3.2.3), however any two designs with the same underlying incidence structure are equivalent by (2.2.6).

■

The case is obviously harder when $t > 2$ but we can show that either we have a construction analogous to (3.2.4) or there exists at most one value of t for which the structure is a $\pi_t - (v, k, \lambda; W)$ point-weight design (under certain technical conditions).

Lemma 4.2.9 *Suppose $\mathcal{S}$ is a point-weight incidence structure and $G \leq Aut\ \mathcal{S}$ is an automorphism group that partitions the point set into two orbits V_1 and V_2, and acts transitively on the block set (so that there exists a single base block β with $\beta^G = \mathcal{B}$). Then there exists no value of t such that $|V_1| \leq t$, $|V_2| \leq t$, G acts t-homogeneously with respect to its orbital decomposition and $\mathcal{S}$ is a $\pi_t - (v, k, \lambda; W)$ point-weight design with $k \neq v$.*

Proof We know that β must contain at least $min\{|V_i|, t\}$ points of V_i for all $1 \leq i \leq 2$. However $min\{|V_i|, t\} = |V_i|$ and so $V_i \subseteq \beta$ for all i. Hence $\beta = V$.

■

Suppose we have that G acts t-homogeneously with respect to its orbital decomposition, where $t \geq 2$, on a set V whose orbital decomposition is $V = V_1 \cup V_2$ and on a single base block $\beta \subseteq V$. Consider the structure given by $(V, \beta^G, \in)$. Furthermore let us suppose that $|V_1| = 1$ and $|V_2| > t$ then there exists two possible structures of sets with t points:

1. The set S which contains the single point of V_1 and $t-1$ points of V_2.
2. The set T which contains t points of V_2.

Note that β must contain at least t points of V_2 and all of V_1.

Hence, assuming the structure has u points and so there are $u-1$ points of V_2:

$$\begin{aligned} \iota(S) &= \frac{|G| \binom{1}{1}\binom{|\beta|-1}{t-1}}{|Stab\ \beta| \binom{1}{1}\binom{u-1}{t-1}}, \\ \iota(T) &= \frac{|G| \binom{1}{0}\binom{|\beta|-1}{t}}{|Stab\ \beta| \binom{1}{0}\binom{u-1}{t}}. \end{aligned}$$

Now let us suppose that $(V, \mathcal{B}, I, w)$ is a $\pi_t - (v, k, \lambda; W)$ point-weight design for some weight function w, that the single point of V_1 has weight n and that the points of V_2 have weight m. Consequently:

$$\begin{aligned} \frac{\iota(S)}{\iota(T)} &= \frac{\lambda n m^{t-1}}{\lambda m^t} \\ &= \frac{n}{m} \\ &= \frac{\binom{|\beta|-1}{t-1}\binom{u-1}{t}}{\binom{u-1}{t-1}\binom{|\beta|-1}{t}} \\ &= \frac{u-t}{|\beta| - t}. \end{aligned}$$

So if we pick n and m to be the unique co-prime integers that satisfy this equation then we can guarantee that the structure will be a row-sum point-weight design. As we would expect, if $n = m$ then $\iota(S) = \iota(T)$ and the underlying structure would be a classical design and so $|W| = 1$, by (2.2.6).

This construction is analogous to the construction of a difficult row-sum point-weight design from a classical design given in (3.2.3). In this case the classical design is constructed from a group G acting t-homogeneously on the set V_2 and a single base block $\beta \setminus V_1$, then we add the single point contained in V_1 to every block.

Lemma 4.2.10 *Suppose $\mathcal{S}$ is a point-weight incidence structure and $G \leq Aut\ \mathcal{S}$ is an automorphism group that partitions the point set into two orbits V_1 and V_2,* *and acts transitively on the block set (so that there exists a single base block β with $\beta^G = \mathcal{B}$). Then there exists no value of t such that $2 \leq |V_1| \leq t$, $t < |V_2|$, G acts t-homogeneously with respect to its orbital decomposition and $\mathcal{S}$ is a $\pi_t-(v, k, \lambda; W)$ point-weight design with $k \neq v$.*

Proof Assume $\mathcal{S}$ is a $\pi_t - (v, k, \lambda; W)$ point-weight design and that the points of the sets V_1 and V_2 have weights n and m respectively. Since $|V_1| \leq t$ we have that $V_1 \subseteq \beta$ and so $|\beta \cap V_2| = |\beta| - |V_2|$.

For all $0 \leq s \leq |V_1|$ we may define a set T_s that contains s points of V_1 and $t - s$ points of V_2. Thus:

$$\iota(T_s) \quad = \quad \frac{|G| \binom{|V_1|}{s}\binom{|\beta|-|V_1|}{t-s}}{|Stab\ \beta| \binom{|V_1|}{s}\binom{|V_2|}{t-s}}.$$

So for all $1 \leq s \leq |V_1|$ we have that:

$$\begin{aligned} \frac{\iota(T_{s-1})}{\iota(T_s)} &= \frac{n}{m} \\ &= \frac{|\beta| - |V_1| - t + s}{|V_2| - t + s}. \end{aligned}$$

Hence, as $t \geq 2$, and since

$$\frac{\iota(T_0)}{\iota(T_1)} = \frac{n}{m} = \frac{\iota(T_1)}{\iota(T_2)}$$

we have that

$$\begin{aligned} \frac{|\beta| - |V_1| - t + 1}{|V_2| - t + 1} &= \frac{|\beta| - |V_1| - t + 2}{|V_2| - t + 2} \\ \text{so } (|\beta| - |V_1| - t + 1)(|V_2| - t + 2) &= (|\beta| - |V_1| - t + 2)(|V_2| - t + 1) \\ \text{and } |\beta| &= |V_1| + |V_2| \end{aligned}$$

and so $\beta = V$.

$\blacksquare$

Lemma 4.2.11 *Suppose $\mathcal{S}$ is a point-weight incidence structure and $G \leq Aut\ \mathcal{S}$ is an automorphism group that partitions the point set into two orbits V_1 and V_2, and acts transitively on the block set (so that there exists a single base block β with $\beta^G = \mathcal{B}$). There exists at most one value of t such that $t < |V_1|$, $t < |V_2|$, G acts t-homogeneously with respect to its orbital decomposition and $\mathcal{S}$ is a $\pi_t-(v,k,\lambda;W)$ point-weight design with $k \neq v$.*

Proof Suppose $\mathcal{S}$ is a $\pi_t-(v,k,\lambda;W)$ point-weight design and that the points of V_1 and V_2 have weights n and m respectively. Let T_s be a set containing s points of V_1 and $t-s$ points of V_2, where $0 \leq s \leq t$. Once again we make use of the fact that for $1 \leq s \leq t$ we have that $\frac{\iota(T_{s-1})}{\iota(T_s)} = \frac{n}{m}$. Note that $\frac{\iota(T_0)}{\iota(T_1)}$ is enough to uniquely determine the co-prime integers n and m.

Now, since we have that $|V_1| > t$,

$$\iota(T_s) = \frac{|G|\ \binom{|\beta\cap V_1|}{s}\binom{|\beta\cap V_2|}{t-s}}{|Stab\ \beta|\ \binom{|V_1|}{s}\binom{|V_2|}{t-s}}$$

Hence, as $t \geq 2$ and since:

$$\frac{\iota(T_0)}{\iota(T_1)} = \frac{\iota(T_{t-1})}{\iota(T_t)}$$

We have that

$$\begin{aligned}
\frac{|V_1|\ \binom{|V_2|}{t-1}\binom{|\beta\cap V_2|}{t}}{|\beta\cap V_1|\ \binom{|\beta\cap V_2|}{t-1}\binom{|V_2|}{t}} &= \frac{|\beta\cap V_2|\ \binom{|V_1|}{t}\binom{|\beta\cap V_1|}{t-1}}{|V_2|\ \binom{|V_1|}{t-1}\binom{|\beta\cap V_1|}{t}} \\
\frac{|V_1|\ (|\beta\cap V_2|\ -t+1)}{|\beta\cap V_1|\ (|V_2|\ -t+1)} &= \frac{|\beta\cap V_2|\ (|V_1|\ -t+1)}{|V_2|\ (|\beta\cap V_1|\ -t+1)}
\end{aligned}$$

$$\begin{aligned}
&|V_1||V_2|\ (|\beta\cap V_1||\beta\cap V_2|\ -(t-1)(|\beta\cap V_1| + |\beta\cap V_2|) + (t-1)^2) \\
&\qquad = |\beta\cap V_1||\beta\cap V_2|\ (|V_1||V_2|\ -(t-1)(|V_1| + |V_2|) + (t-1)^2)
\end{aligned}$$

This is a quadratic expression in $(t-1)$ but the constant terms on both sides of the equation are the same and so we may cancel them out and then, since $t > 1$, divide through by $t-1$ to get a linear expression. So

$$(|V_1||V_2| - |\beta \cap V_1||\beta \cap V_2|)t$$
$$= |V_1||V_2|\ (|\beta \cap V_1| + |\beta \cap V_2| - 1) - |\beta \cap V_1||\beta \cap V_2|\ (|V_1| + |V_2| - 1)$$

We know that $|V_1||V_2| - |\beta \cap V_1||\beta \cap V_2| = 0$ if and only if $|\beta \cap V_1| = |V_1|$ and $|\beta \cap V_2| = |V_2|$. This is because $|\beta \cap V_i| \leq |V_i|$ for $i = 1, 2$. So there exists have a linear polynomial in t and so we can conclude that there exists at most one value of $t \geq 2$ for which $\mathcal{S}$ is a $\pi_t - (v, k, \lambda; W)$ point-weight design.

■

4.3 Generating groups

In this section we will consider the inverse problem of the last section, i.e. given a design $\mathcal{S}$, which groups that are t-homogeneous with respect to its orbital partition coupled with which base blocks generate $\mathcal{S}$. We will do this by developing a new technique in the field of permutation groups. Note that the definitions of two sets having the same structure (4.2.1) and the definition of a group being t-homogeneous with respect to its orbital decomposition (4.2.2) are already phrased in terms of groups acting on sets.

Lemma 4.3.1 *Suppose Ω is a G-space and $|\Omega| \geq t \geq 1$. Then there exists a subgroup $H \leq G$ such that H is t-homogeneous with respect to its orbital partition.*

Proof Certainly the group $H = \{id\}$ is a subgroup of G and the orbits of H are the sets $\{\alpha\}$ for all $\alpha \in \Omega$. So H is trivially t-homogeneous with respect to its orbital decomposition.

