Decoherent Histories and Quantum Maps

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Abstract

Discrete maps play an important role in the investigation of dynamical features of complex classical systems, especially within the theory of chaos. Similarly, *quantum maps* have proven to be a very useful mathematical tool within the study of complex quantum dynamical systems. Using the *decoherent histories formulation* of quantum mechanics we consider a particular framework for studying quantum maps which is motivated by the method of classical symbolic dynamics. Symbolic dynamics is known to be a very powerful method specifically invented for the purpose of representing classical dynamical systems by a discrete model that is suitable for information theoretic studies. Our framework uses the *decoherent histories formalism* which, similarly to classical symbolic dynamics, allows one to introduce information theoretic quantities with respect to system dynamics. Our research within this framework can be viewed as a contribution towards the development of a general theory of "quantum symbolic dynamics".

We start by considering a special but very important example for a quantum map: the *quantum baker's map*, invented for the theoretical investigation of quantum chaos. Here we use the decoherent histories formalism to examine the coarse-grained evolution of this map with regard to the question of how classical predictability of the evolution depends on the character of coarse-graining. We demonstrate that hierarchical coarse-grainings display interesting features with respect to this question.

We proceed with the investigation of decoherence properties of arbitrary unitary quantum maps. A number of interesting results is obtained within the framework of arbitrarily long histories constructed from a fixed projective partition of a finite dimensional Hilbert space. In particular, we derive simple necessary decoherence conditions, which employ only a single iteration of a given unitary quantum map. Furthermore, a surprising result is obtained with regard to the fundamental question of how the choice of the initial state affects decoherence of histories. Within the considered framework we show that if decoherence is established for arbitrary history lengths and all initial

states from the smallest natural set of states that can be associated with the framework, then we get decoherence of such histories for arbitrary initial states. Finally, we make first steps towards proving analogous results for approximate decoherence and suggest various questions for future research.

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Preface

The work presented in this thesis was carried out in the Department of Mathematics at Royal Holloway, University of London, under the supervision of Prof. Rüdiger Schack. This thesis has not been submitted for a degree at any other university.

The research presented in Chap. 4 and Chap. 5 was done in collaboration with my colleagues Andrei Soklakov and Rüdiger Schack.

The results presented in Chap. 5 have been published in [1] , [2] and [3]. The Lemma on uniform quantum recurrence proven in Sec. 5.2 has also been applied in a different context [4]. A journal publication [5] based on results of Chap. 4 is in preparation.

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Chapter 1

Introduction and Outline

Most physical and mathematical problems are usually formulated in the real or complex field. It has become a common practice to use continuous mathematical models to describe physical systems. Continuous spaces are conveniently employed to represent the state space of a system. This seems to be natural for most of physical systems. And it is very convenient: problems formulated in a continuous language can be tackled by utilizing the very convenient and powerful methods of differential and integral calculus. In contrast to that, a discrete description of the system in question is much more appropriate when investigating *information-theoretic* properties. Information-theoretic methods often require, or at least make it desirable, the dynamical system to be represented by a discrete model. The setting of a theory of complexity of dynamical systems is substantially facilitated if it is implemented within a discrete framework. Moreover, most dynamical problems have no known analytic solutions, and thus can be tackled only by means of numerical methods. Indeed, numerical simulations have become a very powerful tool for exploring complex dynamical systems. In a computer simulation all physical quantities have to be replaced by discrete counterparts. This naturally suggests investigations of discrete dynamical models, mathematically represented in terms of *discrete maps*. On the other hand, discrete patterns actually do in fact occur in many relevant physical, chemical and biological systems as well as abstract mathematical models. Examples are magnetical systems, crystals, DNA-chains, and cellular automata. We do not want, here, to address the issue whether discrete or continuous descriptions are more fundamental. This issue is certainly highly debatable, but it is not the subject of this thesis. Rather, here we are interested in developing mathematical methods which are designed for the purpose of transforming a continuous description into a discrete symbolic form. In fact, within the theory of classical dynamical systems, a discrete representation of a given continuous nonlinear system can often be accomplished without loss of relevant information about the dynamics. The general method aimed at achieving this is known as *"symbolic dynamics"* (see [6, 7] and references therein). The method of symbolic dynamics was specifically invented for the purpose of representing classical dynamical systems by a discrete model that is suitable for information-theoretic studies. The basic concepts within the theory of symbolic dynamics are *discrete maps* representing the dynamics after time has been suitably discretized, and a *partitioning of the state space* into "cells" labeled by the letters from an alphabet $\mathbb{A}$. Using these two building blocks the notion of a *coarse-grained trajectory* can be introduced: we list a sequence of symbols corresponding to the labels of the cells visited by the system at times $j = 1, 2, 3\ldots$ during its evolution which started from some initial state. Such a list of symbols, labeling the sequence of cells successively visited by the system, can be viewed as a *coarse-grained trajectory* or, to use another term which later will also be used in the case of quantum dynamical systems, a *coarse-grained history.* By extending these coarse-grained histories infinitely far into the past and into the future it is often possible, by a careful choice of the partition of the state space, to obtain a symbolic description, which retains all the relevant information about the dynamical features of a given dynamical system. For a large class of complex classical dynamical systems such a symbolic representation of the dynamics without loss of relevant information can in fact be accomplished (see [6, 7] and references therein).

The method of symbolic dynamics has proven to be a very powerful tool within the theory of classical dynamical systems. Symbolic dynamics techniques proved extremely useful especially for studying classical chaos. Very convincing *information-theoretic characterizations of chaos* [6, 8, 9] have been proposed using this framework; we will briefly address them in Sec. 2.1 of the next chapter. It has now become a prevailed belief that an information-theoretic approach is also very fruitful with regard to defining and characterizing quantum chaos (see [10] and references therein), after it had been realized that a characterization of quantum chaos in terms of *"sensitivity to initial conditions"* is not possible, due to the unitary nature of quantum evolution of closed quantum systems. Such an information-theoretic approach to quantum chaos would be considerably facilitated if a likewise very powerful symbolic method were available for the quantum case. It is therefore a very desirable goal to develop an analogous symbolic description method in quantum mechanics, which we would call *"quantum symbolic dynamics"*. A certain amount of work has been done into this direction [11, 12, 13, 14, 15]. However, a general framework of "quantum symbolic dynamics" has not yet been established.

The *consistent* (*decoherent*) *histories formalism* of quantum mechanics [16, 17, 18, 19, 20] provides a quantum-theoretic framework which in a sense resembles the classical symbolic representation of system dynamics. Indeed, the decoherent histories formalism, which we expand on below, has very similar building blocks and concepts as the method of classical symbolic dynamics, which we briefly introduced above. It uses the language of coarse-grained (quantum) histories, which in a way resemble the symbolic sequences within the framework of classical symbolic dynamics. A discretization of time naturally arises within the quantum histories framework. A quantum history is typically defined to be a *time-ordered sequence* of quantum-mechanical "propositions" [1], i.e., a sequence of quantum events at a succession of times [2]. The time-evolution between these successive quantum events is conveniently represented — if using the Schrödinger picture — by unitary *quantum maps*. This is exactly how quantum maps emerge from a continuous evolution of a closed quantum system. More-

[1]Mathematically, quantum-mechanical propositions are represented by projectors (see Sec. 2.3). In particular, an exhaustive set of mutually exclusive propositions corresponds to a complete set of mutually orthogonal projectors.

[2]It should here be noted, however, that the framework of consistent histories is general enough to cope with situations where no distinguished external time parameter is available, such as, e.g., in the case of quantum gravity where no privileged set of space-like hyper-surfaces exists. In quantum gravity coarse-grained histories are based on coarse-grainings involving spacetime domains [21].

over, the very basic concept within the quantum histories framework is likewise the concept of coarse-graining. Histories are constructed from coarse-grained projective *partitions* [3] *of the state space* of the system, which is, in quantum mechanics, represented by a Hilbert space [4]. The histories formalism of quantum mechanics is very general, it allows to use different projective partitions of the Hilbert space to represent exhaustive sets of mutually exclusive propositions (events) at different moments of time within the sequence. A very natural way, however, to construct quantum histories is to use the *same* partition of the Hilbert space for all times within the sequence. In this case, the similarity to the method of classical symbolic dynamics becomes most evident. We can label the partition elements of the projective partition of the Hilbert space by letters from some alphabet $\mathbb{A}$. The sequence of quantum events which arises in the course of the quantum evolution can then be labeled by a sequence of symbols — the sequence of symbols corresponding to the labels of the Hilbert space domains (that are associated with the projectors from the partition) "visited" [5] by the system at times $j = 1, 2, 3 \ldots$ during its evolution which started from some initial state. To complete the analogy with the method of classical symbolic dynamics it is then tempting to extend the coarse-grained histories infinitely far into the future (and maybe also into the past) and in this way obtain infinitely long symbolic sequences.

In classical symbolic dynamics information theoretic quantities arise in a very natural way. The method of symbolic dynamics was deliberately developed specifically for this very purpose. Examples include *topological entropy* and *Kolmogorov-Sinai entropy* [6], which constitute measures for the characterization of the complexity of dynamical systems. We will briefly introduce them in Sec. 2.1.5.

Information-theoretic quantities are usually related to probabilities. For example, the Kolmogorov-Sinai entropy defined by means of classical symbolic dynamics is

[3] A projective partition of a Hilbert space is defined to be a complete set of mutually orthogonal projectors on that space (see Chap. 3).

[4] By "state space", here, I mean, of course, only the space of pure states of the system and not the convex set of all possible density operators on the Hilbert space.

[5] In contrast to classical mechanics, however, the "visits" in quantum mechanics are to be understood as *"possible visits"* as opposed to actual *"dynamical visits"* which would in quantum mechanics be interpreted as *"quantum jumps"*.

based on a probability distribution over symbolic sequences [6] (see Sec. 2.1.5). In the very similar fashion we would also like to have probability distributions over quantum histories. However, quantum mechanics prevents us from assigning probabilities to arbitrary sets of quantum histories. Due to quantum interference, one cannot always assign probabilities to a set of histories in a consistent way. For this to be possible, the set of histories must be decoherent. Decoherence of histories is the most fundamental requirement in the quantum histories approach. For, only then the corresponding histories of the system may be assigned probabilities obeying the standard probability sum rules. Only sets of histories that decohere are meaningful within the decoherent (consistent) histories formalism. Sets of histories that do not decohere have no predictive content and are therefore regarded as meaningless [19].

A lot of work has been done analyzing various decoherence conditions in various settings. At one end of the spectrum of the research we have the general development of the decoherent histories formalism and at the other end the analysis of specific physical systems. The research presented in this thesis lies in the middle of this spectrum. Here we use the decoherent histories formalism of quantum mechanics to provide a framework for studying dynamical features of quantum maps. Our framework is motivated by analogies with the method of classical symbolic dynamics. [6] The setting of our framework (cf. Chap. 3) is defined in such a way as to resemble the construction of a symbolic dynamics for classical dynamical systems. In particular, we use a fixed partition of the Hilbert space for all time steps in our histories framework, in analogy with the fact, that in the theory of classical dynamical systems a fixed partition of the classical phase space is the starting point for introducing a symbolic description. Furthermore, we are interested in considering very long histories, and even arbitrarily long histories extended infinitely far into the future. This, too, is motivated by classical symbolic dynamics, which studies infinitely long symbolic sequences. The research within our framework can therefore be viewed as a contribution towards the

[6] In fact, this close analogy has been one of our major motivations for making use of the decoherent histories approach in order to investigate traits of quantum dynamical systems. Examining features of the latter in a similar way as is done for classical dynamical systems using symbolic dynamics techniques would be very desirable.

development of a general theory of "quantum symbolic dynamics".

We start our studies by considering a special but very important example for a quantum map: the *quantum baker's map*, which was specifically invented for the theoretical investigation of quantum chaos. We use our framework within the decoherent histories formalism to examine the coarse-grained behaviour of this map. A family of different coarse-grained descriptions is introduced. We characterize a coarse-grained description within this family by the *number of scales* that are coarse-grained over in the symbolic representation of the quantum baker's map and the *extent* of coarse-graining at every such scale. We establish approximate decoherence for all members within this family of different coarse-grained descriptions. Having established decoherence, we may assign probabilities to the histories. We calculate the probability distributions over sets of coarse-grained histories for each coarse-grained description within the family. A probability distribution allows us to define an entropy measure, which can be used to characterize the predictability of the coarse-grained evolution. Using this entropy measure we examine the issue of how classical predictability of the evolution depends on the character of coarse-graining. We show that the short-time entropy production is determined by the number of scales at which information is lost rather than the extent of coarse-graining on any particular scale. The duration of the short-time regime, however, is determined by the extent of coarse-graining. Multi-scale coarse-grainings display a significantly more unpredictable evolution than 1-scale coarse-grainings with the same degree of prior knowledge.

We then proceed with the investigation of decoherence properties of arbitrary unitary quantum maps. In general it is a very difficult task to decide whether a given set of histories is decoherent. Checking decoherence of very long histories is a particularly difficult problem. With increasing length of the histories, checking decoherence by means of the so-called decoherence functional [19] soon becomes extremely cumbersome. This is especially true when the system dynamics is difficult to simulate as, e.g., in the case of chaotic quantum maps, for which typically only the first iteration is known in closed analytical form. The need of simpler decoherence conditions for checking decoherence of histories for quantum maps has been one of the major motivations for the research presented in this thesis. A number of interesting results has been obtained within the framework of arbitrarily long histories constructed from a fixed projective partition of a finite dimensional Hilbert space. In particular, we derive simple necessary decoherence conditions, which employ only a single iteration of a given unitary quantum map. Furthermore, a surprising result is obtained with regard to the fundamental question of how the choice of the initial state affects decoherence of histories. Within the considered framework we show that if decoherence is established for arbitrary history lengths and all initial states from the smallest natural set of states that can be associated with the framework, then we get decoherence of such histories for arbitrary initial states. These results have been obtained for exact decoherence of histories. In most of physical models, however, decoherence of histories can be established only *approximately* (cf. e.g. [19]). It is therefore desirable to prove analogous results for approximate decoherence. At the end of the thesis we make first steps towards this generalization.

Outline

This thesis is organized as follows.

Chapter 2 provides the background for this thesis. In Sec. 2.1 a comprehensive introduction into the general method of classical symbolic dynamics is provided. In Sec. 2.2 the concept of quantum maps is discussed. Here we first describe Weyl's quantization of the unit square [22], which the definition of various quantum maps is based on. We then direct our attention to the quantum baker's map, whose dynamical features are studied later in the thesis. We introduce the quantum baker's map using the method of Ref. [12]: by exploiting formal similarities between the symbolic dynamics for the classical baker's map on the one hand and the dynamics of strings of quantum bits within quantum computing theory on the other hand. In Sec. 2.3 we then briefly review the general decoherent histories formalism of quantum mechanics.

Chapter 3 defines and motivates the framework for the research presented in this thesis. The motivation for this framework comes from the analogy between concepts of the method of classical symbolic dynamics on the one hand and the decoherent histories formalism of quantum mechanics on the other hand.

In **Chapter 4** we investigate the coarse-grained evolution of the quantum baker's map with regard to the question of how classical predictability of the evolution depends on the character of coarse-graining.

In **Chapter 5** we study decoherence properties of arbitrarily long histories constructed from a fixed projective partition of a finite dimensional Hilbert space. Simple necessary decoherence conditions for such histories are derived and the dependence of decoherence on the initial state is investigated. The obtained results are also discussed in relation to other existing work on decoherent histories. Finally, first steps towards generalizing these results to the case of approximate decoherence are accomplished.