■

So, for every t with $|\Omega| \geq t \geq 1$, there exists at least one subgroup $H \leq G$ such that H is t-homogeneous with respect to its orbital decomposition. We now consider the largest group with this property.

Definition 4.3.2 *Suppose that Ω is a G-space and $1 \leq t \leq |\Omega|$. A subgroup $H \leq G$ is called t-maximal if H is t-homogeneous with respect to its orbital decomposition and there exists no subgroup $H < K \leq G$ which is also t-homogeneous with respect to its orbital decomposition.*

Theorem 4.3.3 *Suppose that Ω is a G-space, $1 \leq t \leq |\Omega|$ and H, K are subgroups of G that are t-homogeneous with respect to their orbital decompositions. Then $\langle HK \rangle$ is t-homogeneous with respect to its orbital decomposition.*

Proof We begin by noting that

$$H, K \leq \langle HK \rangle \leq G$$

Suppose that the orbits of H are $V_1, \ldots, V_d$, the orbits of K are $V'_1, \ldots, V'_\delta$ and the orbits of $\langle HK \rangle$ are $U_1, \ldots, U_\gamma$. Since H and K are subgroups of $\langle HK \rangle$ we must have that for any $1 \leq \eta \leq \gamma$:

$$U_\eta = \bigcup_{\psi \in \Psi} V_\psi \text{ for some } \Psi \subseteq \{1, 2, \ldots, d\}$$
$$U_\eta = \bigcup_{\phi \in \Phi} V'_\phi \text{ for some } \Phi \subseteq \{1, 2, \ldots, \delta\}$$

Let S be any set of $t-1$ elements of Ω and let x and x' be distinct elements of $\Omega \setminus S$ belonging to the same orbit of $\langle HK \rangle$. Then there exists an element $\sigma \in \langle HK \rangle$ such that $x^\sigma = x'$. Furthermore, since $\sigma \in \langle HK \rangle$, there exists group elements $h_1, \ldots, h_m \in H$ and $k_1, \ldots, k_m \in K$ such that $\sigma = h_1 k_1 \ldots h_m k_m$. This means there exists a chain of elements of U_η, $x = x_0, x_1, x_2, \ldots, x_{2m-1}, x_{2m} = x'$ such that for all $0 \leq i \leq 2m-1$ either there exists a ψ such that $x_i, x_{i+1} \in V_\psi$ (if i is even) or there exists a ϕ such that $x_i, x_{i+1} \in V'_\phi$ (if i is odd).

We will show that there exists a $\sigma' \in \langle HK \rangle$ that maps $S \cup \{x\}$ onto $S \cup \{x'\}$. We will proceed using induction on m.

If $m = 1$ then $x_0^{h_1} = x_1$, and so there exists a ψ such that $x_0, x_1 \in V_\psi$. Similarly there exists a ϕ such that $x_1, x_2 \in V'_\phi$. Therefore, providing $x_1 \notin S$, we know that there exists an element $h \in H$ such that $(S \cup \{x_0\})^h = S \cup \{x_1\}$ as H is t-homogeneous with respect to its orbital partition. Similarly there exists a $k \in K$ such that $(S \cup \{x_1\})^k = S \cup \{x_2\}$ as K is also t-homogeneous with respect to its orbital partition. Hence $(S \cup \{x_0\})^{hk} = S \cup \{x_2\}$.

Now if $x_1 \in S$ then there exists $k \in K$ such that $(S \cup \{x_0\})^k = \{x_0, x_2\} \cup S \setminus \{x_1\}$ because $x_1 \in S$, and x_1 and x_2 lie in the same orbit of K. There also exists a $h \in H$ such that $(\{x_0, x_2\} \cup S \setminus \{x_1\})^h = S \cup \{x_2\}$ as x_0 and x_1 lie in the same orbit of H. So $(S \cup \{x_0\})^{kh} = S \cup \{x_2\}$ and $kh = (ek)(he) \in \langle HK \rangle$. Thus the case $m = 1$ is proven.

Now suppose that such a σ exists provided the chain defined above is of length $2m$ or less, and consider a chain of length $2m+2$.

If x_{2m} is not in S then the process is very straightforward. Since there exists a chain of length $2m$ between $x = x_0$ and x_{2m} we know there exists an element $\sigma \in \langle HK \rangle$ such that $(S \cup \{x\})^\sigma = S \cup \{x_{2m}\}$. Furthermore, since there exists a chain of length 2 between x_{2m} and $x' = x_{2m+2}$, we know there exists an element $\sigma' \in \langle HK \rangle$ such that $(S \cup \{x_{2m}\})^{\sigma'} = S \cup \{x_{2m+2}\}$. Hence $(S \cup \{x\})^{\sigma\sigma'} = S \cup \{x'\}$ and our induction is proven.

So suppose $x_{2m} \in S$ and let $S' = S \setminus \{x_{2m}\}$. Note that

$$S \cup \{x\} = S' \cup \{x_0\} \cup \{x_{2m}\}.$$

Since there exists a chain of length 2 between x_{2m} and x_{2m+2} we have that there exists an element $\sigma' \in \langle HK \rangle$ such that

$$\Big((S' \cup \{x_0\}) \cup \{x_{2m}\}\Big)^{\sigma'} = (S' \cup \{x_0\}) \cup \{x_{2m+2}\}.$$

Furthermore, since there exists a chain of length $2m$ between x_0 and x_{2m}, we have that there exists an element $\sigma \in \langle HK \rangle$ such that

$$\Big((S' \cup \{x_{2m+2}\}) \cup \{x_0\}\Big)^{\sigma} = (S' \cup \{x_{2m+2}\}) \cup \{x_{2m}\} = S \cup \{x'\}$$

Hence $(S \cup \{x\})^{\sigma'\sigma} = S \cup \{x'\}$ and our induction is proven.

This will be enough to show that $\langle HK \rangle$ is t-homogeneous with respect to its orbital partition. Suppose we have two sets of points S_1 and S_2 with the same structure with respect to the orbital partition of $\langle HK \rangle$. Then we may enumerate the points of $S_1 = \{s_{1,1}, \ldots, s_{1,t}\}$ and $S_2 = \{s_{2,1}, \ldots, s_{2,t}\}$ so that $s_{1,j}$ and $s_{2,j}$ are in the same orbit of $\langle HK \rangle$ and so find elements of $\sigma_1, \sigma_2, \ldots, \sigma_t \in \langle HK \rangle$ such that:

$$\begin{aligned} S_1^{\sigma_1} &= \{s_{2,1}, s_{1,2}, s_{1,3}, \ldots s_{1,t}\} \\ S_1^{\sigma_1\sigma_2} &= \{s_{2,1}, s_{2,2}, s_{1,3}, \ldots s_{1,t}\} \\ &\vdots \\ S_1^{\sigma_1\sigma_2\ldots\sigma_t} &= \{s_{2,1}, s_{2,2}, s_{2,3}, \ldots s_{2,t}\} \\ &= S_2 \end{aligned}$$

Hence $\langle HK \rangle$ is t-homogeneous.

■

Corollary 4.3.4 *If Ω is a G-space and $1 \leq t \leq |\Omega|$ then there exists a unique t-maximal subgroup $H \leq G$, and any subgroup $K \leq G$ which is t-homogeneous with respect to its orbital partition is a subgroup of H.*

Proof Suppose H and K are both t-maximal subgroups of G. Since $\langle HK \rangle$ is t-homogeneous with respect to its orbital partition and $H \leq \langle HK \rangle$ we have that $H = \langle HK \rangle$. Similarly $K = \langle HK \rangle$, and so $H = K$.

Now suppose that H a t-maximal subgroup of G and $K \leq G$ is t-homogeneous with respect to its orbital partition. If K is not a subgroup of H then H is a proper subgroup of $\langle HK \rangle$. However since H is t-maximal this is a contradiction, hence $K \leq H$.

■

Now that we have shown that these t-maximal subgroups exist and are unique we may begin to investigate some of their properties.

Lemma 4.3.5 *Suppose Ω is a G-space and $1 \leq |\Omega|$. If H is t-homogeneous with respect to its orbital partition and $g \in G$ then gHg^{-1} is also t-homogeneous with respect to its orbital partition.*

Proof If the orbits of Ω under the action of H are $V_1, \ldots, V_d$ then orbits of Ω under the action of gHg^{-1} are $V_1^{g^{-1}}, \ldots, V_d^{g^{-1}}$. Let S and T be two sets of t elements of Ω that have the same structure with respect to the orbits of gHg^{-1}. This means that S^g and T^g have the same structure with respect to the orbits of H, hence there exists a $h \in H$ such that $S^{gh} = T^g$.

We may re-write this as $S^{ghg^{-1}} = T$. Hence there exists an element of gHg^{-1} that maps S onto T, which means that gHg^{-1} is t-homogeneous with respect to its orbital decomposition.

■

Corollary 4.3.6 *Suppose Ω is a G-space and $1 \leq t \leq |\Omega|$. If H is the unique subgroup of G that is t-maximal then H is normal in G.*

Proof Let $H \leq G$ be t-maximal. For any $g \in G$ we have that gHg^{-1} is t-homogeneous with respect to its orbital partition. So $gHg^{-1} \leq H$ as H is t-maximal. This means that $gHg^{-1} = H$ and $H \trianglelefteq G$.

■

The following result is due to Peter Cameron [6].

Result 4.3.7 *Suppose Ω is a G-space and $H \leq G$ is t-homogeneous with respect to its orbital partition. If Ω has d orbits under the action of H and $1 < t \leq \frac{1}{2}|\Omega| - d + 1$ then H is also $(t-1)$-homogeneous with respect to its orbital partition.*

Proof Let $V_1, \ldots, V_d$ be the orbits of H and let S_1 and S_2 be sets of $t-1$ elements of Ω that have the same structure under H. If $|S_1 \cap V_i| > \frac{1}{2}|V_i| - 1$ for all $1 \leq i \leq d$ then

$$\begin{aligned}
t - 1 &= |S_1| \\
&= \sum_{i=1}^{d} |S_1 \cap V_1| \\
&> \sum_{i=1}^{d} (\frac{1}{2}|V_i| - 1) \\
&= \frac{1}{2}|\Omega| - d,
\end{aligned}$$

and so $t > \frac{1}{2}|\Omega| - d + 1$ which is a contradiction. Hence there exists at least one orbit, V_1 say, such that $|S_1 \cap V_1| + 1 \leq \frac{1}{2}|V_1|$.