Chapter 6 concludes with a brief review of our results and suggests questions for future research.

Chapter 2

Background

2.1 Symbolic dynamics of classical dynamical systems

2.1.1 Introduction

The *method of symbolic dynamics* is a mathematical framework that was specifically invented for the purpose of representing classical dynamical systems by a discrete model which is suitable for information-theoretic studies. It is the purpose of this section to review this method which proved to be a very powerful tool within the theory of classical dynamical systems. Parts of the following discussion closely follow Alekseev and Yakobson's classical work in [6] as well as the less abstract and more comprehensible introduction into this subject provided by Badii and Politi in [7].

The *basic idea* underlying the method of symbolic dynamics is to simplify the analysis of dynamical systems by representing points in phase space by symbolic sequences, i.e. strings of letters from a finite alphabet $\mathbb{A} = \{0, \ldots, b-1\}$. The first necessary step towards a construction of such a symbolic representation in the most general case is a *discretization of time*. In Sec. 2.1.2 it will be explained how a discretization of time can be introduced without losing relevant information about the structure of the motion. The next step consists in partitioning the phase space of the given system into a finite number of disjoint cells, E_μ, labeled by the letters from the alphabet $\mathbb{A}$, i.e. $\mu \in \mathbb{A}$. This partitioning determines a coarse-graining on the phase space, see Sec. 2.1.3. Having introduced coarse-graining both in time and on the state space, it becomes clear how the notion of a *coarse-grained trajectory* arises. We list a sequence of symbols corresponding to the cells visited by the exact trajectory of the system at the particular times $j = 1, 2, \ldots$ during its evolution which started from some initial state. Such a list of symbols, labeling the sequence of cells successively visited by the system at the moments of time $j = 1, 2, \ldots$ under the action of the dynamics, can be viewed as a coarse-grained trajectory. One could as well use the term *coarse-grained history.* Of course, the relevant features of the dynamical system must not be lost in this procedure. And in fact, for a large class of interesting systems, including such displaying chaos, it is possible, by a careful choice of the coarse-grained description, to retain all the relevant information about the dynamical features of the given system. "Symbolic dynamics" in the proper meaning of the word is the study of doubly-infinitely long symbolic sequences, extended to both $-\infty$ and $+\infty$ in time. Symbolic dynamics is equivalent to the original continuous dynamics in terms of the real trajectories of the system if every infinitely long symbolic sequence corresponds to a single point (initial condition) in the phase space. This can be achieved for partitions of the phase space that *refine* themselves indefinitely under the dynamics. Such partitions are called *generating partitions.* In terms of symbolic sequences the discretized system dynamics becomes very simple: it is just a *shift* on the symbolic sequences. This will be explained in Sec.2.1.3 and 2.1.4

The method of symbolic dynamics has proven to be a very powerful tool within the theory of classical dynamical systems. This method is especially very effective in those situations in which the deterministic systems being studied reveal an analogy with random processes [6]. Symbolic dynamics techniques proved, e.g., extremely useful in the field of classical chaos. Using the symbolic dynamics representation it is possible to formulate an *information-theoretic characterization of chaos* [6, 8, 9]. In this approach chaos is quantified in terms of the so-called *Kolmogorov-Sinai* (KS) *entropy*, H_{KS}, which measures the rate at which information about the initial phase space point must be supplied in order to enable us to predict the coarse-grained behaviour of a trajectory at a later time [6, 8]. The techniques of symbolic dynamics allow one to calculate the average information, $\langle I \rangle$, necessary to specify a particular coarse-grained trajectory. It has been shown [9] that for classical chaotic systems this average entropy grows linearly in time, $\langle I \rangle(t) \sim H_{\mathrm{KS}} t$, "$\sim$" meaning the leading asymptotic behaviour. The quantity $\langle I \rangle(t)$ tells us how much information, on average, is *missing* about the initial condition in order to be able to predict the coarse-grained history up to time t. With growing time we need more and more information to do so. The rate at which this missing information grows with time is given by H_{KS}. The greater H_{KS} is, the more unpredictable is the corresponding system evolution. This growing unpredictability contributes to the complexity of the given dynamical system. This growth can thus be viewed as an information-theoretic signature of chaos.

The information-theoretic approach to the characterization of chaos turned out to be very promising with regard to the issue of defining chaos for quantum systems. Defining quantum chaos for closed (quantum) systems in the fashion of *"sensitivity to initial conditions"* [1] obviously fails due to the unitary nature of quantum evolution of closed quantum systems, which is linear and preserves the inner product. The unitarity of the linear Schrödinger evolution precludes any sensitivity to initial conditions in the quantum dynamics of state vectors. The information-theoretic approach, on the other hand, provides a framework for the definition of signatures of chaos, which can treat classical and quantum dynamical systems on an equal footing [10]. The main idea is to consider open systems, classical or quantum, and investigate the unpredictability of their evolution which arises due to the interaction with a partially unknown environment. The *"hypersensitivity to perturbation" criterion* [2] proposed by

[1] *"Sensitivity to initial conditions"* is the usual signature of classical chaos. Roughly speaking, this means that phase space points which are initially very close become separated during the evolution by a distance that increases exponentially in time.

[2] Let me briefly explain, what is meant by *"hypersensitivity to perturbation"* (cf. Refs. [9, 10]). Hypersensitivity to perturbation, in either classical or quantum mechanics, is defined completely in terms of information and entropy. The entropy H of an isolated physical system $\mathcal{S}$ — for classical systems it is the Gibbs entropy, for quantum systems the von Neumann entropy — remains constant during the time evolution. If, however, the time evolution of that system is perturbed due to inter-

Schack and Caves [9, 10] provides a characterization of chaos that is applicable to both classical as well as quantum systems. It is by means of the symbolic dynamics techniques that powerful *analytical* results could be obtained for classical systems. Using the method of symbolic dynamics, Schack and Caves showed in [9] that a large class of classical chaotic maps exhibit *exponential hypersensitivity to perturbation.* It is still an open question whether there exist chaotic quantum systems that display exponential hypersensitivity to perturbation as well. We may anticipate that a quantum generalization of the method of classical symbolic dynamics would provide a very powerful tool for examining this issue, and quantum chaos in general. It is therefore a desirable goal to develop a kind of "quantum symbolic dynamics".

Let us now expand on details of this powerful method. We start with a very general continuous description of a dynamical physical system. Such a general description is usually formulated in terms of a system of differential equations,

$$\dot{x} = f(x) \quad , \quad x \in \mathbb{R}^d \quad . \tag{2.1}$$

Here x represents the state of the physical system in the corresponding *phase space*

actions with a partially unknown environment $\mathcal{E}$, we need to average over the perturbations, which leads to an entropy increase $\Delta H_{\mathcal{S}}$. This increase of the system entropy can, in principle, be reduced up to an amount ΔH_{tol}, the "tolerable entropy increase", by obtaining some information about the perturbation through measurements on the environment. One therefore considers the amount of information *needed* to keep track of the perturbed time evolution to a level of accuracy that keeps the increase of system entropy below that "tolerable" level. Let us denote by $\Delta I_{\min}$ the minimum information about the environment needed, on the average, to keep the system entropy below the tolerable level ΔH_{tol}. This information can be compared to the entropy reduction it "buys", i.e., to the difference $(\Delta H_{\mathcal{S}} - \Delta H_{\text{tol}})$ — the difference between the entropy increase that results from averaging over the perturbation and the tolerable entropy increase. The hypersensitivity to perturbation criterion of chaos is characterized in terms of the ratio

$$\chi := \frac{\Delta I_{\min}}{\Delta H_{\mathcal{S}} - \Delta H_{\text{tol}}} \quad .$$

A system is said to display "hypersensitivity to perturbation" if the ratio χ grows rapidly with time, for almost all values of ΔH_{tol}. If this ratio grows exponentially, the system displays "exponential hypersensitivity to perturbation". *Exponential hypersensitivity to perturbation* was proposed by Schack and Caves as a universal criterion for chaos, both for classical and quantum systems.

$\mathbb{R}^d$. The "force" f is a smooth vector field on $\mathbb{R}^d$ and may depend on various control parameters. The components of the vectors x and f will be denoted by $x^{(i)}$ and $f^{(i)}$, respectively. In the construction of a model, the motion of the system is usually required to unfold within a compact subset $\mathbb{S}$ of the phase space $\mathbb{R}^d$. Throughout this section we shall identify $\mathbb{S}$ with the state space of the physical system. The smooth vector field f generates a *flow* $\Phi_t : \mathbb{S} \to \mathbb{S}$, which provides the image $x(t) = \Phi_t(x_0)$ at time t of the initial condition $x_0 = x(0)$. Here we tacitly assume that $\mathbb{S}$ is invariant for the flow Φ_t, i.e., $\Phi_t(x) \in \mathbb{S}$ for all $x \in \mathbb{S}$ and all $t \in \mathbb{R}$. The pair $(\mathbb{S}, f)$ is said to constitute a (classical) dynamical system. For simplicity, the compact set $\mathbb{S} \subseteq \mathbb{R}^d$ within which the motion is circumscribed will be identified with the phase space itself. We now describe the construction of a symbolic representation of the dynamical system $(\mathbb{S}, f)$.

2.1.2 Discretization of time

Discretization of time is the very first step towards a description necessary, or at least very desirable, for an information-theoretic study of dynamical systems. It is also the first step in the construction of a symbolic representation of a given dynamical system. There are two natural ways in which a Hamiltonian flow $\Phi_t : \mathbb{S} \to \mathbb{S}$ induces a *discrete map*, and thus a discretization of time:

For an arbitrary time step τ, a map $\Theta : \mathbb{S} \to \mathbb{S}$ is defined by $\Theta(x) := \Phi_\tau(x)$ for all $x \in \mathbb{S}$. Since $(\Phi_t \circ \Phi_s)(x) = \Phi_{t+s}(x)$ for all times t and s and all $x \in \mathbb{S}$, the map Θ and the flow Φ_t are closely related according to $\Theta^n(x) = \Phi_{n\tau}(x)$ for all $x \in \mathbb{S}$ and all $n \in \mathbb{N}$.

There is an alternative way of introducing a discrete map. A discretization of time can be introduced without losing relevant information about the structure and features of the motion via the *Poincaré surface of section* method, i.e., a suitable choice of a (d-1)-dimensional surface $\Xi \subseteq \mathbb{R}^d$. What means "suitable"? The surface Ξ should be chosen such that it satisfies the following requirements. Firstly, every trajectory in $\mathbb{S}$ must intersect Ξ. More precisely, this means that for any initial condition $x_0 \in \mathbb{S}$ there must exist a moment of time t such that $\Phi_t(x_0) \in \Xi$. A further requirement is needed:

for any $x \in \Xi$, the vector $f(x)$ must not be tangent to Ξ. Whenever the motion takes place in a compact set $\mathbb{S}$, as assumed, it is possible to find such a Poincaré surface. This follows from the simple fact, that, whenever the motion takes place in a compact set, any component $x^{(i)}(t)$ of $x(t)$ must be a bounded function, with the consequence that $\dot{x}^{(i)}(t) = 0$ at some moment t of time. Equation $\dot{x}_i(t) = 0$ can be rewritten as $f^{(i)}(x) = 0$, which defines a possible (d-1)-dimensional surface of section Ξ satisfying all the above requirements.

Now, let the *successive returns* of the trajectory $(\, x(t)\,)_{t\in\mathbb{R}}$ of the system onto the surface Ξ be denoted by x_n. This defines a new discrete time $n \in \mathbb{Z}$, which parameterizes the sequence of the successive returnings, $(x_n)_{n\in\mathbb{Z}}$. The successive returnings are obviously connected by a discrete map, the *"Poincaré map"*:

$$x_{n+1} = \Lambda(x_n) \ , \tag{2.2}$$

where $\Lambda(x_n) := \Phi_{\tau_n}(x_n)$ and τ_n is the time elapsed between the events x_n and x_{n+1}. If the flow Φ_t is a smooth function, Λ is a *diffeomorphism* (i.e., it is one-to-one and both Λ as well as its inverse Λ^{-1} are differentiable). We will denote the n-th composition of Λ with itself by Λ^n with $n \in \mathbb{Z}$. Negative n-values represent iterates of the inverse map.

The discretization of time via a Poincaré surface of section *preserves all the relevant features of the motion.* In fact, the fixed points of Φ_t and Λ obviously coincide, since the surface Ξ passes through them. Furthermore, any periodic orbit of Φ_t is transformed into a finite number of points which are mapped cyclically into one another by Λ. The invariant manifolds present the same degree of intricacy.

Let us define $\mathbb{S}_\Xi := \Xi \cap \mathbb{S}$. Then the *discrete abstract dynamical system* $(\mathbb{S}_\Xi, \Lambda)$ exhibits the same dynamical features as the original continuous dynamical system $(\mathbb{S}\,, f)$.

2.1.3 Coarse graining

We now introduce a *partition* of the state space $\mathbb{S}$ into a finite number b of disjoint domains $P_\lambda \subseteq \mathbb{S}$, labeled by the letters from an b-letter *alphabet* $\mathbb{A} := \{a_1, \ldots, a_b\}$, i.e.:

$$\mathcal{P} := \left\{ P_\lambda \,\middle|\, \lambda \in \mathbb{A} \ , \ \bigcup_{\lambda\in\mathbb{A}} P_\lambda = \mathbb{S} \quad \text{and} \quad P_\lambda \cap P_{\lambda'} = \emptyset \quad \text{for} \quad \lambda \neq \lambda' \right\} . \tag{2.3}$$

This partition induces a partition $\mathcal{E}$ of the space $\mathbb{S}_\Xi = \Xi \cap \mathbb{S}$ into $b < \infty$ disjoint cells,

$$\mathcal{E} := \left\{ E_\lambda \,\middle|\, \lambda \in \mathbb{A} \ , \ \bigcup_{\lambda\in\mathbb{A}} E_\lambda = \mathbb{S}_\Xi \quad \text{and} \quad E_\lambda \cap E_{\lambda'} = \emptyset \quad \text{for} \quad \lambda \neq \lambda' \right\} , \tag{2.4}$$

according to $E_\lambda := P_\lambda \cap \Xi$.