As H acts t-homogeneously with respect to its orbital partition we know that there exists an element $h \in H$ such that $(S_1 \cup \{x_1\})^h = S_2 \cup \{x_2\}$. So h maps $S_1 \setminus V_1$ onto $S_2 \setminus V_1$ but might not map $S_1 \cap V_1$ onto $S_2 \cap V_1$. We are therefore left with the problem of finding an element of $Stab_H(S_2 \setminus V_1)$ that maps $(S_1 \cap V_1)^h$ onto $S_2 \cap V_1$.

Since H is t-homogeneous with respect to its orbital partition, we have that $Stab_H(S_2 \setminus V_1)$ is $(|S_1 \cap V_1| + 1)$-homogeneous on V_1. So, by 1.5.5, we have that $Stab_H(S_2 \setminus V_1)$ is $|S_1 \cap V_1|$-homogeneous on V_1. Hence there exists an element $g \in Stab_H(S_2 \setminus V_1)$ that maps $(S_1 \cap V_1)^h$ onto $S_2 \cap V_1$. Hence $S_1^{hg} = S_2$, and H is $(t-1)$-homogeneous with respect to its orbital partition.

■

We can apply this general theory to automorphism groups of point-weight designs. Suppose $\mathcal{S}$ is a point-weight incidence structure. Then we let the group $G = Aut\ \mathcal{S}$ act on the set of points of $\mathcal{S}$ and find that there exists a unique maximal subgroup $H \trianglelefteq Aut\ \mathcal{S}$ that is t-homogeneous with respect to its orbital partition. To give insight into these results we will analyse the design given in figure 3.5 and use this to construct a new family of designs, given in (4.3.8).

Let $\mathcal{S}$ be the awkward $\pi_2 - (v, k, \lambda; W)$ point-weight design given in Figure 3.5. This has an automorphism group:

$$Aut\ \mathcal{S} \quad = \quad G_1 \times (G_2 \times G_3) \rtimes H$$

where $G_i \cong S_3$ and $H \cong C_2$ (this is actually product of G_1 with the wreath product of G_2 and H). We may partition the point-set V into three parts V_1, V_2 and V_3 as follows. Firstly let V_1 contain the three awkward points, then the remaining six points are partitioned into two sets of three points so that each of the sets is a block. G_i acts on V by permuting the 3 points of V_i and fixing all the other points while H acts on V by fixing the points of V_1 and mapping the points of V_2 onto V_3 and vice versa.

Hence if we suppose that:

$$\begin{aligned} V_1 &= \{x_1, x_2, x_3\}, \\ V_2 &= \{y_1, y_2, y_3\}, \\ V_3 &= \{z_1, z_2, z_3\} \end{aligned}$$

and we take some element $(e, \sigma, e, e) \in Aut\ \mathcal{S}$ then the effect of this group element on a point $p \in V$ is:

$$p \mapsto \begin{cases} p & \text{if } p \notin V_2 \\ y_{i^\sigma} & \text{if } p = y_i \in V_2. \end{cases}$$

Whereas if we take the element $(e, e, e, h) \in Aut\ \mathcal{S}$ where h is the non-identity element of H then the effect of this group element on a point $p \in V$ is:

$$p \mapsto \begin{cases} p & \text{if } p \in V_1 \\ z_i & \text{if } p = y_i \in V_2 \\ y_i & \text{if } p = z_i \in V_3. \end{cases}$$

However the maximal subgroup of *Aut* $\mathcal{S}$ that is 2-homogeneous with respect to its orbital partition is $K = G_1 \times G_2 \times G_3$, and this group generates $\mathcal{S}$ via the three base blocks:

$$\begin{aligned} \beta_1 &= V_1 \cup \{y_1, z_1\}, \\ \beta_2 &= V_2, \\ \beta_3 &= V_3. \end{aligned}$$

Note that $V_1 \cup V_2 \cup V_3$ is the orbital decomposition of V under the action of K and that $\mathcal{S}$ has 3 point orbits and 3 base blocks. We are not required by (4.2.5) to have at least as many base blocks as orbits because $\mathcal{S}$ has more than one awkward point and is therefore not nice. However neither are we forced to have more point orbits than base blocks just because $\mathcal{S}$ isn't nice. Further notice that K is normal in *Aut* $\mathcal{S}$ because K is a subgroup of index 2 in *Aut* $\mathcal{S}$. This is an example of (4.3.6).

We may extend this idea into forming a larger row-sum point-weight design.

Theorem 4.3.8 *For any $n \geq 3$ we can construct an awkward $\pi_2 - (3n^2 - 2n, n^2, n^2; \{1, n\})$ point-weight design.*

Proof Define a point-weight incidence structure $\mathcal{S} = (V, \mathcal{B}, \in, w)$ as follows:

$$\begin{aligned} V_1 &= \{x_1, \ldots x_{n(n-2)}\}, \\ V_2 &= \{y_1, \ldots y_n\}, \\ V_3 &= \{z_1, \ldots z_n\}, \\ V &= V_1 \cup V_2 \cup V_3, \\ \\ \mathcal{B} &= \{V_1 \cup \{y_i, z_j\} : 1 \leq i, j \leq n\} \cup \{V_2, V_3\}, \\ \\ w(p) &= \begin{cases} 1 & \text{if } p \in V_1 \\ n & \text{otherwise.} \end{cases} \end{aligned}$$

In this structure we certainly have that the sum of the weights of all the points is $3n^2 - 2n$ and that the sum of the weights of the points on any block is n^2. A cursory examination will show that $\mathcal{S}$ is a π_2 row-sum point-weight design for $\lambda = n^2$, it therefore remains to show that this design is awkward.

Note that V_1 contains at least 3 points as $n \geq 3$. Hence x_1 and x_2 are points of the same weight and furthermore every block that contains x_1 also contains x_2. This means, by (2.4.6), that x_1 and x_2 are awkward points and so every point of V_1 is awkward too.

Hence $\mathcal{S}$ is an awkward design.

■

Any $\pi_2 - (v, k, \lambda; W)$ point-weight design $\mathcal{S}$ constructed this way will have an automorphism group of the form:

$$Aut\ \mathcal{S} = G_1 \times (G_2 \times G_3) \rtimes H$$

where $G_1 \cong S_{n(n-2)}$ and acts on the points of V_1, $G_2 \cong G_3 \cong S_n$ and act on the points of V_2 (or V_3 respectively) and H swaps the points of V_2 and V_3 over as before. The maximal generating group of $\mathcal{S}$ that is 2-homogeneous with respect to its orbital partition is $G_1 \times G_2 \times G_3$ and generates $\mathcal{S}$ via the base blocks:

$$\begin{aligned} \beta_1 &= V_1 \cup \{y_1, z_1\}, \\ \beta_2 &= V_2, \\ \beta_3 &= V_3. \end{aligned}$$

4.4 Orbital tactical decompositions

This section examines tactical decompositions and in particular the tactical decomposition given by the point and block orbits of an automorphism group acting on row-sum point-weight design. These results will be of particular use in sections 5.2 and 5.3 , where they will be used to derived results about square row-sum and point-sum point-weight designs.

Lemma 4.4.1 *If $\mathcal{S} = (V, \mathcal{B}, I, w)$ is a point-weight incidence structure and $G \leq Aut\ \mathcal{S}$ then the orbital decomposition of V and $\mathcal{B}$ with respect to G form a tactical decomposition and each point class only contains points of one weight.*

Proof It is well known (see [7]) that if $\mathcal{U}$ is an incidence structure and $G \leq Aut\,\mathcal{U}$ then the orbits of G form a tactical decomposition of $\mathcal{U}$. So if $\mathcal{S}$ is as point-weight incidence structure with underlying incidence structure $\mathcal{U}$ then the orbits of any group $G \leq Aut\ \mathcal{U}$ form a tactical decomposition of $\mathcal{U}$.

Therefore the orbits of any group $G \leq Aut\ \mathcal{S} \leq Aut\ \mathcal{U}$ form a tactical decomposition and, by definition of $Aut\ \mathcal{S}$, every orbit will only contain points of the same weight.

■

We find that it is convenient to use orbital decompositions. However all of the following theorems are applicable to any tactical decomposition for which all the points contained in a particular point class are of the same weight.

Definition 4.4.2 *Suppose $\mathcal{S} = (V, \mathcal{B}, I, w)$ is a point-weight incidence structure and that $G \leq Aut\ \mathcal{S}$. Let $V_1, \ldots, V_d$ be the point orbits of G and let $\mathcal{B}_1, \ldots, \mathcal{B}_e$ be the block orbits of G. We may define a function*

$$w : \{V_1, \ldots, V_d\} \to \mathbb{Z}^+$$

by

$$w(V_i) = w(x_i) \text{ where } x_i \in V_i.$$

Let r_{ji} be the number of blocks of $\mathcal{B}_j$ that are incident with any one point of V_i and let k_{ij} be the number of points of V_i that are incident with any one block of $\mathcal{B}_j$.

Define the matrices P, B, C, D by:

$$\begin{aligned} P &= diag(|V_1|, \ldots, |V_d|) \\ B &= diag(|\mathcal{B}_1|, \ldots, |\mathcal{B}_e|) \\ C &= [w(V_i)k_{ij}] \\ D &= [w(V_j)r_{ij}] \end{aligned}$$

Note that we will not differentiate between the two functions $w : V \to \mathbb{Z}^+$ and $w : \{V_1, \ldots, V_d\} \to \mathbb{Z}^+$ due to their similarity.

Lemma 4.4.3 *If $\mathcal{S} = (V, \mathcal{B}, I, w)$ is a point-weight incidence structure and $G \leq Aut\ \mathcal{S}$ then*

$$BC^T = DP.$$

Proof If we consider the incidence structure defined as having a point set V_i and the blocks defined by the blocks of $\mathcal{B}_j$ in the obvious way, then this structure is a $1 - (|V_i|, k_{ij}, r_{ji})$ classical design with $|\mathcal{B}_j|$ blocks. So, by (1.2.3), we have that

$$\begin{aligned} |\mathcal{B}_j|\ k_{ij} &= |V_i|\ r_{ji} \text{ and} \\ w(V_i)\ |\mathcal{B}_j|\ k_{ij} &= w(V_i)\ |V_i|\ r_{ji}. \end{aligned}$$

Hence $BC^T = DP$.

■

Lemma 4.4.4 *If $\mathcal{S} = (V, \mathcal{B}, I, w)$ is a $\pi_2 - (v, k, \lambda; W)$ point-weight design and $G \leq Aut\ \mathcal{S}$ then*

$$CD = \lambda PJ + diag(w(V_1)^2 r_1 - \lambda, \ldots, w(V_d)^2 r_d - \lambda)$$

where r_i is the number of blocks with which any single point of V_i incident.

Proof We examine the entry of CD in row i and column k. If $i \neq k$ then:

$$\begin{aligned} [CD]_{ik} &= \sum_{j=1}^{e} w(V_i)k_{ij}w(V_k)r_{jk} \\ &= w(V_i)w(V_k) \sum_{j=1}^{e} k_{ij}r_{jk}. \end{aligned}$$

Pick a point $y \in V_k$ and consider the sum

$$\sum_{x \in V_i} \iota_{\mathcal{B}_j}(\{x, y\})$$

where $\iota_{\mathcal{B}_j}(\{x, y\})$ is the number of blocks of $\mathcal{B}_j$ that contain both x and y. Now there are r_{jk} blocks of $\mathcal{B}_j$ that are incident with y and each of these blocks is incident with k_{ij} points of V_i. So the total number of flags (x, B) with $x, y \in B \in \mathcal{B}_j$ is $k_{ij} r_{jk}$ and so

$$\sum_{x \in V_i} \iota_{\mathcal{B}_j}(\{x, y\}) \quad = \quad k_{ij} r_{jk}.$$

Hence:

$$\begin{aligned}
w(V_i) w(V_k) \sum_{j=1}^{e} k_{ij} r_{jk} &= w(V_i) w(V_k) \sum_{j=1}^{e} \sum_{x \in V_i} \iota_{\mathcal{B}_j}(\{x, y\}) \\
&= w(V_i) w(V_k) \sum_{x \in V_i} \sum_{j=1}^{e} \iota_{\mathcal{B}_j}(\{x, y\}) \\
&= w(V_i) w(V_k) \sum_{x \in V_i} \iota(\{x, y\}) \\
&= w(V_i) w(V_k) \sum_{x \in V_i} \frac{\lambda}{w(V_i) w(V_k)} \\
&= \sum_{x \in V_i} \lambda \\
&= \lambda \left| V_i \right| .
\end{aligned}$$

The argument is very similar if $i = k$. We may repeat the process up to:

$$\begin{aligned}
w(V_i) w(V_k) \sum_{j=1}^{e} k_{ij} r_{jk} &= w(V_i) w(V_k) \sum_{x \in V_i} \iota(\{x, y\}) \\
&= w(V_i)^2 \Big(\iota(\{y, y\}) + \sum_{x \in V_i \setminus \{y\}} \iota(\{x, y\}) \Big) \\
&= w(V_i)^2 \Big(r_i + (\left| V_i \right| - 1) \frac{\lambda}{w(V_i)^2} \Big) \\
&= w(V_i)^2 r_i + \lambda (\left| V_i \right| - 1) \\
&= \lambda \left| V_i \right| + (w(V_i)^2 r_i - \lambda).
\end{aligned}$$

Therefore, for any appropriate i and k,

$$[CD]_{ik} \quad = \quad \lambda \left| V_i \right| + (w(V_i)^2 r_i - \lambda) \delta(i, k)$$

where δ is the Kronecker delta. So:

$$CD \quad = \quad \lambda P J + diag(w(V_1)^2 r_1 - \lambda, \ldots, w(V_d)^2 r_d - \lambda).$$

■

Combining these two results we get:

Corollary 4.4.5 *If* $\mathcal{S} = (V, \mathcal{B}, I, w)$ *is a* $\pi_2 - (v, k, \lambda; W)$ *point-weight design and* $G \leq Aut\ \mathcal{S}$ *then*

$$CBC^T = \lambda PJP + diag(w(V_1)^2 r_1 - \lambda, \ldots, w(V_d)^2 r_d - \lambda) \cdot P.$$

4.5 Conclusion

This chapter gave some of the basic theory of automorphism groups and although it concentrated mainly on row-sum point-weight designs the principles are applicable to all point-weight incidence structures.

It had already been shown that a $t - (v, k, \lambda; W)$ point-weight design has the same automorphism group as its underlying incidence structure. We exhibited a result of [6] which proved that this was true for almost all $\pi_t - (v, k, \lambda; W)$ point-weight designs too.

We developed the concept of a group acting on a set in such a way that it was t-homogeneous with respect to its orbital partition. This concept is an extension of the idea of a group acting on a set in such a way that it was t-homogeneous and if a group acts t-homogeneously on a set and we pick a suitable set of base blocks then it generates a classical design on that set. This is not the case with a group that acts on a set t-homogeneously with respect to its orbital partial but we derived a formula that allows us to calculate how many blocks a set of points lies upon.

We also examined the abstract concept of a group acting t-homogeneously with respect to its orbital partition and derived some interesting results. We found that, in a G-space, there exists a unique maximal subgroup that is t-homogeneous with respect to its orbital partition and that this subgroup was normal

Lastly we examined some of the matrix properties of the orbits of some group $G \leq Aut\ \mathcal{S}$. These result will be of particular interest in the next chapter.

Chapter 5

Dual and square designs

This chapter is motivated by the fact that a $2-(v,k,\lambda)$ design has $v=b$ if and only if the dual structure is also a classical design. Furthermore the incidence matrix of the dual structure is the transpose of the incidence matrix of the original classical design. It is therefore interesting to ask certain questions such as: "Does a point-weight design have any special properties if the number of points is equal to the number of blocks?" and "What sort of a structure does the transpose of the incidence matrix of a point-weight design define?".

5.1 Dual structures and Underlying Duals

We begin with the latter question and examine the transpose of the incidence matrix of a point-weight incidence structure.

Lemma 5.1.1 *Let $\mathcal{S}$ be a $(V,\mathcal{B},I,w)$ point-weight incidence structure and let M be an incidence matrix for $\mathcal{S}$. M^T is the incidence matrix for a point-weight incidence structure if and only if all the points lying on any block are of the same weight.*

Proof Suppose that there exists a block $B \in \mathcal{B}$ that contains two points of different weights. Consider the column of M that corresponds to the block B, that column contains two non-zero entries of different weights. Hence M^T contains a row which has two non-zero entries of different weights, but in a point-weight incidence matrix all the non-zero entries in a row are equal. So M^T is not the incidence matrix of a point-weight design.

However if $\mathcal{S}$ only has blocks which contains points of equal weights then the rows of M^T all contain non-zero entries of the same weight and so define a point weight incidence structure.

■

Lemma 5.1.2 *If M_1 and M_2 are both incidence matrices for a point-weight incidence $\mathcal{S}$ and no block of $\mathcal{S}$ contains two points of different weights then M_1^T and M_2^T are both incidence matrices of the same point-weight incidence structure.*

Proof Since no block of $\mathcal{S}$ contains two points of different weights we know that both M_1^T and M_2^T both define point-weight incidence structures.

Since M_1 and M_2 are both incidence matrices for the same point-weight incidence structure there exists permutation matrices P and Q such that $M_2 = PM_1Q$. Therefore $M_2^T = Q^T M_1^T P^T$ and the two matrices are incidence matrices for the point-weight incidence structure as P^T and Q^T are also permutation matrices.

■

We are now in a position where we may define the dual of a point-weight incidence structure.

Definition 5.1.3 *Let $\mathcal{S}$ be a point-weight incidence structure with an incidence matrix M. If there exists no block of $\mathcal{S}$ that contains two points of different weights then we define dual$(\mathcal{S})$ to be the structure defined by the point-weight incidence matrix M^T. If there exists a block of $\mathcal{S}$ that contains two points of different weights then dual$(\mathcal{S})$ does not exist.*

The previous two lemmas show that this definition is independent of which incidence matrix was chosen. We may now examine the duals of point-weight designs.

Lemma 5.1.4 *If $\mathcal{S}$ is a $t-(v,k,\lambda;W)$ point-weight design with $|W|\geq 2$ and dual$(\mathcal{S})$ is the dual structure of $\mathcal{S}$ then $t=1$ and dual$(\mathcal{S})$ is not a $t'-(v',k',\lambda';W)$ point-weight design for any $t'>0$.*

Proof If $t \geq 2$ then for any two points there exists at least one block that contains both of them and so, by definition, $dual(S)$ does not exist. Hence, in this case, we must have that $t=1$ and no block of $\mathcal{S}$ contains two points of different weights.

Consequently it is possible to partition the point set of $\mathcal{S}$ into components such that each component has the underlying incidence structure of a $1-(u_i,k_i,\lambda)$ design, an incidence matrix A_i and there exists no block that is incident with points of more than one component. Hence M must be of the form:

$$\begin{bmatrix} w_1A_1 & 0 & \dots & 0 \\ 0 & w_2A_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & w_nA_n \end{bmatrix}$$

Now suppose that $dual(\mathcal{S})$ is a $t' - (v', k', \lambda'; W)$ point-weight design. Obviously $dual(dual(\mathcal{S})) = \mathcal{S}$ so $dual(S)$ must have the property that any block only contains points of equal weights and so $t' = 1$.

Let B be a block of $\mathcal{S}$ that is incident with the point x. Since every point incident with B must have the same weight B must contain $\frac{k}{w(x)}$ points. This means that in $dual(\mathcal{S})$, B corresponds to a point which is incident with $\frac{k}{w(x)}$ blocks. However in $dual(\mathcal{S})$ every point is incident with λ' blocks and so $\lambda' = \frac{k}{w(x)}$ for all points x of $\mathcal{S}$, which is a contradiction as $|W| \geq 2$.