Under the action of dynamics, the system trajectory visits various elements of $\mathcal{E}$. Starting from some initial state $x_0 \in \mathbb{S}_\Xi$ the successive action of the Poincaré map, applied K times, produces some discretized orbit $\mathcal{O}_K(x_0) \equiv \{x_0, x_1, x_2, \ldots, x_K\}$, which touches several partition elements of $\mathcal{E}$. Let us denote by the *symbol* "s_j" the index (label) of the cell $E_{s_j} \in \mathcal{E}$ visited by the system at time "j". Then the "itinerary" of the orbit $\mathcal{O}_K(x_0)$ can be associated with the *symbolic sequence* $S_K(x_0) \equiv \{s_0, s_1, \ldots, s_K\}$, where $x_j \in E_{s_j}$ for all $j = 0, 1, \ldots, K$. We may think of s_j as a coarse-grained value of the exact state x_j visited at time "j" and regard the *symbolic sequence* $S_K(x_0) \equiv \{s_0, s_1, \ldots, s_K\}$ as a *coarse-grained trajectory* of the system. It is desirable that the original orbit $\mathcal{O}_K(x_0)$ can be retraced to some extent from the knowledge of the symbolic sequence $S_K(x_0)$, of the partition $\mathcal{E}$ and of the dynamical law given by the map Λ. This is indeed possible by a careful choice of the partition $\mathcal{E}$. In fact, the knowledge of $S_K(x_0)$ can be used to recover the history of the system to a better precision than given by the coarse-graining. For example, the observation of the first two symbols s_0 and s_1 in $S_K(x_0)$ implies that the (unknown) exact initial state x_0 was, at the initial time $j = 0$, not only in E_{s_0} but also in the first preimage $\Lambda^{-1}(E_{s_1})$. Hence, the joint measurement at times $j = 0$ and $j = 1$ reduces the uncertainty on the (exact) initial state x_0, provided that $\Lambda^{-1}(E_{s_1}) \cap E_{s_0} \subseteq E_{s_0}$. The sequence $\{s_0, s_1, \ldots, s_K\}$ can thus be produced only by such *initial* states x_0 which belong to the intersection

$$\mathbb{E}_{s_0 \ldots s_K} := \bigcap_{j=0}^{K} \Lambda^{-j}(E_{s_j}) \equiv E_{s_0} \cap \Lambda^{-1}(E_{s_1}) \cap \Lambda^{-2}(E_{s_2}) \cap \cdots \cap \Lambda^{-K}(E_{s_K}) \ . \tag{2.5}$$

With increasing length K of the symbolic sequences $S_K(x_0)$ smaller and smaller intersections $\mathbb{E}_{s_0 \ldots s_K}$ are identified. On the other hand, the size of $\mathbb{E}_{s_0 \ldots s_K}$, determines the accuracy to which x_0 must be known in order to enable a reconstruction of the coarse-grained trajectory $S_K(x_0)$. Obviously, the amount of information to specify a specific intersection $\mathbb{E}_{s_0 \ldots s_K}$ as a part of $\mathbb{S}_\Xi$ increases with K. Hence, more and more information about the initial condition x_0 must be supplied if we want to be able to predict the system evolution for a longer period of time. This means knowing the initial condition x_0 with a better and better precision. Usually — in most practical situations — the available precision is limited, with the consequence that we lose the ability to predict the future behaviour of the system. How fast this predictability is lost depends on the complexity of the dynamical system $(\mathbb{S}_\Xi, \Lambda)$. Roughly speaking, the more rapidly the predictability is lost, the more chaotic is the dynamical system.[3]

The finite symbolic sequences $s_0 \cdots s_K$ are sometimes called *"symbolic words"*. The symbolic word $s_0 \cdots s_K$ is *admissible* if $\mathbb{E}_{s_0 \ldots s_K}$ contains at least one point. Using all the nonempty sets $\mathbb{E}_{s_0 \ldots s_K}$ corresponding to all admissible symbolic words, we can introduce a new, finer-grained, partition of the state space $\mathbb{S}_\Xi$ as:

$$\mathcal{E}^K := \left\{ \mathbb{E}_{s_0 \ldots s_K} \,\middle|\, (s_0, s_1, \ldots, s_K) \in \mathbb{A}^{K+1} \quad \text{such that} \quad \mathbb{E}_{s_0 \ldots s_K} \neq \emptyset \right\} . \tag{2.6}$$

The new partition $\mathcal{E}^K$ is called a *"refinement of $\mathcal{E}$ under Λ"*.[4] It is worth emphasizing that the refinements $\{\mathcal{E}^K \mid K = 1, 2, \ldots\}$ are *dynamical refinements*, i.e., they are produced by the dynamics itself.

The study of infinitely long symbolic sequences $\{S_\infty(x_0) \mid x_0 \in \mathbb{S}_\Xi\}$ is equivalent to the study of the real trajectories of the original system (2.1) if for every such infinitely long sequence $S_\infty(x_0) = (s_j)_{j \in \mathbb{N}_0}$ there exists only one initial condition $x_0 \in \mathbb{S}_\Xi$ that generates $(s_j)_{j \in \mathbb{N}_0}$ under the action of the map Λ, i.e., if every symbolic sequence

[3] The complexity of a dynamical system — in both classical as well as quantum mechanics — can be quantified by the minimal information which is necessary to specify a coarse-grained trajectory (or history) given the knowledge of the initial condition up to some fixed accuracy [13]. The complexity of dynamical systems can grow for different reasons. Sensitivity to initial conditions is one possible reason. Perturbations due to interactions with some incompletely known environment in the case of open systems is another possibility. In any case, the rate of the growth quantifies the resources needed to be able to predict the system evolution, and as such it indicates chaotic behaviour.

[4] More generally, given two partitions $\mathcal{E}$ and $\mathcal{E}'$ of a set (space) $\mathcal{X}$, the partition $\mathcal{E}'$ is said to be a refinement of $\mathcal{E}$ if any element of $\mathcal{E}$ is a union of some elements of $\mathcal{E}'$.

$(s_j)_{j\in\mathbb{N}_0} \in \{S_\infty(x_0)\,|\,x_0 \in \mathbb{S}_\Xi\}$ corresponds to one and only one initial point x_0. This will be achieved whenever the original partition $\mathcal{E}$ is *generating*, i.e., if $\mathcal{E}$ refines itself indefinitely under the dynamics. If a generating partition exists, there are infinitely many of them. In particular, every refinement $\mathcal{E}'$ of a generating partition $\mathcal{E}$ is, *a fortiori*, also generating. The construction of a single generating partition is a highly nontrivial problem within the theory of discrete dynamical systems, though. The difficult task consists in finding one with the minimum number of elements.

In the case of an invertible map Λ, the backward iterates of x_0 are also taken into account, by extending each infinite symbolic sequence $S_\infty(x_0)$ to a doubly-infinite symbolic sequence

$$\ldots s_{-3}s_{-2}s_{-1}\,.\,s_0s_1s_2\ldots \quad , \tag{2.7}$$

where the symbols to the left of the *"dot"* represent the backward iterates of x_0.

2.1.4 Shift dynamics

The study of infinite symbolic sequences is usually referred to as *symbolic dynamics*. The reason for this name becomes evident by means of the observation that a one-step iterate of the map Λ in (2.2) corresponds to the reading of the next symbol in the associated symbolic sequence. Thus, the system dynamics given in terms of the discrete map $\Lambda : \mathbb{S}_\Xi \to \mathbb{S}_\Xi$ is equivalent to a *shift* of the *"dot"* to the right for the forward iterations of Λ and to the left for its backward iterations. This mechanism can be formalized by introducing a shift transformation σ on the set of all bi-infinite symbolic sequences. [5] Let us from now on denote a bi-infinite sequence of letters $\omega_j \in \mathbb{A}$ by $\omega \equiv (\omega_j)_{j\in\mathbb{Z}} = \cdots\omega_{-1}\omega_0\omega_1\omega_2\cdots$ and the set of all such bi-infinite symbolic sequences by Σ, i.e., $\Sigma \equiv \mathbb{A}^{\mathbb{Z}}$.

For each $x \in \mathbb{S}_\Xi$ we can define the set $\Sigma_x \subseteq \Sigma$ as follows:

$$\Sigma_x \equiv \left\{\omega \in \Sigma \;\middle|\; x \in \bigcap_{n=-\infty}^{\infty} \Lambda^{-n}(E_{\omega_n})\right\} . \tag{2.8}$$

Equivalently, one can say that $\omega \in \Sigma_x \iff \Lambda^n(x) \in E_{\omega_n}$ for all n. We are interested

[5] The introduction provided in this section closely follows [6] and the brief discussion given in [9].

in the set of all symbolic sequences corresponding to at least one point in $\mathbb{S}_\Xi$:

$$\Sigma_{\mathcal{E}} := \bigcup_{x \in \mathbb{S}_\Xi} \Sigma_x \subseteq \Sigma \ . \tag{2.9}$$

This set is called the set of all *admissible sequences.* Obviously, the partition $\mathcal{E}$ will be a *generating partition* if for each $\omega \in \Sigma_{\mathcal{E}}$ the intersection

$$\bigcap_{n=-\infty}^{\infty} \Lambda^{-n}(E_{\omega_n}) \tag{2.10}$$

consists of only one point, i.e., if each admissible symbolic sequence defines a unique point in $\mathbb{S}_\Xi$. In general, even for generating partitions, the set Σ_x may consist of more than one element. This means that a point $x \in \mathbb{S}_\Xi$ may be represented by several symbolic sequences $\omega \in \Sigma_x$. For a generating partition, the picture one should have is that the set $\Sigma_{\mathcal{E}}$ of all admissible sequences is the union of disjoint subsets Σ_x, which may have more than one member.

We are now in a position to introduce the so-called *shift map* $\sigma : \Sigma \to \Sigma$ on the set of symbolic sequences, which is induced by the map $\Lambda : \mathbb{S}_\Xi \to \mathbb{S}_\Xi$. The shift map is defined as

$$(\,\sigma(\omega)\,)_n = \omega_{n+1} \quad \text{for all } n \in \mathbb{Z}\ ; \tag{2.11}$$

i.e., σ shifts the entire symbolic sequence to the left, or, equivalently, the *"dot"* in (2.7) to the right. In the literature, the shift map is also called *"Bernoulli shift"* [7]. The set of admissible sequences is invariant under the shift map, i.e.,

$$\sigma(\Sigma_{\mathcal{E}}) = \Sigma_{\mathcal{E}} \ . \tag{2.12}$$

Furthermore, for a generating partition $\mathcal{E}$, the map $\pi : \Sigma_{\mathcal{E}} \to \mathbb{S}_\Xi$ defined by

$$\pi(\omega) = \bigcap_{n=-\infty}^{\infty} \Lambda^{-n}(E_{\omega_n}) \tag{2.13}$$

[i.e., $\pi(\omega) = x \Longleftrightarrow \omega \in \Sigma_x$] is single-valued and continuous [6].[6]

[6] If the domains E_i forming the partition $\mathcal{E}$ are not mutually exclusive, as required in (2.4), then the map π is *not* one-to-one. The overlap between different sets E_i, however, is usually of measure zero.

The relation between Λ and σ can be summarized by a *commutation diagram*:

$$\begin{array}{ccc} \Sigma_{\mathcal{E}} & \xrightarrow{\sigma} & \Sigma_{\mathcal{E}} \\ \pi \downarrow & & \downarrow \pi \\ \mathbb{S}_{\Xi} & \xrightarrow{\Lambda} & \mathbb{S}_{\Xi} \end{array} \tag{2.14}$$

In this way the action of the map Λ on measurable subsets of $\mathbb{S}_{\Xi}$ is being faithfully represented by the action of σ on measurable sets of symbolic sequences.

2.1.5 Information-theoretic measures: complexity of a dynamical system and entropy

We shall now show how *information-theoretic measures* can be defined using the symbolic dynamics representation of a given dynamical system as described above — measures in order to characterize the complexity of the dynamical features. The discussion given in this section closely follows [6].

Let us again consider finite symbolic sequences, i.e. *symbolic words*, composed of letters from the alphabet $\mathbb{A}$. We have learnt above that a word $s_0 \cdots s_K \in \mathbb{A}^{K+1}$ of length $K+1$ corresponds to the point $x \in \mathbb{S}_{\Xi}$ if $x \in \mathbb{E}_{s_0 \dots s_K} \equiv \bigcap_{j=0}^{K} \Lambda^{-j}(E_{s_j}) = E_{s_0} \cap \Lambda^{-1}(E_{s_1}) \cap \Lambda^{-2}(E_{s_2}) \cap \cdots \cap \Lambda^{-K}(E_{s_K})$, and we have called such a word *"admissible"* if it corresponds to at least one point out of $\mathbb{S}_{\Xi}$. The more different admissible words exist, the more complex is the dynamical system. Hence, the asymptotic behaviour of the total number of all possible admissible words, i.e., the behaviour of the cardinality $|\,\mathcal{E}^K\,|$ in the limit $K \to \infty$, can be used to quantify the complexity of the dynamical system. Let us define [7]

$$h(\Lambda|\,\mathcal{E}) := \lim_{K\to\infty} \frac{\log|\,\mathcal{E}^K\,|}{K}\,. \tag{2.15}$$

Yet, this quantity still depends on the choice of partitioning $\mathcal{E}$ of $\mathbb{S}_{\Xi}$, which can be highly non-unique. To remove this dependence, one defines

$$h(\Lambda) := \sup_{\mathcal{E}} h(\Lambda|\,\mathcal{E})\,. \tag{2.16}$$

[7]This limit has been shown to exist (see lemma 1.18 in Ref. [23]).

The quantity $h(\Lambda)$ is known as the *"topological entropy"* of the mapping $\Lambda : \mathbb{S}_\Xi \to \mathbb{S}_\Xi$ (cf. Ref. [6]). It quantifies the maximum complexity of dynamics generated by the map Λ.

An alternative way of characterizing the complexity of dynamical systems is provided by the so-called *"Kolmogorov-Sinai entropy"*. This quantity can be introduced if the state space of a given physical system is a *measurable space*, i.e., "equipped" with a measure. This is usually the case in almost all physical models. A familiar example is the measure "$dp\,dq$" on the usual phase space of a Hamiltonian system, with respect to which integration is performed.

Let therefore on the space $\mathbb{S}_\Xi$ a normalized Borel measure μ be defined, which is invariant with respect to the map $\Lambda : \mathbb{S}_\Xi \to \mathbb{S}_\Xi$, while $\mathcal{E}$ is a measurable partition of $\mathbb{S}_\Xi$. Such a dynamical system $(\mathbb{S}_\Xi, \mu, \Lambda)$ thus consists of a measurable space $\mathbb{S}_\Xi$ with a normalized measure μ and a measure-preserving automorphism Λ on $\mathbb{S}_\Xi$, i.e., $\mu(\mathbb{S}_\Xi) = 1$ and $\mu(\,\Lambda(B)\,) = \mu(B)$ for all measurable $B \in \mathbb{S}_\Xi$.

We again consider symbolic words $s_0 \cdots s_K \in \mathbb{A}^{K+1}$ of length $K+1$. Since $\mathcal{E}$ is assumed to be a measurable partition of $\mathbb{S}_\Xi$, the measure μ on $\mathbb{S}_\Xi$ induces a probability distribution $\wp$ on the set of all admissible words of length $K+1$ (associated with the refinement $\mathcal{E}^K$), *via*

$$\wp(s_0 \ldots s_K) := \mu(\mathbb{E}_{s_0 \ldots s_K}) = \mu(E_{s_0} \cap \Lambda^{-1}(E_{s_1}) \cap \cdots \cap \Lambda^{-K}(E_{s_K})\,)\ . \tag{2.17}$$

A finite symbolic sequence (word) $(s_0, s_1, \ldots, s_K)$ represents a coarse-grained trajectory $S_K(x_0)$ (see Sec. 2.1.3), where the initial point x_0 belongs to a set of measure $\mu(\mathbb{E}_{s_0 \ldots s_K})$, which is equal to the "probability" of the symbolic sequence, $\wp(s_0 \ldots s_K)$. Using the probability distribution (2.17) we can introduce an entropy, which quantifies the *average uncertainty* about the coarse-grained trajectory of the system:

$$\begin{aligned} H(\mu, \mathcal{E}, K) \;&:=\; -\sum_{s_0, s_1, \ldots, s_K} \wp(s_0 \ldots s_K)\, \log \wp(s_0 \ldots s_K) \\ &=\; -\sum_{\mathbb{E}_{s_0 \ldots s_K} \in \mathcal{E}^K} \mu(\mathbb{E}_{s_0 \ldots s_K})\, \log\, \mu(\mathbb{E}_{s_0 \ldots s_K})\ . \end{aligned} \tag{2.18}$$

The *Kolmogorov-Sinai entropy* ("KS-entropy") is defined as the asymptotic rate at which the uncertainty about the dynamics accumulates with time:

$$H_{\mathrm{KS}}(\mu) := \sup_{\mathcal{E}} \lim_{K\to\infty} \frac{H(\mu,\mathcal{E},K)}{K}\,, \tag{2.19}$$

where as in Eq. (2.16) the supremum is taken to remove any dependence on the partitioning $\mathcal{E}$. The KS-entropy is sometimes alternatively called *"metric entropy"*.