■

Lemma 5.1.5 *If $\mathcal{S}$ is a $t - (v, k, \lambda; W)$ point-weight design with $|W| \geq 2$ and $dual(\mathcal{S})$ is the dual structure of $\mathcal{S}$ then $t = 1$ and $dual(\mathcal{S})$ is not a $\pi_{t'} - (v', k', \lambda'; W)$ point-weight design for any $t' \geq 1$.*

Proof Again we note that since $dual(\mathcal{S})$ exists and $dual(dual(\mathcal{S})) = \mathcal{S}$ there are no blocks in $\mathcal{S}$ or $dual(\mathcal{S})$ that contain two points of different weights. Hence $t = 1$ and if $dual(\mathcal{S})$ is a $\pi_{t'} - (v', k', \lambda'; W)$ point-weight design then $t' = 1$.

Assume that $dual(S)$ is a $\pi_1 - (v', k', \lambda'; W)$ design. Every point in $\mathcal{S}$ is incident with exactly λ blocks, so every block in $dual(\mathcal{S})$ is incident with exactly λ points. This means that the sum of the weights of the points on any block is $k' = \lambda w$ where w is the unique weight of the points on that block. This is not constant if $|W| \geq 2$, hence $dual(\mathcal{S})$ is not a $\pi_1 - (v', k', \lambda'; W)$ point-weight design.

■

Since $dual(dual(\mathcal{S})) = \mathcal{S}$ the above lemma also shows that there exists no $\pi_t - (v, k, \lambda; W)$ point-weight design with $|W| \geq 2$ whose dual structure is a $t - (v, k, \lambda; W)$ point-weight design. Hence it only remains to show:

Lemma 5.1.6 *Suppose $\mathcal{S}$ is a $\pi_t - (v, k, \lambda; W)$ point-weight design with $|W| \geq 2$ and $dual(\mathcal{S})$ is the dual structure of $\mathcal{S}$. If $dual(\mathcal{S})$ is a $\pi_{t'} - (v', k', \lambda'; W)$ point-weight design then $t = 1$ and $dual(\mathcal{S})$ is, in fact, a $\pi_1 - (\frac{v\lambda}{k}, \lambda, k; W)$ point-weight design.*

Proof Since the $dual(\mathcal{S})$ exists we know that no block of $\mathcal{S}$ contains two points of different weights, hence $t = 1$. Now suppose $dual(\mathcal{S})$ is a $\pi_{t'} - (v', k'\lambda'; W)$ point-weight design, again since $dual(dual(\mathcal{S})) = \mathcal{S}$ exists we have that no block of $dual(\mathcal{S})$ contains two points of differing weights and so $t' = 1$.

Let x_B is a point of $dual(\mathcal{S})$ that corresponds to the block B in $\mathcal{S}$. Then x_B has weight equal to the unique weight of all the points that B contains. Let us denote this as $w(x_B)$ and assume that, since it should always be explicit which structure we are talking about, we hope this will not cause confusion to the reader. Note that $w(x_B) = w(z)$ for all points z in B.

Since each block B of $\mathcal{S}$ is incident with $\frac{k}{w(x_B)}$ points we have that each point x_B of $dual(\mathcal{S})$ is incident with $\frac{k}{w(x_B)}$ blocks. Hence a value $\lambda' = k$ is correct.

Suppose B_x is the block of $dual(\mathcal{S})$ that corresponds to the point x in $\mathcal{S}$. Since the point x is incident with $\frac{\lambda}{w(x)}$ blocks we have that the block B_x is incident with $\frac{\lambda}{w(x)}$ points, each of weight $w(x)$. Hence the sum of the weights of the points on any block is λ.

Therefore $dual(\mathcal{S})$ is a $\pi_1 - (v', \lambda, k; W)$ point-weight design. It therefore only remains to show that $v' = \frac{v\lambda}{k}$. We know from (5.1.1) that $\mathcal{S}$ is actually the disjoint union of n classical designs $\mathcal{S}_1, \ldots, \mathcal{S}_n$ where each point of $\mathcal{S}_i$ has weight w_i in $\mathcal{S}$ $(1 \leq i \leq n)$. Let $\mathcal{S}_i$ be a $1 - (u_i, k_i, \lambda_i)$ design where $k_i = \frac{\lambda}{w_i}$ and $\lambda_i = \frac{\lambda}{wi}$. By (1.2.3) we know that $\mathcal{S}_i$ contains $\frac{u_i \lambda_i}{k_i} = \frac{u_i \lambda}{k}$ blocks and in each of these contributes a weight w_i to v'. So,

$$\begin{aligned} v' &= \sum_{B \in \mathcal{B}} w(x_B) \\ &= \sum_{i=1}^{n} w_i \frac{u_i \lambda}{k} \\ &= \frac{\lambda}{k} \sum_{i=1}^{n} w_i u_i \\ &= \frac{\lambda}{k} v. \end{aligned}$$

So $dual(\mathcal{S})$ is a $\pi_1 - (\frac{v\lambda}{k}, \lambda, k; W)$ point-weight design.

■

So we see that the obvious definition of a dual structure is not very useful. In an attempt to alleviate this problem we define the following relationship between structures:

Definition 5.1.7 *Suppose $\mathcal{S}$ and $\mathcal{T}$ are point-weight incidence structures. $\mathcal{S}$ is an underlying dual of $\mathcal{T}$ if the underlying incidence structure of $\mathcal{T}$ is the dual of the underlying incidence structure of $\mathcal{S}$.*

This means that several non-equivalent point-weight incidence structures might be underlying duals of the same structure. However this mightn't be of much use as the following result of Röhmel shows. For further details see [3] and presumably [16].

Definition 5.1.8 *A (point-weight) incidence structure is called normal if any point is on at least one block but not all blocks and any block contains at least one point but not all points.*

Result 5.1.9 *If $\mathcal{S} = (V, \mathcal{B}, I)$ is a normal incidence structure for which there exists $s, t > 1$ and $\lambda, \mu > 0$ such that*

- *any t point are incident with exactly λ blocks*
- *and any s blocks intersect at exactly μ points*

then $\mathcal{S}$ is either a symmetric classical design or a degenerate projective plane (see (1.7.11)).

Hence we have that:

Theorem 5.1.10 *If $\mathcal{S}$ is a $t - (v, k, \lambda; W)$ point-weight design with $k < v$ and $|W| > 1$ then an underlying dual of $\mathcal{S}$ is a $s - (v', k', \lambda'; W')$ point-weight design if and only if $\mathcal{S}$ is a degenerate projective plane or possibly if $t = s = 1$ and the underlying incidence structure of $\mathcal{S}$ is a classical design.*

Proof Suppose $t = 1$ and consider the underlying incidence structure $\mathcal{U}$ of $\mathcal{S}$. Since each point of $\mathcal{U}$ is incident with the same number of blocks, we have that each block of $dual(\mathcal{U})$ is incident with the same number of points. Hence $dual(\mathcal{U})$ is a classical design, and in particular we have that each point of $dual(\mathcal{U})$ is incident with the same number of blocks. Thus every block of $\mathcal{U} = dual(dual(\mathcal{U})$ is incident with the same number of points and $\mathcal{U}$ is a classical design.

Next we note that if $k' = v'$ then every block of $dual(\mathcal{U})$ contains every point, which means that every point of $\mathcal{U}$ is incident with every block and so $k = v$, which is a contradiction. So we know that $k' < v'$ and therefore $s = 1$ by (1.7.8).

Therefore if s or t equals 1 then $s = t = 1$ and both $\mathcal{U}$ and $dual(\mathcal{U})$ are classical designs.

So if $t > 1$ then we must have $s > 1$. However in that case we may apply (5.1.9) and we have that $\mathcal{U}$ is either a degenerate projective plane or a symmetric classical design. If $\mathcal{U}$ is a symmetric classical design then $|W| = 1$ by (1.7.8) which is a contradiction, hence $\mathcal{S}$ is a degenerate projective plane.

■

5.2 Square designs

A square design is one in which there are as many points as blocks. The properties of a square classical design are mostly derived from the idea that given information about MM^T, which is relatively abundant, we may derive properties of M. In this sense it is sensible to examine the properties of square row-sum point-weight design, for which there is also a relatively large amount of information known about MM^T.

Definition 5.2.1 *A point-weight incidence structure is square if there are as many points as blocks (i.e. $u = b$).*

A trivial consequence of this is:

Lemma 5.2.2 *If $\mathcal{S}$ is a square $\pi_1 - (v, k, \lambda; W)$ point-weight design then $\lambda = k$ and w divides k for all $w \in W$.*

Proof Since $\mathcal{S}$ is square we have that $u = b$ but, by (2.1.4), we have that $u\lambda = bk$, hence $\lambda = k$. We also know, by (2.3.4), that for every $x \in V$ we have that $w(x)$ divides λ. Thus w divides $k = \lambda$ for all $w \in W$

■

However we are aiming for more substantial results on square designs. We start by giving two lemmas from [3] that need only trivial modification in order to be applicable to point-weight incidence structures. The first is a specific example of Brauer's Permutation Lemma.

Lemma 5.2.3 (Brauer's Permutation Lemma) *Suppose $\mathcal{S} = (V, \mathcal{B}, I, w)$ is a square point-weight incidence structure and $\alpha \in Aut\ \mathcal{S}$. If the incidence matrix of $\mathcal{S}$ is non-singular then the number of fixed points of α equals the number of fixed blocks.*

Proof Let M be the incidence matrix of $\mathcal{S}$ then the effect of α on M is to permute the rows and columns. So there exists two permutation matrices P and Q such that $PMQ = M$, where P describes the action of α on the points and Q describes the action of α on the blocks.

The number of fixed points of α is $trace(P)$ and, similarly, the number of fixed blocks of α is $trace(Q)$, but $Q = M^{-1}P^{-1}M$ so $trace(Q) = trace(P^{-1}) = trace(P^T) = trace(P)$.

■

Lemma 5.2.4 *Let $\mathcal{S} = (V, \mathcal{B}, I, w)$ be a square point-weight incidence structure with a non-singular incidence matrix M and let $G \leq Aut\ \mathcal{S}$ be an automorphism subgroup of $\mathcal{S}$. Then the number of point orbits of G is equal to the number of block orbits of G.*

Proof We have shown, (5.2.3), that for any $\alpha \in G$

$$fix_V(\alpha) = fix_{\mathcal{B}}(\alpha)$$

where $fix_U(\sigma)$ is equal to the number of elements of U that are fixed under the action of σ on U.