There exists a relationship between the topological and metric entropies:

$$h(\Lambda) = \sup_{\mu} H_{\mathrm{KS}}(\mu)\,, \tag{2.20}$$

where the supremum is taken over the set of all normalized Borel measures, which are invariant with respect to Λ.

Both the topological entropy as well as the Kolmogorov-Sinai entropy considered above characterize the complexity of a dynamical system *as a whole*. Alternatively, one might be interested in evaluating the complexity of an individual trajectory. Such a dynamical complexity for an individual trajectory can in fact be constructed using Kolmogorov's notion of algorithmic complexity. The corresponding definition and details can be found in [6].

2.2 Quantum Maps

2.2.1 Introduction

A *quantum map* is usually associated with some unitary transformation on a Hilbert space, or, more generally, a sequence of unitary transformations, which may correspond to classical canonical transformations if the system has a classical analog [24]. Quantum maps are usually defined via quantization of their classical counterparts. A general procedure of quantization can be broken into two distinct steps [22]. The first step consists in introducing the kinematics which includes the definition of the state space of the system and of the operators that describe the system. This leads to a Hilbert space together with a pair of conjugate variables representing the "position" q and "momentum" p. After the kinematics has been established, the second step consists in the introduction of a dynamics, which is to induce the time evolution of the system. Both of these steps can be highly non-unique. Indeed, there is no unique quantization procedure [25]. A rough guide is given by the *correspondence principle*, i.e. the requirement that the quantized version of the classical system should in some sense resemble the classical system in the "classical limit" $\hbar \to 0$.

In what follows I first describe Weyl's quantization of the unit square [22], on which the definition of a number of quantum maps is based. Weyl's quantization procedure is the most commonly used way to provide the kinematics for quantum maps whose classical counterparts are defined on the unit square of the phase space. The short introduction given below in Sec. 2.2.2 closely follows [22] and [26]. Special attention will then be given to the quantum baker's map, whose dynamical properties we are going to expand on in Chap. 4. After a short review of the classical baker's transformation [27] in Sec. 2.2.3, we shall discuss its quantum counterpart in Sec. 2.2.4. Here we introduce the quantum baker's map in the style of Ref. [12], where a whole class of quantum baker's maps was defined on the basis of formal similarities between the symbolic dynamics representation for the classical baker's map on the one hand and the dynamics of strings of quantum bits within quantum computing theory on the other hand.

2.2.2 Quantized unit square

In this section we briefly describe Weyl's method to quantize the unit square [22]. Our introduction is in the style of [26] and closely follows the short discussion provided in [10].

To represent the unit square in a D-dimensional Hilbert space, we start with unitary *"displacement"* operators $\hat{U}$ and $\hat{V}$, which produce displacements in the "momentum" and "position" directions, respectively, and which obey the commutation relation [22]

$$\hat{U}\hat{V} = \hat{V}\hat{U}\epsilon \,, \tag{2.21}$$

where $\epsilon^D = 1$. One may choose $\epsilon = e^{2\pi i/D}$. We further assume that $\hat{V}^D = \hat{U}^D = \eta = \pm 1$. The case $\eta = 1$ corresponds to periodic boundary conditions, whereas the case $\eta = -1$ corresponds to anti-periodic boundary conditions. According to [22, 26] the operators $\hat{U}$ and $\hat{V}$ can then be written as

$$\hat{U} = e^{2\pi i\hat{q}} \qquad \text{and} \qquad \hat{V} = e^{-2\pi i\hat{p}} \ . \tag{2.22}$$

Here $\hat{q}$ and $\hat{p}$ represent the "position" and "momentum" operators. Both of them have eigenvalues $(\beta + j)/D$, where the integer j runs over a sequence of D integers and β is such that $\eta = e^{2\pi i\beta}$, i.e.,

$$\beta = \begin{cases} 0, & \text{if } \eta = 1 \\ \frac{1}{2}, & \text{if } \eta = -1 \ . \end{cases} \tag{2.23}$$

This quantization procedure restricts the eigenvalues of $\hat{q}$ and $\hat{p}$ to lie within some unit interval. But it leaves open which unit interval. We have therefore the freedom in choosing the unit interval conveniently depending on situation. Using this freedom we can write the eigenvalues of $\hat{q}$ and $\hat{p}$ as

$$q_j = \frac{\beta + j}{D} = p_j \ , \quad j = D_0, \ldots, D_0 + D - 1 \ , \tag{2.24}$$

where the integer D_0 can be chosen freely. For instance, in the case $D_0 = 0$ we have $q_j, p_j \in [0, 1)$, whereas the choice $D_0 = -D/2$ leads to $q_j, p_j \in [-\frac{1}{2}, \frac{1}{2})$. Let us denote by $|q_j\rangle$ and $|p_j\rangle$ the eigenvectors of $\hat{q}$ and $\hat{p}$ corresponding to the eigenvalues q_j and p_j, respectively. The position basis $\{|q_j\rangle\}_j$ and the momentum basis $\{|p_j\rangle\}_j$ form orthonormal bases of the Hilbert space. Operators on the Hilbert space can be represented by matrices with respect to the position basis or the momentum basis. For consistency of units, the quantum scale on phase space is given by $2\pi\hbar = 1/D$.

Finally, a transformation between the position basis $\{|q_j\rangle\}_j$ and the momentum basis $\{|p_j\rangle\}_j$ is effected by means of a discrete Fourier transformation, which is given in terms of the *Fourier transform* operator F defined by $|p_k\rangle = F|q_k\rangle$. In the position representation, F has the following matrix elements:

$$(F)_{kj} = \langle q_k|F|q_j\rangle = \langle q_k|p_j\rangle = \sqrt{2\pi\hbar}\, e^{ip_j q_k/\hbar} = \frac{1}{\sqrt{D}} e^{2\pi i(\beta+j)(\beta+k)/D} \ . \tag{2.25}$$

2.2.3 Classical baker's map

The classical baker's transformation [27] maps the unit square

$$\mathbb{S} := \{(q, p) \,|\, 0 \le q, p \le 1\} \tag{2.26}$$

onto itself according to

$$(q,p) \longmapsto \begin{cases} \left(2q, \frac{1}{2}p\right) , & \text{if } 0 \le q \le \frac{1}{2} \\ \left(2q-1, \frac{1}{2}(p+1)\right) , & \text{if } \frac{1}{2} < q \le 1 \ . \end{cases} \tag{2.27}$$

This corresponds to compressing the unit square in the p direction and stretching it in the q direction, while preserving the area, then cutting it vertically, and finally stacking the right part on top of the left part — in analogy to the way a piece of dough is transformed when a baker kneads it.

The classical baker's map is most easily described in terms of its symbolic dynamics. We start with partitioning the phase space. We choose the partition $\mathcal{E} := \{E_0, E_1\}$, where E_0 and E_1 denote the left and the right halves of the unit square, respectively,

$$\begin{aligned} E_0 &:= \{(q,p) \,|\, 0 \le q \le 1/2 \,;\, 0 \le p \le 1\} \\ E_1 &:= \{(q,p) \,|\, 1/2 < q \le 1 \,;\, 0 \le p \le 1\} \ . \end{aligned} \tag{2.28}$$

This partition can be shown to be generating and it has a minimal number of elements. Using this partition we can construct an alternative but equivalent representation of the baker's transformation (2.27) in terms of its symbolic dynamics. In this representation the action of the map is given by a Bernoulli shift σ_B on bi-infinite symbolic sequences of zeros and ones,

$$s = (\cdots s_{-2}s_{-1} \,.\, s_0 s_1 \cdots) \in \{0,1\}^{\mathbb{Z}} \ , \tag{2.29}$$

as $\sigma_B : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$, $\sigma_B(s) = s'$, where $s'_k = s_{k+1}$. In other words, at each time step, the entire string is shifted one place to the left while the dot remains fixed.

Since the partition (2.28) is a generating partition, each symbolic string (2.29) corresponds to exactly one point $(q,p) \in \mathbb{S}$ of the phase space. In the present case this correspondence is given by the following relation:

$$q = \sum_{k=0}^{\infty} s_k 2^{-k-1} \ , \quad p = \sum_{k=1}^{\infty} s_{-k} 2^{-k} \ . \tag{2.30}$$

This relation between points (q,p) in phase space (unit square) and symbolic sequences $s = (s_j)_{j\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$ is a particular feature of the (classical) baker's transformation.

2.2.4 Quantum baker's map

The quantum baker's map [28, 26] is a prototypical quantum map invented for the theoretical investigation of quantum chaos. It has been studied extensively during the last fifteen years (see, e.g., [14] and references therein).

The quantum baker's map was introduced as a quantized version of the classical baker's transformation [27] discussed above. There is, however, no unique quantization procedure [25]. The original definition of the map [28, 26] is based on Weyl's quantization of the unit square. Essentially the same map has been derived using different methods (cf. [14, 12] and references therein). In [12] a whole class of quantum baker's maps has been defined by exploiting formal similarities between the symbolic dynamics [6] for the classical baker's map on the one hand and the dynamics of strings of quantum bits within quantum computing theory on the other hand. These maps admit a symbolic description in terms of shifts on strings of qubits similar to classical symbolic dynamics [6]. Their symbolic description has then been further developed in [14].

Let me, very briefly, mention the original definition of the quantum baker's map, before expanding on the alternative definition in the style of a symbolic description.

The original definition of the map is based on Weyl's quantization of the unit square. It was introduced by Balazs and Voros in [28] and put into a more symmetrical form by Saraceno in [26]. We assume anti-periodic boundary conditions ($\eta = -1$, $\beta = \frac{1}{2}$) and $D_0 = 0$, which leads, as discussed in Sec. 2.2.2 above, to discrete position and momentum eigenvalues at half-integer values $q_j = (\frac{1}{2} + j)/D = p_j$, $j = 0, \ldots, D - 1$. The quantum baker's map is then defined by a product of two non-commuting matrices,

$$B := G_D^{-1} \begin{pmatrix} G_{D/2} & 0 \\ 0 & G_{D/2} \end{pmatrix} , \tag{2.31}$$

where the operator G_D is the inverse discrete Fourier transform on D sites, defined by the matrix elements (cf. Eq. (2.25))

$$(G_D)_{kj} = \langle q_k | G_D | q_j \rangle = \langle p_k | q_j \rangle = \sqrt{2\pi\hbar}\, e^{-ip_k q_j/\hbar} = \frac{1}{\sqrt{D}} e^{-2\pi i(\beta+k)(\beta+j)/D} . \tag{2.32}$$

In Eq. (2.31) D is assumed to be even and the matrix elements are to be understood relative to the position basis.

Let us now expand on the alternative definition of the quantum baker's map in terms of a symbolic description. The following introduction is given in the style of [12], where a whole class of quantum baker's maps is defined using symbolic representation techniques. All these quantum baker's maps are defined on the D-dimensional Hilbert space of the quantized unit square. For consistency of units, we let the quantum scale on "phase space" be $2\pi\hbar = 1/D$. Following Ref. [26], we choose half-integer eigenvalues $q_j = (j + \frac{1}{2})/D$, $j = 0, \ldots, D-1$, and $p_k = (k + \frac{1}{2})/D$, $k = 0, \ldots, D-1$, of the discrete "position" and "momentum" operators $\hat{q}$ and $\hat{p}$, respectively, corresponding to antiperiodic boundary conditions. We further assume that $D = 2^N$, which is the dimension of the Hilbert space of N qubits.

The $D = 2^N$ dimensional Hilbert space modeling the unit square can be identified with the tensor product space of N qubits via

$$|q_j\rangle = |\xi_1\rangle \otimes |\xi_2\rangle \otimes \cdots \otimes |\xi_N\rangle \ , \tag{2.33}$$

where $j = \sum_{l=1}^{N} \xi_l 2^{N-l}$, $\xi_l \in \{0, 1\}$, and where each qubit has orthonormal basis states $|0\rangle$ and $|1\rangle$. We can write q_j as a binary fraction, $q_j = 0.\xi_1\xi_2\ldots\xi_N 1$. Using the basis $\{|q_j\rangle\}_j$ we can identify the position operator $\hat{q}$ with the matrix $\mathrm{diag}(q_1, \ldots, q_{2^N})$ with respect to this basis. Furthermore, we see that the eigenvalues of $\hat{q}$ each satisfy the inequality $0 < q_j < 1$ in analogy with the values of "position" variable x in the classical baker's map. Let us further define the symbolic notation

$$|.\xi_1\xi_2\ldots\xi_N\rangle = e^{i\pi/2}|q_j\rangle \ , \tag{2.34}$$

which is closely analogous to Eq. (2.29), where the bits to the right of the dot specify the position variable; see Ref. [12] for the reason for the phase factor $e^{i\pi/2}$. Momentum and position eigenstates are related through the quantum Fourier transform operator F [26], i.e., $F|q_k\rangle = |p_k\rangle$. See Eq. (2.25) for its matrix elements. Here we have:

$$
\begin{aligned}
e^{-i\pi(.\xi_1\xi_2\ldots\xi_N 1)}F|.\xi_{1:N}\rangle \;\; \equiv \;\; & \sqrt{1/2}\{|0\rangle + \exp[2\pi i(0.\xi_N 1)]|1\rangle\}\otimes \\
& \sqrt{1/2}\{|0\rangle + \exp[2\pi i(0.\xi_{N-1}\xi_N 1)]|1\rangle\} \otimes \cdots \otimes \\
& \sqrt{1/2}\{|0\rangle + \exp[2\pi i(0.\xi_1\xi_2 \ldots \xi_N 1)]|1\rangle\} \; , \qquad (2.35)
\end{aligned}
$$

We can use this as a definition of the momentum eigenstates $|p_k\rangle$ and again, in analogy to Eq. (2.29), we define the notation

$$
|\xi_1\xi_2 \ldots \xi_N.\rangle \equiv |p_k\rangle \; , \qquad (2.36)
$$

where $p_k = 0.\xi_N \ldots \xi_2\xi_1 1$. Since F is a unitary operator, each number p_k is an eigenvalue of $\hat{p}$ corresponding to the eigenvector $|p_k\rangle$.

By applying a *partial* quantum Fourier transform operator [12] to the n rightmost bits of the position eigenstate $|.\xi_{n+1} \ldots \xi_N\xi_n \ldots \xi_1\rangle$, one obtains the family of states [12]

$$
\begin{aligned}
|\xi_1 \ldots \xi_n.\xi_{n+1} \ldots \xi_N\rangle \;\; \equiv \;\; & 2^{-n/2}e^{i\pi(0.\xi_n\ldots\xi_1 1)}|\xi_{n+1}\rangle \otimes \cdots \otimes |\xi_N\rangle\otimes \\
& (|0\rangle + e^{2\pi i(0.\xi_1 1)}|1\rangle) \otimes (|0\rangle + e^{2\pi i(0.\xi_2\xi_1 1)}|1\rangle)\otimes \\
& (|0\rangle + e^{2\pi i(0.\xi_3\xi_2\xi_1 1)}|1\rangle) \otimes \cdots \otimes \\
& (|0\rangle + e^{2\pi i(0.\xi_n\ldots\xi_1 1)}|1\rangle) \; , \qquad (2.37)
\end{aligned}
$$

where $1 \le n \le N-1$. For fixed values of n and N we will use the notation

$$
|\xi_1 \ldots \xi_N\rangle_n \equiv |\xi_1 \ldots \xi_n.\xi_{n+1} \ldots \xi_N\rangle \, . \qquad (2.38)
$$

These states form an orthonormal basis of the Hilbert space. The state (2.37) is localized in both position and momentum: it is strictly localized within a position region of width $1/2^{N-n}$, centered at position $q = 0.\xi_{n+1} \ldots \xi_N 1$, and it is crudely localized within a momentum region of width $1/2^n$, centered at momentum $p = 0.\xi_n \ldots \xi_1 1$.