Let $o_U(G)$ denote the number of orbits of the set U when the groups G acts upon it. Then we have, by an application of Burnside's lemma:

$$\begin{aligned} o_V(G) &= \frac{1}{|G|}\sum_{\alpha \in G} fix_V(\alpha) \\ &= \frac{1}{|G|}\sum_{\alpha \in G} fix_{\mathcal{B}}(\alpha) \\ &= o_{\mathcal{B}}(G). \end{aligned}$$

■

Corollary 5.2.5 *If $\mathcal{S}$ is a nice, square $\pi_t - (v, k, \lambda; W)$ point-weight design and $G \leq Aut\ \mathcal{S}$ then $\mathcal{S}$ has as many point orbits under G as block orbits.*

We follow the lines of [7] to prove a result about the existence of square designs. The technique is based upon the results about Hilbert symbols (1.6.3) and Hasse symbols (1.6.5). We use the notation developed in (4.4.2), namely if $\mathcal{S} = (V, \mathcal{B}, I, w)$ is a point-weight incidence structure and $G \leq Aut\ \mathcal{S}$ then $V_1, \ldots, V_d$ are the point orbits of G and $\mathcal{B}_1, \ldots, \mathcal{B}_e$ are the block orbits of G. We let r_{ji} be the number of blocks of $\mathcal{B}_j$ that are incident with any one point of V_i, r_i be the number of blocks incident with any one point of V_i in total and let k_{ij} be the number of points of V_i that are incident with any one block of $\mathcal{B}_j$.

Lastly we define, as we did in 4.4, the matrices P, B, C, D by:

$$\begin{aligned} P &= diag(|V_1|, \ldots, |V_d|) \\ B &= diag(|\mathcal{B}_1|, \ldots, |\mathcal{B}_e|) \\ C &= [w(V_i)k_{ij}] \\ D &= [w(V_j)r_{ij}] \end{aligned}$$

Lemma 5.2.6 *Suppose there exists a matrix relation*

$$CBC^T = \lambda PJP + NP$$

where

- *B and P are non-singular diagonal matrices of the form given above,*
- *N is the non-singular diagonal matrix $diag(n_1, \ldots, n_d)$,*
- *C is a non-singular square matrix of the form given above,*
- *and λ is a non-zero constant.*

Then the matrices

$$P' = \left[\begin{array}{c|c} NP & 0 \\ \hline 0 & \lambda \end{array}\right]$$

and

$$B' = \left[\begin{array}{c|c} B & 0 \\ \hline 0 & \lambda n(\lambda \sum_{i=1}^{d} \frac{n|V_i|}{n_i} + n) \end{array}\right]$$

where $n = \prod n_i$, are congruent.

Proof Since

$$CBC^T = \lambda PJP + NP$$

we know there exists a non-singular matrix X such that

$$XNPX^T = B - \lambda XPJPX^T$$

Let us attempt to find a solution to the equation $YP'Y^T = B'$ where Y has the form:

$$Y = \left[\begin{array}{c|c} X & \underline{b} \\ \hline \underline{a} & c \end{array}\right]$$

Hence:

$$\begin{aligned} YP'Y^T &= \left[\begin{array}{c|c} XNPX^T + \lambda \underline{b}\,\underline{b}^T & XNP\underline{a}^T + c\lambda \underline{b} \\ \hline \underline{a}NPX^T + c\lambda \underline{b}^T & \underline{a}NP\underline{a}^T + c^2\lambda \end{array}\right] \\ &= \left[\begin{array}{c|c} B - \lambda XPJPX^T + \lambda \underline{b}\,\underline{b}^T & XNP\underline{a}^T + c\lambda \underline{b} \\ \hline \underline{a}NPX^T + c\lambda \underline{b}^T & \underline{a}NP\underline{a}^T + c^2\lambda \end{array}\right] \end{aligned}$$

Now there exists a non-singular matrix K such that

$$KJK^T = \left[\begin{array}{c|cccc} 1 & 0 & \ldots & 0 \\ \hline 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{array}\right]$$

so if $XPK^{-1} = [\alpha_{ij}]$ then setting $b_i = \alpha_{i1}$ gives $XPJPX^T = \underline{b}\,\underline{b}^T$ as $P = P^T$.

So we have two further matrix equations to solve:

$$XNP\underline{a}^T + c\lambda\underline{b} = 0, \tag{5.1}$$

$$\underline{a}NP\underline{a}^T + c^2\lambda = \lambda n(\lambda \sum_{i=1}^{d} \frac{n \mid V_i \mid}{n_i} + n). \tag{5.2}$$

We may re-arrange (5.1) as:

$$XNP\underline{a}^T = -c\lambda\underline{b}$$

and multiply it by its own transpose gives:

$$\begin{aligned} XNP\underline{a}^T\underline{a}PNX^T &= c^2\lambda^2\underline{b}\,\underline{b}^T \\ &= c^2\lambda^2 XPJPX^T. \end{aligned}$$

Therefore, and noting that NP=PN as both matrices are diagonal,

$$\underline{a}^T\underline{a} = c^2\lambda^2 N^{-1}JN^{-1}$$

which gives us that $a_i = \frac{c\lambda}{n_i}$ as $N^{-1} = diag(n_1^{-1}, \ldots, n_d^{-1})$.

Lastly, from (5.2), we have:

$$\begin{aligned} \underline{a}NP\underline{a}^T + c^2\lambda &= c^2\lambda + c^2\lambda^2 \sum_{i=1}^{d} \frac{|V_i|}{n_i} \\ &= c\lambda(c + \lambda \sum_{i=1}^{d} \frac{c\,|V_i|}{n_i}) \\ &= n\lambda(n + \lambda \sum_{i=1}^{d} \frac{n\,|V_i|}{n_i}) \end{aligned}$$

providing we set $c = n$. Hence the result is proven.

■

Corollary 5.2.7 *If $\mathcal{S}$ is a $\pi_2 - (v, k, \lambda; W)$ square point-weight design and $G \leq Aut\ \mathcal{S}$ then the matrices:*

$$P' = \left[\begin{array}{c|c} diag(n_1, \ldots, n_d)P & 0 \\ \hline 0 & \lambda \end{array}\right]$$

and

$$B' = \left[\begin{array}{c|c} B & 0 \\ \hline 0 & \lambda n(\lambda \sum_{i=1}^{d} \frac{n|V_i|}{n_i} + n) \end{array}\right]$$

where $n_i = w(V_i)^2 r_i - \lambda$ and $n = \prod_{i=1}^{d} n_i$ are congruent providing C is non-singular and $\mathcal{S}$ contains no awkward points.

Proof This is an obvious consequence of the previous lemma, (5.2.6), where $N = diag(n_1, \ldots, n_d)$ (this is non-singular as $\mathcal{S}$ contains no awkward points, hence $n_i \neq 0$ for all $1 \leq i \leq d$) and the matrix relation shown in (4.4.5).

■

Note that B' and P' are integer symmetric matrices and hence have the same Hasse symbol for all primes p providing that C is non-singular. It is therefore prudent to examine the determinant of C.

Lemma 5.2.8 *If $\mathcal{S} = (V, \mathcal{B}, I, w)$ is a $\pi_2-(v, k, \lambda; W)$ point-weight design. Consider the tactical decomposition given by the orbits of $G \leq Aut\ \mathcal{S}$ then:*

$$\begin{aligned} det(CD) &= \Big(1 + \lambda \sum_{i=1}^{d} \frac{|V_i|}{n_i}\Big) \prod_{i=1}^{d} n_i \\ &= \Big(1 + \lambda \sum_{x \in V} \frac{1}{n_x}\Big) \prod_{i=1}^{d} n_i. \end{aligned}$$

Proof Using basic theory of determinants we have that:

$$\begin{aligned} det(CD) &= det(\lambda PJ + diag(n_1, \ldots, n_d)) \text{ by } (4.4.4) \\ &= \prod_{i=1}^{d} |V_i|\ det(\lambda J + diag(\frac{n_1}{|V_1|}, \ldots, \frac{n_d}{|V_d|})) \\ &= \prod_{i=1}^{d} |V_i| \Big(1 + \lambda \sum_{i=1}^{d} \frac{|V_i|}{n_i}\Big) \prod_{i=1}^{d} \frac{n_i}{|V_i|} \text{ by } (1.4.4) \\ &= \Big(1 + \lambda \sum_{i=1}^{d} \frac{|V_i|}{n_i}\Big) \prod_{i=1}^{d} n_i \\ &= \Big(1 + \lambda \sum_{x \in V} \frac{1}{n_x}\Big) \prod_{i=1}^{d} n_i. \end{aligned}$$

■

Corollary 5.2.9 *If $\mathcal{S}$ is a nice $\pi_2 - (v, k, \lambda; W)$ point-weight design then CD is non-singular, i.e. $det(CD) \neq 0$.*

Proof If $\mathcal{S}$ is nice then one of the following three cases holds:

Case 1: $\mathcal{S}$ is neither awkward nor difficult

Hence $n_x > 0$ for all $x \in V$, hence a simple inspection of the determinant shows that it is greater than zero.

Case 2: $\mathcal{S}$ contains a single awkward point

So there exists a single point $x \in V$ such that $n_x = w(x)^2 r_x - \lambda = 0$. This point must form a singleton point class in the tactical decomposition, so let $V_1 = \{x\}$. We may expand the determinant in a manner similar to (2.4.1), thus:

$$det(CD) = \prod_{i=1}^{d} n_i + \lambda \sum_{i=1}^{d} \prod_{i \neq j=1}^{d} n_j$$

and the term $\prod_{j=2}^{d} n_j$ is non-zero in this sum. Hence CD is non-singular.

Case 3: $\mathcal{S}$ contains a single difficult point but is still nice

Since $\mathcal{S}$ is nice we must have that $1 + \lambda \sum_{x \in V} n_x$ was non-zero in the original calculation of $det(MM^T)$, where M is the point-weight incidence matrix associated with $\mathcal{S}$. We still have that $n_i \neq 0$ by (2.4.10) so CD is non-singular.

■

In fact unless $\mathcal{S}$ is difficult there always exists a group $G \leq Aut\ \mathcal{S}$ such that CD is non-singular as even if $\mathcal{S}$ is awkward with more than one awkward point then there exists a subgroup of $Aut\ \mathcal{S}$ with an orbital decomposition that has all the awkward points in one orbit. The proof of case 2 above shows that CD is non-singular here as there still exists at least one non-zero term in the sum. So, when G is chosen with care, the techniques below are applicable to all square designs except those that are both difficult and not nice.