Having introduced the quantum kinematics on the unit square we are now in the position to introduce the dynamics of the baker's map. For each fixed n, $0 \le n \le N-1$, a quantum baker's map B_n can be defined by

$$
B_n|\xi_1 \ldots \xi_n.\xi_{n+1} \ldots \xi_N\rangle = |\xi_1 \ldots \xi_{n+1}.\xi_{n+2} \ldots \xi_N\rangle \; , \qquad (2.39)
$$

i.e.

$$B_n|\xi_1 \ldots \xi_N\rangle_n = |\xi_1 \ldots \xi_N\rangle_{n+1} \; . \tag{2.40}$$

The action of the map B_n on the basis states (2.37) is thus given by a *shift of the dot by one position.* In phase-space language, the map B_n takes a state localized at $(q,p) = (0.\xi_{n+1} \ldots \xi_N 1, 0.\xi_n \ldots \xi_1 1)$ to a state localized at $(q',p') = (0.\xi_{n+2} \ldots \xi_N 1, 0.\xi_{n+1} \ldots \xi_1 1)$, while it stretches the state by a factor of two in the q direction and squeezes it by a factor of two in the p direction. This analogy with the classical baker's map motivates calling the maps (2.39) *"quantum baker's maps"*. For $n = N-1$, the map is the original quantum baker's map as defined in Ref. [26]. In [13, 14] it has been shown that all the maps B_n in (2.39) reduce to the classical baker's map in the limit $\hbar \to 0$.

In Chap. 4 it will be convenient to simplify our notation slightly. Throughout Chap. 4 the parameters n and N are kept fixed. So we omit the index n and denote the quantum baker's map simply by B always keeping in mind that we are dealing with the special baker's map B_n for the given value of n.

2.3 Decoherent Histories Formalism

It is the purpose of this section to give a brief introduction into the general *decoherent histories formalism*, without expanding on the conceptual issues and problems of this approach, which have been highly debated during the last fifteen years. I shall make some remarks on the general interpretation, though.

2.3.1 Introduction

The formalism of decoherent histories was introduced to provide a self-contained description of closed quantum systems that does not rely on either the external observer nor on the existence of classical devices [16, 17, 18, 19, 20]. [8] It has been successfully applied in various fields of quantum theory. Applications include, e.g., quantum cosmology [29], derivation of effective classical dynamics from the fundamental quantum dynamical laws [19, 20], in particular a derivation of the equations of classical hydrodynamics [30], and the study of the coarse-grained evolution of iterated quantum maps [15]. Recently the formalism of decoherent histories has also been applied for investigating classicality in quantum information processing [31].

The main motivation for introducing consistent or decoherent histories has already been mentioned above. The most common aspect is the desire to find a language which is also suitable for closed quantum systems, including all observers and measurement apparata, but where *no reference to measurements or observers is needed.* This formalism thus makes no distinction between microscopic and macroscopic sys-

[8]In the orthodox Copenhagen interpretation of quantum mechanics all properties of a quantum system are defined with respect to measurements performed by an external observer using classical measuring devices. This interpretation, however, cannot be used in the case of closed quantum systems, such as the Universe as a whole. In this case any observer must be a part of the system itself. A self-contained theory of closed quantum systems that does not rely on either the external observer nor on the existence of classical devices is still under development. The decoherent histories approach is probably the most promising candidate for such a theory. Ultimately, of course, only the entire Universe is a strictly closed system. It is for this reason why the decoherent histories formalism is most often encountered within the framework of quantum cosmology.

tems. A separate classical domain like in the Copenhagen interpretation is therefore not assumed, although it may arise as an emergent feature. Moreover, it makes no use of the notion of the collapse of the wave function, although this notion may be discussed within this approach. Another motivation is given by the interest to implement appropriate coarse-grainings in time. A further motivation is provided by the hope, that the decoherent histories approach is general enough to be also applicable in quantum gravity, where no distinguished external time parameter is available and one is aiming at constructing decoherent histories for space-time domains [21].

In its very core the decoherent histories formalism *assigns probabilities to quantum histories*. The concept of *histories* is central to this approach. A quantum history is the very basic building block of the corresponding formalism. As already mentioned in the introduction, a quantum history is defined to be a time-ordered sequence of quantum mechanical "propositions", i.e., a sequence of quantum events at a succession of times. Mathematically, these propositions are represented by projection operators. In particular, an exhaustive set of mutually exclusive propositions corresponds to a complete set of mutually orthogonal projectors. Generalizations to *"effect histories"* have also been proposed [32, 33], with the propositions of the histories being represented by *effects* of a POVM (positive operator valued measure) instead of projection operators. It is the main feature of this approach that it focuses on a *succession of events* of a closed system, rather than events at a fixed moment of time. Such sequences of events represent the most general class of physical situations, especially in any experiment.

The decoherent histories formulation of quantum mechanics has proven to be a very appropriate framework especially for investigating *classicality* in quantum theory [9], particularly with regard to closed quantum systems [19, 20]. Within this ap-

[9]The investigation of "classicality" in quantum theory, i.e., the explanation of the appearance of a classical world of everyday experience within a world governed at a fundamental level by quantum-physical laws, has been studied extensively during the last three decades (see, e.g., [46] and references therein). The emergence of quasiclassical behaviour in quantum theory, the laws of which are so much different from the deterministic, classical laws, is to be explained by quantum mechanics itself. One of the main goals of this research area is the derivation of effective, phenomenological classical equations of motion from the fundamental quantum-dynamical laws [20].

proach to quantum theory a quantum mechanical system is said to exhibit classical behaviour when the probability is high for histories having correlations in time implied by classical deterministic laws [20]. Classicality arises, if and only if the probability distribution for the various histories is strongly peaked about such classical histories.

In general, however, quantum mechanics prevents us from assigning probabilities to arbitrary sets of quantum histories. Due to quantum interference one cannot always assign probabilities to a set of histories in a consistent way. For this to be possible, the set of histories must be decoherent. Decoherence of histories is therefore the most fundamental requirement. It is a prerequisite for classical behaviour. For, only then the corresponding histories of the system may be assigned probabilities obeying the standard *probability sum rules*. Only decoherent sets of histories have a meaningful predictive content within the quantum histories framework.

The central aim within the quantum histories framework is thus to find sets of histories for a given closed quantum system which exhibit vanishing (or negligible) interference with respect to each other. Having found such a set, one can then formally assign, at least approximately, probabilities to its members, which obey the usual sum rules of probability theory. This consistent assignment of a probability distribution to a set of quantum histories leads to the notion of *"consistent histories"*. Usually consistency of histories is achieved by a physical process called decoherence (see, e.g., [46]). It is often possible to divide the given closed system into some subsystems called "system" and "environment". If the latter has a decohering influence on the "system", a consistent set of histories arises. The notion of consistent histories was connected with decoherence by Gell-Mann and Hartle [18] who showed how the emergence of classical behaviour can be formulated in the language of consistent histories. It is due to this connection between the mathematical consistency condition (i.e., consistent assignment of probabilities) and the physical process of decoherence (i.e., disappearance of interference) that consistent histories are very often also called *"decoherent histories"*. In this thesis we regard these two terms as synonyms. We will, however, mainly use the term "decoherent histories". Similarly, the mathematical consistency conditions that will be discussed below, will also be called "decoherence conditions".

2.3.2 Formalism

Let us now briefly introduce the *general formalism* of the decoherent histories approach to quantum mechanics. Further details can be found in the original papers [18, 19, 20].

Let us start with the notion of a *quantum history*. As already stated above, a "quantum history" is defined to be a *time-ordered sequence of quantum mechanical "propositions"*. In quantum mechanics, propositions about the attributes of a system at a fixed moment of time are represented by some set of projection operators, $\{P_\alpha\}$, on the Hilbert space $\mathcal{H}$ of the system. In order to represent an *exhaustive* set of *mutually exclusive* quantum-mechanical propositions we need to employ a *complete* set of *mutually orthogonal* projectors. This means the set $\{P_\alpha\}$ is such that

$$\forall\, P_{\alpha'},\, P_{\alpha''} \in \{P_\alpha\} \;:\; P_{\alpha'} P_{\alpha''} = \delta_{\alpha'\alpha''} P_{\alpha'} \quad \text{and} \quad \sum_\alpha P_\alpha = \mathbb{1}_{\mathcal{H}} \;, \tag{2.41}$$

where $\mathbb{1}_{\mathcal{H}}$ denotes the unit operator on the Hilbert space $\mathcal{H}$ of the given system. A projector is said to be completely *fine-grained* if it corresponds to precise specification of a complete set of commuting observables; in this case it projects on a one-dimensional subspace of $\mathcal{H}$ and can thus be represented as $P_\alpha = |\alpha\rangle\langle\alpha|$, where $\{|\alpha\rangle\}_\alpha$ forms an orthonormal basis of $\mathcal{H}$. Otherwise the projector is called *coarse-grained*. A coarse-grained projector corresponds to imprecise specification of a complete set of commuting observables or a precise specification of an incomplete set (cf. [19]).

In the literature on the decoherent histories formalism the formal definition of a history is usually given in terms of a time-ordered chain of *time-dependent* projectors in the *Heisenberg picture*,

$$C_{\boldsymbol{\alpha}} = P^k_{\alpha_k}(t_k) P^{k-1}_{\alpha_{k-1}}(t_{k-1}) \cdots P^1_{\alpha_1}(t_1) \;, \tag{2.42}$$

with time ordering $t_k > t_{k-1} > \cdots > t_1$. The bold letter $\boldsymbol{\alpha}$ is a shorthand notation for the sequence (string) of alternatives α_1 to α_k, i.e., $\boldsymbol{\alpha} \equiv (\alpha_1, \ldots \alpha_k)$. The superscript "$j$" in $P^j_{\alpha_j}(t_j)$ denotes the *particular set* of projectors that has been chosen at time t_j, whereas α_j denotes the particular alternative which has been chosen from within this particular set. One thus allows for the possibility of using different types of complete sets of orthogonal projectors at different times within the chain. These different sets are labeled by the superscript j. It should also be noted that, in general, the number n_j of mutually exclusive alternatives α_j, i.e., the number of elements of the complete set $\{P^j_{\alpha_j}(t_j)\}_{\alpha_j\in\{1,\ldots,n_j\}}$, is different for different times t_j, $j \in \{1, \ldots, k\}$. The histories are also allowed to be explicitely *branch-dependent*. That is, we allow the possibility that the set of alternatives at time t_j explicitely depends on the realisation of earlier alternatives $\alpha_1, \ldots, \alpha_{j-1}$. In this case a better notation than given by Eq. (2.42) for the chain of projectors would be the following:

$$C_{\boldsymbol{\alpha}} = P^k_{\alpha_k}(t_k; \alpha_{k-1}, \ldots, \alpha_1) P^{k-1}_{\alpha_{k-1}}(t_{k-1}; \alpha_{k-2}, \ldots, \alpha_1) \cdots P^1_{\alpha_1}(t_1) \ . \tag{2.43}$$

Clearly, this introduces an *arrow of time* into the description, unless the branch-dependence is extended to future alternatives as well.

The formal definition of quantum histories in terms of time-ordered chains of *time-dependent* projection operators in the Heisenberg picture includes the unitary time-evolution of the system. The Heisenberg picture projection operators $P^j_{\alpha_j}(t_j)$ are related to the (time-independent) Schrödinger picture projection operators $P^j_{\alpha_j}$ at the initial time t_0 according to

$$P^j_{\alpha_j}(t_j) = U^\dagger(t_j, t_0) P^j_{\alpha_j} U(t_j, t_0) \ , \tag{2.44}$$

where $U(t'', t')$ is the unitary time evolution operator of the system. The framework considered in this thesis (see Chap. 3), however, uses a different definition of quantum histories. In Chap. 3 we define histories as time-ordered sequences of *Schrödinger picture projection operators*, so that the unitary dynamics of the system is *not* included in the definition of histories. This notion of histories bears a closer resemblance to symbolic sequences in the theory of classical symbolic dynamics.

Starting with a set of chain operators $\{C_{\boldsymbol{\alpha}}\}$ for all possible strings[10] $\boldsymbol{\alpha}$ we can introduce a *coarser-grained* set of histories in the following way. We first partition the set $\{\boldsymbol{\alpha}\}$ of all possible strings $\boldsymbol{\alpha}$ into *disjoint classes* $[\boldsymbol{\alpha}]$, [11]

$$\{\boldsymbol{\alpha}\} = \dot{\bigcup_{\boldsymbol{\alpha}' \in \{\boldsymbol{\alpha}\}}} [\boldsymbol{\alpha}'] \ , \tag{2.45}$$

[10] The set of all possible strings $\boldsymbol{\alpha}$ is given by $\{\boldsymbol{\alpha}\} = \{\alpha_1\}_{\alpha_1} \times \{\alpha_2\}_{\alpha_2} \times \cdots \times \{\alpha_k\}_{\alpha_k}$, and consists of $\prod_{j=1}^k n_j$ elements, where n_j is the cardinality of the set of alternatives $\{\alpha_j\}_{\alpha_j}$ at time t_j.

[11] Thus, $\forall \boldsymbol{\alpha}', \boldsymbol{\alpha}'' \in \{\boldsymbol{\alpha}\}$ either $[\boldsymbol{\alpha}'] = [\boldsymbol{\alpha}'']$ or $[\boldsymbol{\alpha}'] \cap [\boldsymbol{\alpha}''] = \emptyset$.

such that every $\boldsymbol{\alpha}' \in \{\boldsymbol{\alpha}\}$ belongs to just one and only one class $[\boldsymbol{\alpha}]$. We then can regard the set of all such classes, $\{[\boldsymbol{\alpha}]\}$, as a coarse-graining of the original set of strings given by $\{\boldsymbol{\alpha}\}$. This new set of alternatives, $\{[\boldsymbol{\alpha}]\}$, defines a set of histories which is a coarse-graining of the original set. The elements of this coarse-grained set of histories are represented by the operators $C_{[\boldsymbol{\alpha}]}$, which are given by

$$C_{[\boldsymbol{\alpha}]} = \sum_{\boldsymbol{\alpha}' \in [\boldsymbol{\alpha}]} C_{\boldsymbol{\alpha}'} = \sum_{(\alpha'_1,\ldots,\alpha'_k) \in [\boldsymbol{\alpha}]} P^k_{\alpha'_k}(t_k) P^{k-1}_{\alpha'_{k-1}}(t_{k-1}) \cdots P^1_{\alpha'_1}(t_1) \ . \tag{2.46}$$

Note that from (2.41) it follows that

$$\sum_{[\boldsymbol{\alpha}]} C_{[\boldsymbol{\alpha}]} = \sum_{[\boldsymbol{\alpha}]} \sum_{\boldsymbol{\alpha}' \in [\boldsymbol{\alpha}]} C_{\boldsymbol{\alpha}'} = \sum_{\boldsymbol{\alpha}' \in \{\boldsymbol{\alpha}\}} C_{\boldsymbol{\alpha}'} = \mathbb{1}_{\mathcal{H}} \ . \tag{2.47}$$

The set $\{C_{[\boldsymbol{\alpha}]}\}$ is a coarse-graining of the original set of histories $\{C_{\boldsymbol{\alpha}}\}$. If the original set $\{C_{\boldsymbol{\alpha}}\}$ is already a coarse-grained set of histories, then $\{C_{[\boldsymbol{\alpha}]}\}$ is even *coarser*-grained. Operators that are given by sums of chains of projectors are also called *class operators*. Note that it is in general not possible to represent the class operators themselves by chains of projectors.

The main task of the decoherent histories formalism is to predict probabilities for quantum histories. As explained in the introduction, a consistent assignment of probabilities to quantum histories is not always possible — due to quantum interference. In what follows we "derive" the condition which is necessary and sufficient for a consistent assignment of a probability distribution to a set of quantum histories.

Let us first discuss how one would calculate the probabilities of quantum histories in the case when a consistent assignment is possible. The probability for a single event P_α in quantum mechanics (in the sense of the "collapse interpretation" within the measurement theory) is given by the usual formula $p_\alpha = \mathrm{Tr}\ [P_\alpha \rho_0 P_\alpha] = \mathrm{Tr}\ [P_\alpha \rho_0 P_\alpha^\dagger]$, where ρ_0 is the initial state of the system. The obvious generalisation to a history $C_{\boldsymbol{\alpha}}$ would be

$$p(\boldsymbol{\alpha}) = \mathrm{Tr}\ [C_{\boldsymbol{\alpha}} \rho_0 C_{\boldsymbol{\alpha}}^\dagger] \ . \tag{2.48}$$

Indeed, this would be the probability for a sequence of *measured* alternatives $\alpha_1, \ldots, \alpha_k$ in a situation with an initial state ρ_0 and unitary evolution between measurements. However, the decoherent histories approach concerns *closed* quantum systems. It makes no reference to measurements or collapse. The projectors characterizing the histories mustn't be interpreted as dynamical events such as quantum jumps in a sequence of measurements. Nevertheless, it is tempting to postulate the formula (2.48) as a reasonable candidate probability for a history within a consistent set of histories.

Consider now an arbitrary exhaustive set of mutually exclusive histories $\{C_{\boldsymbol{\alpha}}\}$. A consistent assignment of probabilities to the individual members of this set is possible if and only if the probability sum rules are satisfied for *all possible coarse-grainings* of this set. So let $\{C_{[\boldsymbol{\alpha}]}\}$ be an *arbitrary coarse-graining* of the original set $\{C_{\boldsymbol{\alpha}}\}$ as described above. Then, the probability sum rules to be satisfied are that the probability of *each* coarser-grained history out of $\{C_{[\boldsymbol{\alpha}]}\}$ should be equal to the sum of the probabilities of the finer-grained histories out of $\{C_{\boldsymbol{\alpha}}\}$ of which it is comprised. So consider *any* coarser-grained history $C_{[\boldsymbol{\alpha}]}$ out of the set $\{C_{[\boldsymbol{\alpha}]}\}$. Its probability, $\mathrm{p}([\boldsymbol{\alpha}])$, should then be equal to the sum of the probabilities of all its constituent finer-grained histories of which it is comprised:

$$p([\boldsymbol{\alpha}]) = \sum_{\boldsymbol{\alpha}' \in [\boldsymbol{\alpha}]} p(\boldsymbol{\alpha}') \ . \tag{2.49}$$

This probability sum rule, however, will *in general not be satisfied.* Calculating the probability $p([\boldsymbol{\alpha}])$ according to Eq. (2.48) yields:

$$\begin{aligned} p([\boldsymbol{\alpha}]) &= \mathrm{Tr}\,[C_{[\boldsymbol{\alpha}]}\,\rho_0\,C^\dagger_{[\boldsymbol{\alpha}]}] \\ &= \mathrm{Tr}\left[\Big(\sum_{\boldsymbol{\alpha}' \in [\boldsymbol{\alpha}]} C_{\boldsymbol{\alpha}'}\Big)\,\rho_0\,\Big(\sum_{\boldsymbol{\alpha}'' \in [\boldsymbol{\alpha}]} C^\dagger_{\boldsymbol{\alpha}''}\Big)\right] \\ &= \sum_{\boldsymbol{\alpha}' \in [\boldsymbol{\alpha}]}\ \sum_{\boldsymbol{\alpha}'' \in [\boldsymbol{\alpha}]} \mathrm{Tr}\,[C_{\boldsymbol{\alpha}'}\,\rho_0\,C^\dagger_{\boldsymbol{\alpha}''}] \\ &= \sum_{\boldsymbol{\alpha}' \in [\boldsymbol{\alpha}]} \mathrm{Tr}\,[C_{\boldsymbol{\alpha}'}\,\rho_0\,C^\dagger_{\boldsymbol{\alpha}'}] + \sum_{\substack{\boldsymbol{\alpha}',\boldsymbol{\alpha}'' \in [\boldsymbol{\alpha}] \\ \boldsymbol{\alpha} \neq \boldsymbol{\alpha}''}} \mathrm{Tr}\,[C_{\boldsymbol{\alpha}'}\,\rho_0\,C^\dagger_{\boldsymbol{\alpha}''}] \\ &= \sum_{\boldsymbol{\alpha}' \in [\boldsymbol{\alpha}]} p(\boldsymbol{\alpha}') + \sum_{\substack{\boldsymbol{\alpha}',\boldsymbol{\alpha}'' \in [\boldsymbol{\alpha}] \\ \boldsymbol{\alpha} \neq \boldsymbol{\alpha}''}} \mathrm{Tr}\,[C_{\boldsymbol{\alpha}'}\,\rho_0\,C^\dagger_{\boldsymbol{\alpha}''}] \\ &\neq \sum_{\boldsymbol{\alpha}' \in [\boldsymbol{\alpha}]} p(\boldsymbol{\alpha}') \quad \underline{\text{in general}} \ . \end{aligned} \tag{2.50}$$

Thus, the probability sum rule is satisfied if, and only if, the quantum mechanical interference term on the right hand side vanishes. The presence of this interference term generally prevents us from identifying the terms Tr $[C_{\boldsymbol{\alpha}'}\,\rho_0\,C^\dagger_{\boldsymbol{\alpha}'}]$ with the probabilities of the histories $C_{\boldsymbol{\alpha}'}$.

This analysis motivates the introduction of the so-called *decoherence functional*:

$$\mathcal{D}_{\rho_0}[\,\boldsymbol{\alpha}',\boldsymbol{\alpha}''\,] := \mathrm{Tr}\;[C_{\boldsymbol{\alpha}'}\,\rho_0\,C^\dagger_{\boldsymbol{\alpha}''}]\ . \tag{2.51}$$

It is the mathematical object which tells us whether or not probabilities may be assigned to histories in a consistent way, and what those probabilities are. It has the following elementary properties:

$$\mathcal{D}_{\rho_0}[\,\boldsymbol{\alpha}',\boldsymbol{\alpha}''\,] = \mathcal{D}^*_{\rho_0}[\,\boldsymbol{\alpha}'',\boldsymbol{\alpha}'\,]\quad \text{(Hermiticity)}\ , \tag{2.52}$$

$$\sum_{\boldsymbol{\alpha}'}\sum_{\boldsymbol{\alpha}''}\mathcal{D}_{\rho_0}[\,\boldsymbol{\alpha}',\boldsymbol{\alpha}''\,] = \mathrm{Tr}\;[\rho_0] = 1\ , \tag{2.53}$$

$$\mathcal{D}_{\rho_0}[\,\boldsymbol{\alpha},\boldsymbol{\alpha}\,] \geq 0\ , \tag{2.54}$$

$$\sum_{\boldsymbol{\alpha}}\mathcal{D}_{\rho_0}[\,\boldsymbol{\alpha},\boldsymbol{\alpha}\,] = 1\ . \tag{2.55}$$

From the Hermiticity property it follows that only the real part of the decoherence functional contributes to the interference term in (2.50). A sufficient condition for consistency, therefore, is:

$$\forall\boldsymbol{\alpha}',\boldsymbol{\alpha}'' \in \{\boldsymbol{\alpha}\}\quad \text{such that}\quad \boldsymbol{\alpha}' \neq \boldsymbol{\alpha}''\ :\ \ \mathrm{Re}\Big[\mathcal{D}_{\rho_0}[\,\boldsymbol{\alpha}',\boldsymbol{\alpha}''\,]\Big] = 0 \tag{2.56}$$

This condition is also a necessary condition for consistency because the sum over the off-diagonal terms $\mathcal{D}_{\rho_0}[\,\boldsymbol{\alpha}',\boldsymbol{\alpha}''\,]$ in (2.50) must vanish for *all possible* coarser grainings of the original set $\{C_{\boldsymbol{\alpha}}\}$, implying that all possible sums of the off-diagonal terms must vanish. The condition (2.56) is usually referred to as *"weak decoherence"*. Gell-Mann and Hartle [20] imposed a stronger condition which demands that the non-diagonal elements of the whole decoherence functional be zero:

$$\forall\boldsymbol{\alpha}',\boldsymbol{\alpha}'' \in \{\boldsymbol{\alpha}\}\quad \text{such that}\quad \boldsymbol{\alpha}' \neq \boldsymbol{\alpha}''\ :\ \ \mathcal{D}_{\rho_0}[\,\boldsymbol{\alpha}',\boldsymbol{\alpha}''\,] = 0\ . \tag{2.57}$$

This condition is usually referred to as *"medium decoherence"*. Further consistency conditions, stronger and weaker than the two above, have been discussed in the literature. For a review of them see [37]. If condition (2.56) or (2.57) holds, probabilities may be assigned to the histories of the set $\{C_{\boldsymbol{\alpha}}\}$ in a consistent way, and are given by the diagonal elements of the decoherence functional,

$$p(\boldsymbol{\alpha}) = \mathcal{D}_{\rho_0}[\,\boldsymbol{\alpha}, \boldsymbol{\alpha}\,] \ . \tag{2.58}$$

In most physical models decoherence of histories can be established only *approximately* [19]. Gell-Mann and Hartle [18] argue that it is natural to consider sets of histories for which the probability sum rules are *slightly* violated, by pointing out that, if the violation is sufficiently small, no experiment ever would detect the discrepancy, and that in any case one can always remove this small sum rule violation by an ad hoc, but equally undetectable, renormalization of the probabilities for the histories. Moreover, using naive but very plausible counting arguments, Dowker and Kent demonstrate in [34] that, in the neighborhood of generic approximately consistent sets of histories, an exactly consistent set can be found. This means, as suggested by Dowker and Kent in [34], that there is no need to consider approximately consistent sets in any fundamental discussions on the conceptual issues of the theory.

In the next chapter we employ the general decoherent histories formalism of quantum mechanics to provide the formal framework for the research presented in this thesis.

Chapter 3

Our framework

In this chapter we define and motivate our special framework within the general decoherent histories formulation of quantum mechanics. Our particular mathematical framework is tailored for studying unitary quantum maps within the quantum histories formalism, and is motivated by the method of symbolic dynamics of Sec. 2.1. A formulation of the quantum histories formalism is provided which resembles the symbolic representation techniques of classical dynamical systems. The research presented in this thesis is based on this framework and can thus be regarded as a contribution towards the development of a general theory of "quantum symbolic dynamics".

Definition 1: (*"Projective partitions"*)
A set of projectors $\{P_\mu\}$ on a Hilbert space $\mathcal{H}$ is called a projective *partition* of $\mathcal{H}$, if $\forall\, \mu, \mu' :\ P_\mu P_{\mu'} = \delta_{\mu\mu'} P_\mu$ and $\sum_\mu P_\mu = \mathbb{1}_\mathcal{H}$. Here, $\mathbb{1}_\mathcal{H}$ denotes the unit operator. We will call a projective partition *fine-grained* if all projectors are one-dimensional, i.e., $\forall\, \mu\ \dim\big(\mathrm{supp}(P_\mu)\big) = 1$, [1] and *coarse-grained* otherwise. □

Definition 2: (*"Histories"*)
Given a projective partition $\{P_\mu\}$ of a Hilbert space $\mathcal{H}$, we denote by $\mathcal{K}[\{P_\mu\}\,;\,k\,] := \big\{h_{\boldsymbol{\alpha}} \,|\, h_{\boldsymbol{\alpha}} = (P_{\alpha_1}, P_{\alpha_2}, \ldots, P_{\alpha_k}) \in \{P_\mu\}^k\big\}$ the corresponding exhaustive set of mutually exclusive histories of length k. Histories are thus defined to be ordered sequences of

[1] The *support* of a Hermitian operator A is defined to be the vector space spanned by the eigenvectors of A corresponding to its non-zero eigenvalues.

projection operators, corresponding to quantum-mechanical propositions. Note that we restrict ourselves to histories constructed from a *fixed* projective partition: the projectors P_{α_j} within the sequences are all chosen from the same partition for all "times" $j = 1, \ldots, k$. A set of histories $\mathcal{K}[\{P_\mu\}\,;\,k\,]$ will be called *fine-grained* iff it is constructed from a fine-grained projective partition, it is said to be *coarse-grained* iff the corresponding projective partition $\{P_\mu\}$ is coarse-grained. A single history $h_{\boldsymbol{\alpha}} \in \mathcal{K}[\{P_\mu\}\,;\,k\,]$ is said to be *fine-grained*, iff it is represented by a sequence of one-dimensional projectors (*all* of them being of rank 1), otherwise the history is called *coarse-grained*. Pictorially, coarse-grained histories can also be viewed as bunches of bundled fine-grained histories that are constructed from a fine-grained partition obtained by the process of refinement from the original coarse-grained partition, i.e., the projectors of the latter are partial sums over the projectors of the former. □

Definition 3: (*"classical states"*)
A state represented by the density operator ρ is called *classical with respect to (w.r.t.) a partition* $\{P_\mu\}$ of the Hilbert space $\mathcal{H}$, if it is block-diagonal w.r.t. $\{P_\mu\}$, i.e., if

$$\rho = \sum_\mu P_\mu \, \rho \, P_\mu \ . \tag{3.1}$$

This is equivalent to saying that ρ can be written in the form

$$\rho = \sum_k p_k \rho_k \ , \quad \text{where} \quad \forall k \, \exists \mu \quad \text{such that} \quad \mathrm{Tr}\,[P_\mu \rho_k] = 1 \ . \tag{3.2}$$

The last statement means that for every ρ_k in the convex decomposition $\rho = \sum_k p_k \rho_k$ there exists a $P_\mu \in \{P_\nu\}$ such that $\mathrm{supp}(\rho_k) \subseteq \mathrm{supp}(P_\mu)$. For a given Hilbert space $\mathcal{H}$ and a projective partition $\{P_\mu\}$ of $\mathcal{H}$ we denote by $\mathcal{S}(\mathcal{H})$ the set of all density operators on $\mathcal{H}$, i.e. $\mathcal{S}(\mathcal{H}) \equiv \mathcal{T}_1^+(\mathcal{H})$,[2] and by $\mathcal{S}^{\mathrm{cl}}_{\{P_\mu\}}(\mathcal{H})$ the set of all density operators that are classical w.r.t. $\{P_\mu\}$. Furthermore, we will denote by $\mathcal{S}_{\{P_\mu\}}(\mathcal{H})$ the discrete subset of

[2]In the mathematical literature on spaces of linear operators on Hilbert spaces, $\mathcal{T}_1^+(\mathcal{H})$ is the conventional notation for the closed convex set of all positive trace-class operators $\rho \geq 0$ on the Hilbert space $\mathcal{H}$ with trace equal to 1, $\mathrm{Tr}[\rho] = 1$. This set represents the "set of all physical states" of a system with associated Hilbert space $\mathcal{H}$. Throughout the rest of the thesis this set will be denoted by $\mathcal{S}(\mathcal{H})$.