Lemma 5.2.10 *If $det(CD) \neq 0$ then $det(C) \neq 0$*

Proof This is a trivial consequence of $det(CD) = det(C)det(D)$.

■

Putting all of these results together we find that:

Corollary 5.2.11 *If $\mathcal{S}$ is a nice, square $\pi_2 - (v, k, \lambda; W)$ point-weight design with no awkward points then $det(C) \neq 0$ and so P' is congruent to B'.*

and so,

Theorem 5.2.12 *If $\mathcal{S}$ is a nice, square $\pi_2 - (v, k, \lambda; W)$ point-weight design with no awkward points and a tactical decomposition that is the orbital partition of a group $G \leq Aut\ \mathcal{S}$ then*

$$(-1, \lambda\kappa\beta_d\nu_d)_p(\beta_d, \kappa)_p(\nu_d, \lambda)_p \prod_{1 \leq s < t \leq d} (|\mathcal{B}_s|, |\mathcal{B}_t|)_p \prod_{1 \leq s < t \leq d} (n_s\ |V_s|, n_t\ |V_t|)_p = 1$$

where

$$\begin{aligned}\beta_j &= \prod_{i=1}^{j} |\mathcal{B}_i| \ , \\ \nu_j &= \prod_{i=1}^{j} n_i \ |V_i| \quad \textit{and} \\ \kappa &= n\lambda(n + \lambda \sum_{i=1}^{d} \frac{n \ |V_i|}{n_i}).\end{aligned}$$

Proof Since P' and B' are congruent integer symmetric matrices they have the same Hasse symbol for all primes p.

$$\begin{aligned}H_p(B') &= (-1, -\beta_d\kappa)_p(\beta_d, -\beta_d\kappa)_p \prod_{j=1}^{d-1}(\beta_j, -\beta_{j+1})_p \\ &= (-1, -1)_p(-1, \beta_d\kappa)_p(\beta_d, -\beta_d)_p(\beta_d, \kappa)_p \prod_{j=1}^{d-1}(\beta_j, -\beta_{j+1})_p \\ &= (-1, -1)_p(-1, \beta_d\kappa)_p(\beta_d, \kappa)_p \prod_{j=1}^{d-1}(\beta_j, -\beta_{j+1})_p.\end{aligned}$$

Now

$$\begin{aligned}(\beta_j, -\beta_{j+1})_p &= (\beta_j, -\beta j)_p(\beta_j, |\mathcal{B}_{j+1}|)_p \\ &= (\beta_j, |\mathcal{B}_{j+1}|)_p \\ &= \prod_{s=1}^{j}(|\mathcal{B}_s|, |\mathcal{B}_{j+1}|)_p\end{aligned}$$

and so

$$H_p(B') = (-1, -1)_p(-1, \beta_d\kappa)_p(\beta_d, \kappa)_p \prod_{1\le s<t\le d}(|\mathcal{B}_s|, |\mathcal{B}_t|)_p.$$

Similarly,

$$\begin{aligned}H_p(P') &= (-1, -\nu_d\lambda)_p(\nu_d, -\nu_d\lambda)_p \prod_{j=1}^{d-1}(\nu_j, -\nu_{j+1}) \\ &= (-1, -1)_p(-1, \nu_d\lambda)_p(\nu_d, -\nu_d)_p(\nu_d, \lambda)_p \prod_{j=1}^{d-1}(\nu_j, -\nu_{j+1}) \\ &= (-1, -1)_p(-1, \nu_d\lambda)_p(\nu_d, \lambda)_p \prod_{j=1}^{d-1}(\nu_j, -\nu_{j+1}) \\ &= (-1, -1)_p(-1, \nu_d\lambda)_p(\nu_d, \lambda)_p \prod_{1\le s<t\le d}(n_s \ |V_s|, n_t \ |V_t|)_p\end{aligned}$$

but $H_p(B') = H_p(P')$, which is equivalent to saying $H_p(B')H_p(P') = 1$ as Hasse symbols can only take the values ± 1, and so,

$$\begin{aligned} 1 &= (-1,-1)_p(-1,\nu_d\lambda)_p(\nu_d,\lambda)_p(-1,-1)_p(-1,\beta_d\kappa)_p(\beta_d,\kappa)_p \\ &\qquad \cdot \prod_{1\le s<t\le d}(n_s\,|V_s|, n_t\,|V_t|)_p \prod_{1\le s<t\le d}(|\mathcal{B}_s|,|\mathcal{B}_t|)_p \\ &= (-1,\nu_d\lambda)_p(-1,\beta_d\kappa)_p(\beta_d,\kappa)_p(\nu_d,\lambda)_p \\ &\qquad \cdot \prod_{1\le s<t\le d}(n_s\,|V_s|, n_t\,|V_t|)_p \prod_{1\le s<t\le d}(|\mathcal{B}_s|,|\mathcal{B}_t|)_p \\ &= (-1,\lambda\kappa\beta_d\nu_d)_p(\beta_d,\kappa)_p(\nu_d,\lambda)_p \prod_{1\le s<t\le d}(n_s\,|V_s|, n_t\,|V_t|)_p \prod_{1\le s<t\le d}(|\mathcal{B}_s|,|\mathcal{B}_t|)_p. \end{aligned}$$

■

In particular:

Corollary 5.2.13 *If $\mathcal{S}$ is a nice, square $\pi_2-(v,k,\lambda;W)$ point-weight design with u points (none of which are awkward) then*

$$(-1,n\kappa\lambda)_p(n,\lambda)_p \prod_{1\le s<t\le u}(n_s,n_t)_p = 1$$

where

$$\kappa = n\lambda\Big(n+\lambda\sum_{i=1}^{u}\frac{n}{n_i}\Big)$$

and $n_1,\ldots,n_u$ are the orders of the points of $\mathcal{S}$.

Proof If we let $G = \{id\}$ then each point or block form there own point or block class. Hence

$$\begin{aligned} |V_i| &= 1 \text{ for all } 1\le i\le u \\ |\mathcal{B}_i| &= 1 \text{ for all } 1\le i\le u \\ \beta_i &= 1 \text{ for all } 1\le i\le u \\ \nu_u &= \prod_{j=1}^{u} n_j = n \end{aligned}$$

If we substitute these values into the previous equation and note that $(1,\gamma)_p = 1$ for all non-zero p-adic numbers γ then we obtain the result.

■

Corollary 5.2.14 *If $\mathcal{S} = (V, \mathcal{B}, I, w)$ is a nice, square $\pi_2 - (v, k, \lambda; W)$ point-weight design with the property that for any weight the number of points of that weight is divisible by four (except for maybe one weight for which the number of points of that weight is congruent to 1 modulo 4, the order of those points is square and those points are not awkward) then:*

$$(n + \lambda \sum_{x \in V} \frac{n}{n_x}, -1)_p = 1.$$

Consequently the equation:

$$\Big(n + \lambda \sum_{x \in V} \frac{n}{n_x}\Big) X^2 - Y^2 = Z^2$$

has a non-trivial integer solution for X,Y and Z provided det$(MM^T) > 0$, where M is an incidence matrix for $\mathcal{S}$.

Proof Suppose there exists u points of $\mathcal{S}$ and the orders of these points are $n_1, \ldots, n_u$. Suppose further that $|W|= d$ and that $V'_1, \ldots, V'_d$ is the partition of V into classes of points of the same weight. Let u_i be equal to the number of points in V'_i, hence u_i is divisible by four except for maybe in one case, say V'_1, where the $u_1 \equiv 1 (mod\ 4)$ points have square order n'_1 and none of these point is awkward.

Note that since $\mathcal{S}$ is nice it can contain at most one difficult point, by (2.4.8), or at most one awkward point, by (2.4.11). Moreover there can exist no other point with the same weight as the awkward or difficult point. Now, unless V'_1 is the set of points of minimal weight, the number of points with that are awkward/difficult must be divisible by four and so $V'_2, \ldots, V'_d$ does not contain any awkward points. Furthermore we know that V'_1 doesn't contain any awkward points, so $\mathcal{S}$ doesn't have any awkward points.

If the points x and y have the same weight then we know by (2.3.1) that $r_x = r_y$ and so $n_x = n_y$. Let n'_i be the order of the points of V'_i. So,

$$\begin{aligned} n &= \prod_{x \in V} n_x \\ &= \prod_{i=1}^{d} {n'_i}^{u_i} \\ &= {n'_1}^{u_1} \prod_{i=2}^{d} {n'_i}^{u_i}. \end{aligned}$$

Now u_i is even for each $2 \leq i \leq d$ so $\prod_{i=2}^{d} {n'_i}^{u_i}$ is square. Either u_1 is even or n'_1 is square, in either case ${n'_1}^{u_1}$ is a square number too. Hence n is a square and so $(n, \lambda)_p = (1, \lambda)_p = 1$ as we may disregard square factors.

Furthermore

$$\begin{aligned}
\prod_{1\le s<t\le u}(n_s,n_t)_p &= \prod_{1\le i<j\le d}(n'_i,n'_j)_p^{u_iu_j}\cdot\prod_{i=1}^{d}(n_i,n_i)_p^{\binom{u_i}{2}} \\
&= \prod_{1\le i<j\le d}(n'_i,n'_j)_p^{u_iu_j}\cdot\prod_{i=1}^{d}(n_i,n_i)_p^{\frac{1}{2}u_i(u_i-1)} \\
&= 1
\end{aligned}$$

as $(n_i,n_j)_p$ can only take the values 1 or -1, and u_i is divisible by four for $2\le i\le d$ and either u_1 or u_1-1 is divisible by four. If we apply this to (5.2.13) then we get:

$$(-1,n\kappa\lambda)_p = 1$$

which is the same as

$$(-1,n^2\lambda^2(n+\lambda\sum_{x\in V}\frac{n}{n_x}))_p = 1$$

and

$$(-1,(n+\lambda\sum_{x\in V}\frac{n}{n_x}))_p = 1$$

as we may disregard square factors.