$\mathcal{S}^{\rm cl}_{\{P_\mu\}}(\mathcal{H})$ of *"partition states"* induced by the partition $\{P_\mu\}$ via normalization:

$$\mathcal{S}_{\{P_\mu\}}(\mathcal{H}) := \left\{ \frac{P_\nu}{\mathrm{Tr}\,[P_\nu]} \;:\; P_\nu \in \{P_\mu\} \right\}. \tag{3.3}$$

In order to keep the notation as simple as possible we will often simply write $\mathcal{S}$, $\mathcal{S}^{\rm cl}_{\{P_\mu\}}$ and $\mathcal{S}_{\{P_\mu\}}$ instead of $\mathcal{S}(\mathcal{H})$, $\mathcal{S}^{\rm cl}_{\{P_\mu\}}(\mathcal{H})$ and $\mathcal{S}_{\{P_\mu\}}(\mathcal{H})$, respectively, as soon as there is just one fixed Hilbert space and no confusion possible. $\Box$

An initial state $\rho \in \mathcal{S}$ and a unitary dynamics generated by a unitary map $U : \mathcal{H} \to \mathcal{H}$ induce a probabilistic structure on the event algebra associated with $\mathcal{K}[\{P_\mu\}\,;\,k]$, if certain consistency conditions are fulfilled. These are given in terms of properties of the *decoherence functional* $\mathcal{D}_{U,\rho}\,[\cdot,\cdot]$ on $\mathcal{K}[\{P_\mu\}\,;\,k\,] \times \mathcal{K}[\{P_\mu\}\,;\,k\,]$, defined by

$$\mathcal{D}_{U,\rho}\,[h_{\boldsymbol\alpha}, h_{\boldsymbol\beta}] := \mathrm{Tr}\left[C_{\boldsymbol\alpha}\, \rho\, C^\dagger_{\boldsymbol\beta} \right] , \tag{3.4}$$

where

$$\begin{aligned} C_{\boldsymbol\alpha} &:= \left(U^{\dagger\,k} P_{\alpha_k} U^k\right) \left(U^{\dagger\,k-1} P_{\alpha_{k-1}} U^{k-1}\right) \ldots \left(U^\dagger P_{\alpha_1} U\right) \\ &= U^{\dagger\,k} P_{\alpha_k} U P_{\alpha_{k-1}} U \ldots P_{\alpha_2} U P_{\alpha_1} U\;. \end{aligned} \tag{3.5}$$

The set $\mathcal{K}[\{P_\mu\}\,;\,k\,]$ is said to be *decoherent* or *consistent* with respect to a given unitary map $U : \mathcal{H} \to \mathcal{H}$ and a given initial state $\rho \in \mathcal{S}$, if

$$\mathcal{D}_{U,\rho}\,[h_{\boldsymbol\alpha}, h_{\boldsymbol\beta}] \propto \delta_{\boldsymbol{\alpha\beta}} \equiv \prod_{j=1}^{k} \delta_{\alpha_j \beta_j} \tag{3.6}$$

for all $h_{\boldsymbol\alpha}, h_{\boldsymbol\beta} \in \mathcal{K}[\{P_\mu\}\,;\,k\,]$. This is the consistency condition. If it is fulfilled, probabilities may be assigned to the histories and are given by the diagonal elements of the decoherence functional, $p[h_{\boldsymbol\alpha}] = \mathcal{D}_{U,\rho}\,[h_{\boldsymbol\alpha}, h_{\boldsymbol\alpha}]$. According to [18, 35, 36] one therefore can define an entropy of the given set of histories:

$$\begin{aligned} H[\{h_{\boldsymbol\alpha}\}] &:= -\sum_{\boldsymbol\alpha} p[h_{\boldsymbol\alpha}] \log_2 p[h_{\boldsymbol\alpha}] \\ &= -\sum_{\boldsymbol\alpha} \mathcal{D}_{U,\rho}\,[h_{\boldsymbol\alpha}, h_{\boldsymbol\alpha}] \log_2 \left(\mathcal{D}_{U,\rho}\,[h_{\boldsymbol\alpha}, h_{\boldsymbol\alpha}]\right)\,. \end{aligned} \tag{3.7}$$

An analogy is worth mentioning here — the analogy between the entropy of a set of histories of length k in quantum mechanics (Eq. (3.7)) and the average uncertainty about the coarse-grained trajectory of length k (cf. Eq. (2.18)) when introducing the Kolmogorov-Sinai entropy within the symbolic dynamics framework of classical dynamical systems (cf. Sec. 2.1.5).

What we have just described is obviously a slightly simplified version of the general decoherent histories formalism as introduced in Sec. 2.3. In general, both the partition and the unitary may depend on the "time"-parameter $j = 1, \ldots, k$. The mathematical framework used here is very similar to the formalism of symbolic dynamics [6, 7], which we introduced in Sec. 2.1. As in the theory of classical dynamical systems we start by partitioning the space of possible system states, using a fixed partition for all times. We proceed by looking for a probability measure over the set of histories — again in close analogy with symbolic dynamics. This analogy has been exploited before in a symbolic dynamics approach to the quantum baker's map [15, 14], which we will also meet and deal with in Chap. 4. Furthermore, as described in Sec. 2.3, several consistency conditions of different strength can be found in the literature [37]. The condition given above is known as *medium decoherence* [20]. Throughout the rest of the thesis we will always employ the medium decoherence condition. It has recently been shown that consideration of the weaker decoherence conditions can be problematic [38].

Although, at a fundamental level, the decoherent histories approach does not need the notion of a measurement, this notion can be very helpful for visualizing the properties of quantum states. For example, a projective partition can be regarded as defining a projective measurement on the system. One can see that classical states are not perturbed by such measurements. Indeed, according to definition 3, a state $\rho \in \mathcal{S}$ is classical with respect to the partition $\{P_\mu\}$ if, and only if, it is invariant under the trace-preserving quantum operation of the corresponding projective measurement: $\sum_\mu P_\mu \, \rho \, P_\mu = \rho$. This typically classical invariance property motivates the name "classical states". Throughout this thesis, especially in the theorems stated below in Chap. 5, we will always choose classical states as the initial states for the histories. This choice is motivated by the fact that only classical states $\rho \in \mathcal{S}^{\rm cl}_{\{P_\mu\}}$ can be "prepared" by the projective measurement defined by $\{P_\mu\}$.

Chapter 4

Classical predictability and coarse-grained evolution of the quantum baker's map

In this chapter we use the decoherent histories formalism to investigate the issue of how classical predictability of the coarse-grained evolution of the quantum baker's map depends on the character of coarse-graining. Using symbolic representation of the quantum baker's map we characterize a coarse-grained description by the *number of scales* at which information is discarded and the *extent* of coarse-graining at every scale. We show that the short-time entropy production is determined by the number of scales at which information is lost rather than the extent of coarse-graining on any particular scale. The duration of the short-time regime, however, is determined by the extent of coarse-graining. Multi-scale coarse-grainings display a significantly more unpredictable evolution than 1-scale coarse-grainings with the same degree of prior knowledge.

4.1 Introduction

Coarse-graining plays a decisive role in deriving classical predictability from the original fundamental quantum-mechanical equations of motion [20, 39]. The form of the effective classical equations of motion is as much influenced by the character of the coarse-graining as by the fundamental quantum-mechanical equations of motion themselves [20]. A systematic way to study coarse-grained quantum evolution is provided by the decoherent histories formalism of quantum mechanics [16, 17, 18, 19, 20]. As already discussed in Sec. 2.3, within this approach to quantum theory a quantum mechanical system is said to exhibit classical behaviour when the probability is high for histories displaying correlations in time implied by classical deterministic laws [20, 39].

Coarse-grained descriptions are utilized also in classical physics. Whenever the number of degrees of freedom is very large and the fundamental regularities at a detailed fine-grained level practically impossible to apply, coarse-graining is introduced involving a much smaller number of variables in the description of the evolution. Effective equations of motion for these coarse-grained variables emerge from this procedure. Whether these effective equations in any way resemble (or can be referred to) some phenomenological equations of motion, e.g., hydrodynamical equations, depends very crucially on the character of coarse-graining.

The character of coarse-graining is important. A given physical system may be described by many alternative sets of coarse-grained variables. Some coarse-grained descriptions, however, are more useful for prediction than others. We normally employ some set of "practical" coarse-grained variables, in terms of which the observables of our interest are simple and slowly varying functions. The way of coarse-graining is especially crucial in quantum theory. An important question arises in this context: what distinguishes "classical" coarse-grainings leading to predictable, deterministic effective classical evolutions from other coarse-grainings? It is a peculiarity of quantum mechanics that, in general, arbitrarily many sets of alternative coarse-grained histories do decohere and so can be assigned probabilities. Moreover, two such decoherent sets of histories are in general mutually incompatible. Which of these many possible coarse-grainings all leading to decoherent sets of histories do in addition involve useful predictability for the evolution of the coarse-grained variables, i.e., useful regularities in time governed by effective, phenomenological equations of motion? This kind of questions have been raised and analyzed by T. Brun and J. B. Hartle in [39]. They investigated the origin of classical predictability by considering the simplest linear system with a continuum description — the linear (one-dimensional) harmonic chain regarded as a closed quantum mechanical system. In their analysis a chain of $\mathcal{N}$ atoms is divided up into groups of N atoms each. Each such group is then itself further subdivided into N/d equally spaced clumps of d atoms each, the clumps being separated from each other by the occurance of clumps (one each) from all other groups, i.e. by the distance $(\mathcal{N}/N) \cdot d$. A coarse-grained description is introduced by restricting attention to the average positions (displayments) of the atoms in a group, these constituting the relevant (or collective) variables defining the "system" under consideration, and ignoring the "internal" coordinates within each group, these constituting its "environment". Further coarse-graining is involved by an imprecise specification of the values of the relevant variables, and this only at certain moments of time. They coarse-grain by equal ranges Δ of the values of the average positions of the atoms in a group at discrete moments of time equally spaced by Δt. In this way a whole family of different coarse-grained descriptions is introduced, parameterized by the four parameters $(N, d, \Delta, \Delta t)$. In the case $d = N$ the chain is divided into local groups, as the N atoms of each group are all neighbors. The corresponding coarse-grained description is therefore entirely *local*. As d decreases from N to 1 the coarse-grained description becomes more and more *nonlocal*. In the extreme case $d = 1$ the N atoms of each group are distributed non-locally over the whole chain, the corresponding coarse-graining being therefore highly *nonlocal*. By comparing several properties concerning classical predictability, in particular decoherence, the intensity of noise and computational complexity, of the members within this family of different coarse-grained descriptions of the chain, they concluded that local coarse-grainings in their family were more useful for deterministic predictability than nonlocal ones.

The purpose of this chapter is to investigate the same issue for a quantum dynamical system displaying chaotic behaviour. We consider coarse-grained evolutions of the quantum baker's map which has been introduced in Sec. 2.2.4. By comparing different coarse-grainings with regard to predictability of the evolution we examine the importance of the character of the coarse-graining for the classical predictability, in close analogy with the analysis in [39]. Predictability of the coarse-grained evolution will be characterized and quantified by the entropy increase during the evolution: the greater the entropy increase is the more unpredictable the evolution becomes. In contrast to the linear dynamical system of [39] the quantum dynamical model investigated here displays chaotic behaviour. It is interesting whether chaoticity of the dynamical system involves a stronger constraint on the character of the coarse-graining than in the case of quantum mechanical systems with linear dynamics.

The rest of this chapter is organized as follows. We start with considering coarse-grained evolution for the quantum baker's map using the decoherent histories formalism. In close analogy with [39] we introduce a family of different coarse-grained descriptions (Sec. 4.2.1) and compare its different members with respect to predictability of the evolution. The main results are summarized in Sec. 4.2.2, their derivations and detailed illustrations are then provided in Sec. 4.2.3. We finally conclude in Sec. 4.3 with a brief discussion of our results.

4.2 Predictability of different coarse-grained descriptions for the quantum baker's map

Our concern is to investigate the issue of how the predictability of the coarse-grained evolution of the quantum baker's map is affected by the character of coarse-graining. Predictability of the coarse-grained evolution will be characterized and quantified by the entropy increase of the evolution: the greater the entropy increase is the more unpredictable the evolution becomes. A systematic way of describing coarse-grained quantum evolution is provided by the *decoherent histories formalism* of quantum mechanics. As described in Sec. 2.3 as well as in Chap. 3, in this formalism a coarse-grained description is given in terms of sets of decoherent coarse-grained histories, which can be thought of as bunches of bundled fine-grained histories and are represented by time-ordered sequences of projection operators with at least one of them being of rank > 1. The entropy of a set of coarse-grained histories has been defined and analyzed in [18, 35, 36]. For its definition, see Eq. (3.7) of Chap. 3.

We start our investigation with defining different coarse-grained descriptions for the quantum baker's map in Sec. 4.2.1. This section contains two parts: in part A we first introduce a family of different coarse-grained descriptions in terms of different types of coarse-grained projective partitions of the Hilbert space; in part B a family of different sets of coarse-grained histories is constructed using these different types of coarse-grained projective partitions. In Sec. 4.2.2 we summarize our main results of this chapter, which are then derived and illustrated in detail in Sec. 4.2.3.

4.2.1 Different coarse-grained descriptions

A. Coarse-grained partitions

Let us first introduce two different types of coarse-grained projective partitions of the 2^N-dimensional Hilbert space modeling the unit square, which later will be regarded as special cases of a family of more general coarse-grained descriptions. We refer to the definitions and notations of Sec. 2.2.4. In particular we use the orthonormal basis (2.38) of the Hilbert space to construct the partitions. It will be convenient to simplify our notation slightly, though. Throughout this chapter n and N are fixed. So we may omit the index n and denote the quantum baker's map simply by B always keeping in mind that we are dealing with the special baker's map B_n for the given value of n.