Now the expression $n+\lambda\sum_{x\in V}\frac{n}{n_x}$ is equal to $det(MM^T)$ and we have assumed that $det(MM^T)>0$. So we may attempt to apply (1.6.8) to the equation

$$\left(n+\lambda\sum_{x\in V}\frac{n}{n_x}\right)X^2-Y^2=Z^2$$

We have already shown that this has a solution in $\mathbb{Q}_p$ for all primes p so it remains to show that this equation has a solution in $\mathbb{R}$. However if $det(MM^T)>0$ then the above equation has the solution

$$X=\frac{1}{\sqrt{n+\lambda\sum_{x\in V}\frac{n}{n_x}}}\quad Y=0\quad Z=1$$

in $\mathbb{R}$. So the equation has a solution in $\mathbb{Z}$.

■

5.3 Point-sum point-weight designs

The last section concentrated on row-sum point-weight designs. However any of the techniques used are applicable to point-sum point-weight designs because of the following result:

Result 5.3.1 *If $\mathcal{S}$ is a $2-(v,k,\lambda;W)$ point-sum point-weight design with u points where the ith point is incident with r_i blocks and $M_{\mathcal{U}}$ is the incidence matrix of the underlying incidence structure then*

$$M_{\mathcal{U}}M_{\mathcal{U}}^T = \lambda(J-I) + diag(r_1,\ldots,r_u).$$

Hence there exists an inherent connection between the underlying incdience structures of point-sum point-weight designs and row-sum point-weight designs. We may now examine square point-weight designs in a manner similar to the last section. Since we will be using both the incidence matrix of $\mathcal{S}$ and the incidence matrix of the underlying incidence structure of $\mathcal{S}$, $\mathcal{U}$, we will use subscripts to differentiate between them in what we hope will be an obvious manner.

Lemma 5.3.2 *If $\mathcal{S}$ is a square $t-(v,k,\lambda;W)$ point-weight design with a non-singular incidence matrix $M_{\mathcal{S}}$ and $G \leq Aut\ \mathcal{S}$ is an automorphism subgroup of $\mathcal{S}$, then $\mathcal{S}$ has as many point orbits under G as block orbits.*

Proof This is a direct consequence of (5.2.4).

■

Result 5.3.3 *Suppose $\mathcal{S}$ is a square $t-(v,k,\lambda;W)$ point-weight design and let $\mathcal{U}$ be the underlying incidence structure of $\mathcal{S}$. Suppose $M_{\mathcal{U}}$ is a non-singular incidence matrix of $\mathcal{U}$ and that $G \leq Aut\ \mathcal{U}$. Then $\mathcal{U}$ has as many point orbits under G as block orbits.*

Proof This is a known result of classical design theory shown in [3], [12] and [7]. It can also be seen as a corollary of the previous lemma as $Aut\ \mathcal{S} = Aut\ \mathcal{U}$ and an incidence matrix for $\mathcal{S}$ is singular if and only if the incidence matrix for $\mathcal{U}$ is singular.

■

Suppose that $\mathcal{S}$ is a square $t-(v,k,\lambda;W)$ point-weight design with a non-singular incidence matrix $M_{\mathcal{S}}$ and $G \leq Aut\ \mathcal{S}$ is an automorphism subgroup of $\mathcal{S}$. Suppose that the point orbits of G are $V_1,\ldots,V_d$ and that the block orbits are $\mathcal{B}_1,\ldots,\mathcal{B}_e$. Suppose further that any block of $\mathcal{B}_j$ is incident with k_{ij} points of V_i and that any point of V_i is incident with r_{ji} blocks of $\mathcal{B}_j$. It is shown in [7] that if

$$\begin{aligned}
P &= diag(|V_1|,\ldots,|V_d|)\\
B &= diag(|\mathcal{B}_1|,\ldots,|\mathcal{B}_e|)\\
C &= [k_{ij}]\\
D &= [r_{ij}]
\end{aligned}$$

then the matrix relation

$$CBC^T = \lambda PJP + diag(r_1 - \lambda, \ldots, r_d - \lambda) \cdot P$$

holds, where r_i is the number of blocks with which a point of V_i is incident.

Again we will use the idea of the order of a point and let $n_x = r_x - \lambda$. This should be thought of as the order of the point x in $\mathcal{U}$ rather than the order of the point in $\mathcal{S}$ which could be more consistently be defined as $w(x)^2 r_x - \lambda$. We note that if $n_x = 0$ for any point x then we have that $r_x = \lambda$. Consequently, if $t > 1$, then for any other point z we have that there exists at least $r_x = \lambda$ blocks that contain both x and z. So every block that contains x also contains z but our choice of z was arbitrary so we know that any block that contains x must contain every point. Hence $k = v$. Therefore the matrix $diag(n_1, \ldots, n_d)$ is non-singular provided $t > 1$ and $k < v$, if either of these conditions fail then this matrix is the zero matrix.

Let n_i be the order of any point of V_i and $n = \prod_{i=1}^{d} n_i$ as before. So

Lemma 5.3.4 *If $\mathcal{S}$ is a square $2-(v,k,\lambda;W)$ point-weight design with $k < v$ and a tactical decomposition which is the orbital partition of a group $G \leq Aut\ \mathcal{S}$ then the matrices*

$$P' = \left[\begin{array}{c|c} diag(n_1, \ldots, n_d)P & 0 \\ \hline 0 & \lambda \end{array}\right]$$

and

$$B' = \left[\begin{array}{c|c} B & 0 \\ \hline 0 & \lambda n(\lambda \sum_{i=1}^{d} \frac{n|V_i|}{n_i} + n) \end{array}\right]$$

are congruent providing C is non-singular.

Proof This is a direct corollary of (5.2.6) where $N = diag(n_1, \ldots, n_d)$ and using the matrix relation given above. N is non-singular as $k < v$.

■

Unfortunately we do not have the same information about point-sum point-weight designs that we do about row-sum point-weight designs so we cannot reduce the condition that C must be non-singular to any specific properties of the structure $\mathcal{S}$. However as long as we remember that proviso we have that:

Theorem 5.3.5 *If $\mathcal{S}$ is a square $2-(v,k,\lambda;W)$ point-weight design with $k < v$, a tactical decomposition that is the orbital partition of a group $G \leq Aut\ \mathcal{S}$ and suppose that C is non-singular for that tactical decomposition then:*

$$(-1, \lambda\kappa\beta_d\nu_d)_p(\beta_d, \lambda)_p(\nu_d, \kappa)_p \prod_{1\leq s<t\leq d} (|\mathcal{B}_s|, |\mathcal{B}_t|)_p \prod_{1\leq s<t\leq d} (n_s\,|V_s|, n_t\,|V_t|)_p = 1$$

where

$$\begin{aligned}\beta_j &= \prod_{i=1}^{j} |\mathcal{B}_i| \\ \nu_j &= \prod_{i=1}^{j} n_i\, |V_i| \\ \kappa &= n\lambda(n + \lambda \sum_{i=1}^{d} \frac{n\, |V_i|}{n_i})\end{aligned}$$

This is proven in exactly the same way as (5.2.12).

Corollary 5.3.6 *If $\mathcal{S}$ is a square $2-(v,k,\lambda;W)$ point-weight design with u points and $k < v$ then:*

$$(-1, n\kappa\lambda)_p (n,\kappa)_p \prod_{1\le s<t\le u} (n_s, n_t)_p \quad = \quad 1$$

where

$$\kappa \quad = \quad n\lambda\Big(n + \lambda \sum_{i=1}^{u} \frac{n}{n_i}\Big)$$

and $n_1, \ldots, n_u$ are the orders of the points of $\mathcal{S}$.

Proof This is proven in exactly the same way as (5.2.13). We note, however, that in this case C is the incidence matrix of the underlying incidence structure of $\mathcal{S}$, $M_{\mathcal{U}}$. Hence the condition that C be non-singular is the same as the condition that $det(M_{\mathcal{U}}) \neq 0$. However it is shown in [9] that $rank(M_{\mathcal{S}}) = u$ for all point-sum point-weight designs with $k < v$ and $t > 1$. So $det(M_{\mathcal{U}}) \neq 0$ because $M_{\mathcal{S}}$ is a $u \times u$ matrix of rank u and $M_{\mathcal{U}}$ is singular if and only if $M_{\mathcal{S}}$ is singular.

■

So finally we have that

Corollary 5.3.7 *If $\mathcal{S}$ is a square $2-(v,k,\lambda;W)$ point-weight design with $k < v$ and the property that for any weight the number of points of that weight is divisible by four (except for maybe one weight for which the number of points of that weight is congruent to 1 modulo 4 and the order of those points is square) then:*

$$(n + \lambda \sum_{x\in V} \frac{n}{n_x}, -1)_p = 1$$

Consequently the equation:

$$\Big(n + \lambda \sum_{x\in V} \frac{n}{n_x}\Big) X^2 - Y^2 = Z^2$$

has a non-trivial integer solution for X,Y and Z.

Proof This is proven in exactly the same way as (5.2.14). We may disregard the condition that $det(M_{\mathcal{U}}M_{\mathcal{U}}^T)$ be greater than zero, where $M_{\mathcal{U}}$ is an incidence matrix for the underlying incidence structure, because by (5.3.1) and (1.4.4) we have

$$\det(M_{\mathcal{U}}M_{\mathcal{U}}^T) = \left(1 + \lambda \sum_{j=1}^{u} \frac{1}{r_j - \lambda}\right) \prod_{j=1}^{u} (r_j - \lambda)$$

and this is always greater than zero here because $r_j > \lambda$.

■

5.4 Conclusion

In classical design theory there is a link between classical designs that admit a dual design and square designs however we have been unable to prove such a link exists for point-weight designs. In fact we have not been able to satisfactorily extend the concept of a dual to the field of point-weighted designs. We have shown that the most obvious definition (5.1.1) is almost completely useless for all point-weight designs. We attempted to improve this definition in (5.1.7) but our success was limited: we have shown that very few point-sum point-weight designs can admit an underlying dual and we could not provide any examples of row-sum point-weight designs that admit an underlying dual either.

Our study of square point-weight incidence structures was more successful. Drawing on, amongst others, the work of section 4.4 we managed to find non-existence results for square row-sum and point-sum point-weight designs based on the solubility of equations in p-adic number fields, subject to certain matrix properties. We also managed to extend these to a non-existence result based on the solubility of a Diophantine equation if we are allowed to control the number of points of each weight there are.

Unfortunately we have not been able to exhibit any examples of square row-sum point-weight designs and consequently their existence is still an open problem.

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