For a fixed binary string $\boldsymbol{y} = y_1 \dots y_{N-l-r} \in \{0,1\}^{N-l-r}$ we define the "local" projection operators by

$$P_{\boldsymbol{y}}^{(l,r)} \equiv \sum_{\substack{a_1,\dots,a_l \\ b_1,\dots,b_r}} |a_1 \dots a_l \; \boldsymbol{y} \; b_1 \dots b_r\rangle_{n\,n}\langle a_1 \dots a_l \; \boldsymbol{y} \; b_1 \dots b_r| \equiv \sum_{\substack{\boldsymbol{a}\in\{0,1\}^l \\ \boldsymbol{b}\in\{0,1\}^r}} |\boldsymbol{a} \; \boldsymbol{y} \; \boldsymbol{b}\,\rangle_{n\,n}\langle \boldsymbol{a} \; \boldsymbol{y} \; \boldsymbol{b}|\,, \tag{4.1}$$

and for fixed strings $\boldsymbol{y}^1 \in \{0,1\}^{s_1}$ and $\boldsymbol{y}^2 \in \{0,1\}^{s_2}$ we define the "nonlocal" projection operators by

$$P_{\boldsymbol{y}^1,\boldsymbol{y}^2}^{(l,m_l,m_r,r)} := \sum_{\boldsymbol{a}\in\{0,1\}^l} \sum_{\boldsymbol{b}\in\{0,1\}^r} \sum_{\boldsymbol{\xi}\in\{0,1\}^{m_l+m_r}} |\boldsymbol{a}\; \boldsymbol{y}^1\; \boldsymbol{\xi}\; \boldsymbol{y}^2\; \boldsymbol{b}\,\rangle_{n\,n}\langle \boldsymbol{a}\; \boldsymbol{y}^1\; \boldsymbol{\xi}\; \boldsymbol{y}^2\; \boldsymbol{b}\,| \tag{4.2}$$

$\equiv$ (see next page)

$$\equiv \sum_{\substack{a_1,\dots,a_l \\ b_1,\dots,b_r}} \sum_{\substack{\xi_1\cdots\xi_{m_l} \\ \xi_{m_l+1}\cdots\xi_{m_l+m_r}}} |a_1\dots a_l\ \boldsymbol{y}^1\ \xi_1\dots\xi_{m_l}\cdot\xi_{m_l+1}\dots\xi_{m_l+m_r}\boldsymbol{y}^2\ b_1\dots b_r\rangle \times$$

$$\langle a_1\dots a_l\ \boldsymbol{y}^1\ \xi_1\dots\xi_{m_l}\cdot\xi_{m_l+1}\dots\xi_{m_l+m_r}\boldsymbol{y}^2\ b_1\dots b_r| \tag{4.3}$$

What the terms "*local*" and "*non-local*" mean in this context, will be explained below. Throughout this chapter, bold variables denote binary strings. Furthermore, lower indices label individual bits of a string, whereas upper indices will label different strings. It will be convenient to abbreviate a substring $\alpha_\kappa\dots\alpha_\sigma$ of a string $\boldsymbol{\alpha} = \alpha_1\dots\alpha_\kappa\alpha_{\kappa+1}\dots\alpha_\sigma\alpha_{\sigma+1}\dots\alpha_\gamma$ by $\boldsymbol{\alpha}_{\kappa:\sigma}$. Concatenation of strings is defined in the usual way. Taking the just mentioned example we can, for instance, express the string $\boldsymbol{\alpha}$ as a concatenation of three substrings, $\boldsymbol{\alpha} = \boldsymbol{\alpha}_{1:\kappa-1}\boldsymbol{\alpha}_{\kappa:\sigma}\boldsymbol{\alpha}_{\sigma+1:\gamma}$. The length of a string $\boldsymbol{\alpha}$ will be denoted by $|\boldsymbol{\alpha}|$.

For simplicity, we will always assume in the following that $l < n$ and $r < N-n$ in the first case, and $l + s_1 \leq n$ and $r + s_2 \leq N-n$ in the second case. In both cases l and r acquire the specific meaning as the number of "momentum" and "position" bits ignored in the coarse-graining. In the second case, in addition m_l most significant momentum bits and m_r most significant position bits are coarse-grained over.

The operator $P_{\boldsymbol{y}}^{(l,r)}$ is a projector on a 2^{l+r}-dimensional subspace labeled by the string $\boldsymbol{y}$. The projector $P_{\boldsymbol{y}^1,\boldsymbol{y}^2}^{(l,m_l,m_r,r)}$ projects on a $2^{l+m_l+m_r+r}$-dimensional subspace labeled by the pair of strings $(\boldsymbol{y}^1,\boldsymbol{y}^2)$. In both cases we are dealing with complete sets of mutually orthogonal projectors, i.e., with projective partitions, as

$$P_{\boldsymbol{y}}^{(l,r)} P_{\boldsymbol{y}'}^{(l,r)} = \delta_{\boldsymbol{y},\boldsymbol{y}'} P_{\boldsymbol{y}}^{(l,r)} \quad \text{and} \quad \sum_{\boldsymbol{y}} P_{\boldsymbol{y}}^{(l,r)} = \mathbb{1}\ , \tag{4.4}$$

$$P_{\boldsymbol{y}^1,\boldsymbol{y}^2}^{(l,m_l,m_r,r)} P_{\boldsymbol{y}'^1,\boldsymbol{y}'^2}^{(l,m_l,m_r,r)} = \delta_{\boldsymbol{y}^1,\boldsymbol{y}'^1}\delta_{\boldsymbol{y}^2,\boldsymbol{y}'^2} P_{\boldsymbol{y}^1,\boldsymbol{y}^2}^{(l,m_l,m_r,r)} \quad \text{and} \quad \sum_{\boldsymbol{y}^1,\boldsymbol{y}^2} P_{\boldsymbol{y}^1,\boldsymbol{y}^2}^{(l,m_l,m_r,r)} = \mathbb{1}\ . \tag{4.5}$$

Let us explain what is meant by "*local*" and "*nonlocal*" regarding the above projection operators. The projectors $P_{\boldsymbol{y}}^{(l,r)}$ and $P_{\boldsymbol{y}^1,\boldsymbol{y}^2}^{(l,m_l,m_r,r)}$ project on subspaces of the Hilbert space associated with phase-space regions of the unit square in which the corresponding eigenstates of eigenvalue 1 are localized. In the case of the projectors $P_{\boldsymbol{y}}^{(l,r)}$ these regions are *connected* cells whose location within the unit square of the phase space is determined by the specified most significant position and momentum bits given by the binary string $\boldsymbol{y} = y_1 \dots y_{N-l-r}$. The size of these cells depends on the significance of the scales which are not resolved and therefore ignored, i.e., coarse-grained over. In the case of the projectors $P^{(l,m_l,m_r,r)}_{\boldsymbol{y}^1,\boldsymbol{y}^2}$, on the other hand, coarse-graining is involved also at the most significant scales: a number of the most significant position and momentum bits are not specified. The associated phase space domains must therefore consist of *disconnected parts* spread over the whole unit square, the number depending on how many most significant position and momentum bits are coarse-grained over, i.e. on the parameter $m := m_l + m_r$. Examples are discussed in Fig. 4.1.

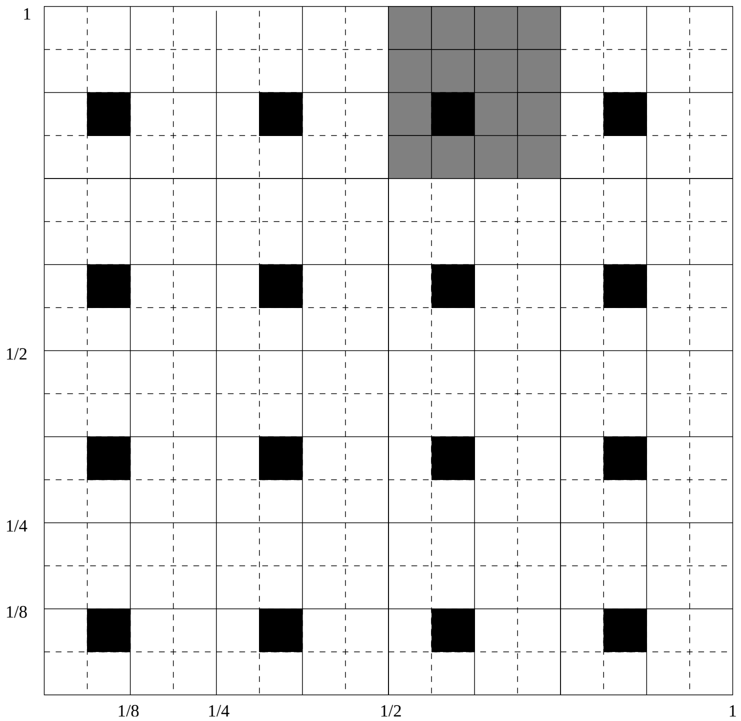

Figure 4.1: A *crude* illustration of the projectors $P^{(l,r)}_{\boldsymbol{y}}$ and $P^{(l,m_l,m_r,r)}_{\boldsymbol{y}^1,\boldsymbol{y}^2}$. *Example (a):* For $n - l = 2$ and $N - l - r = 4$ let $\boldsymbol{y} = y_1 \dots y_4 = 1110$. The projector $P^{(l,r)}_{\boldsymbol{y}=1110}$ is *crudely* associated with a phase space region depictured by the big *grey block* within the unit square. *Example (b)* : Let $n - l = 4$, $m_l = 2$, $m_r = 2$, $\boldsymbol{y}^1 = 10$ and $\boldsymbol{y}^2 = 01$. The projector $P^{(l,2,2,r)}_{\boldsymbol{y}^1=10,\,\boldsymbol{y}^2=01}$ can be *crudely* associated with a phase space region depictured by *disconnected black cells* spread over the unit square.

We will also use the following diagram notation for the introduced projectors:

$$P_{\boldsymbol{y}}^{(l,r)} \equiv (\underbrace{\Box\Box\ldots\Box}_{l}\ \boldsymbol{y}\ \underbrace{\Box\Box\ldots\Box}_{r})\ , \tag{4.6}$$

$$P_{\boldsymbol{y}^1,\boldsymbol{y}^2}^{(l,m_l,m_r,r)} \equiv (\underbrace{\Box\Box\ldots\Box}_{l}\ \boldsymbol{y}^1\ \underbrace{\Box\Box\ldots\Box}_{m_l}\ .\ \underbrace{\Box\Box\ldots\Box}_{m_r}\ \boldsymbol{y}^2\ \underbrace{\Box\Box\ldots\Box}_{r})\ , \tag{4.7}$$

where the empty boxes indicate the bits which are coarse-grained over. Obviously we can write the projectors of the second type as sums over projectors of the first type:

$$P_{\boldsymbol{y}^1,\boldsymbol{y}^2}^{(l,m_l,m_r,r)} = \sum_{\boldsymbol{\xi}\in\{0,1\}^{m_l+m_r}} P_{\boldsymbol{y}^1\boldsymbol{\xi}\boldsymbol{y}^2}^{(l,r)}\ , \tag{4.8}$$

where $\boldsymbol{y}^1\boldsymbol{\xi}\boldsymbol{y}^2$ means the concatenation of the three strings $\boldsymbol{y}^1$, $\boldsymbol{\xi}$ and $\boldsymbol{y}^2$. Remember that in the definition of the projectors $P_{\boldsymbol{y}^1,\boldsymbol{y}^2}^{(l,m_l,m_r,r)}$ we assume that $l+|\boldsymbol{y}^1|\leq n$ and $r+|\boldsymbol{y}^2|\leq N-n$.

The projectors (4.6) and (4.7) are special cases of the family of all projection operators, which define the scales at which information is lost in the symbolic representation. In general such projectors exhibit structure on many different scales, and the most general projector of this type would be one with *multi-scale coarse-graining*:

$$P_{\boldsymbol{y}^1,\boldsymbol{y}^2,\ldots,\boldsymbol{y}^\lambda}^{(l,m_1,m_2,\ldots,m_{\lambda-1},r)} =$$
$$= \Big(\underbrace{\Box\ldots\Box}_{l}\ \boldsymbol{y}^1\ \underbrace{\Box\ldots\Box}_{m_1}\ \boldsymbol{y}^2\ \underbrace{\Box\ldots\Box}_{m_2}\ \ldots\ldots\ \boldsymbol{y}^{\lambda-1}\ \underbrace{\Box\ldots\Box}_{m_{\lambda-1}}\ \boldsymbol{y}^\lambda\ \underbrace{\Box\ldots\Box}_{r}\Big)\ . \tag{4.9}$$

The projector (4.9) defines a coarse-graining in which information is lost on several different scales. We will call this a "*multi-scale coarse-graining*" or "*hierarchical coarse-graining*". Accordingly, the special cases (4.6) and (4.7) will be called 1-scale and 2-scale coarse-graining, respectively. The 2-scale coarse-graining (4.7) we introduced above is a special 2-scale coarse-graining, as we assumed that the coarse-grained "island" of size m_l+m_r between the specified strings $\boldsymbol{y}^1$ and $\boldsymbol{y}^2$ lies around the dot separating the momentum and position bits in the symbolic representation. The first step towards a generalization is to combine the two parameters "m_l" and "m_r" (i.e. the number of most significant momentum and position bits that are coarse-grained over in the symbolic representation) to a single parameter "$m := m_l+m_r$" and allow the corresponding coarse-grained island of size m between the specified strings $\boldsymbol{y}^1$ and $\boldsymbol{y}^2$ to lie "*anywhere*", not necessarily at the most significant "region" around the dot. The next step is then obviously to introduce several coarse-grained islands of this kind, on several scales, thereby separating several specified bit strings $\boldsymbol{y}^1$, $\boldsymbol{y}^2$, $\boldsymbol{y}^3$, ..., by which the coarse-grained event is characterized. Let the number of the latter be λ. So an event will be specified by λ bit strings $\boldsymbol{y}^1$, $\boldsymbol{y}^2$, ..., $\boldsymbol{y}^\lambda$ of length $|\boldsymbol{y}^i| = s_i$ at a time, separated by $(\lambda - 1)$ coarse-grained "islands" of "size" m_i each, and we arrive at (4.9).

More precisely, the most general family of coarse-grained descriptions is represented by sets of projection operators defined as follows:

$$\begin{aligned} P^{(l,m_1,m_2,\ldots,m_{\lambda-1},r)}_{\boldsymbol{y}^1,\boldsymbol{y}^2,\ldots,\boldsymbol{y}^\lambda} &:= \sum_{\boldsymbol{a}\in\{0,1\}^l} \sum_{\boldsymbol{b}\in\{0,1\}^r} \sum_{\boldsymbol{\xi}^1\in\{0,1\}^{m_1}} \cdots \sum_{\boldsymbol{\xi}^{\lambda-1}\in\{0,1\}^{m_{\lambda-1}}} \\ & \quad |\boldsymbol{a}\ \boldsymbol{y}^1\ \boldsymbol{\xi}^1\ \boldsymbol{y}^2\ \boldsymbol{\xi}^2\ \ldots \boldsymbol{\xi}^{\lambda-1}\boldsymbol{y}^\lambda \boldsymbol{b}\ \rangle_{n\,n}\langle \boldsymbol{a}\ \boldsymbol{y}^1\ \boldsymbol{\xi}^1\ \boldsymbol{y}^2\ \boldsymbol{\xi}^2\ \ldots \boldsymbol{\xi}^{\lambda-1}\boldsymbol{y}^\lambda \boldsymbol{b}\ | \\ &= \sum_{\boldsymbol{\xi}^1\in\{0,1\}^{m_1}} \cdots \sum_{\boldsymbol{\xi}^{\lambda-1}\in\{0,1\}^{m_{\lambda-1}}} P^{(l,r)}_{\boldsymbol{y}^1\boldsymbol{\xi}^1\boldsymbol{y}^2\boldsymbol{\xi}^2\ldots\boldsymbol{y}^{\lambda-1}\boldsymbol{\xi}^{\lambda-1}\boldsymbol{y}^\lambda}\ , \end{aligned} \tag{4.10}$$

where $\boldsymbol{y}^1\boldsymbol{\xi}^1\boldsymbol{y}^2\boldsymbol{\xi}^2\ldots\boldsymbol{y}^{\lambda-1}\boldsymbol{\xi}^{\lambda-1}\boldsymbol{y}^\lambda$ means the concatenation of the particular strings $\boldsymbol{y}^1$, $\boldsymbol{\xi}^1$, $\boldsymbol{y}^2, \ldots, \boldsymbol{\xi}^{\lambda-1}$, $\boldsymbol{y}^\lambda$. We still assume $l < n$ and $r < N - n$. Eq. (4.9) is a diagram notation of Eq. (4.10). It is easily seen that for fixed $m_1, \ldots, m_{\lambda-1}$ the set $\{P^{(l,m_1,m_2,\ldots,m_{\lambda-1},r)}_{\boldsymbol{y}^1,\boldsymbol{y}^2,\ldots,\boldsymbol{y}^\lambda}\}$ forms a projective partition of the Hilbert space, as

$$\begin{aligned} P^{(l,m_1,m_2,\ldots,m_{\lambda-1},r)}_{\boldsymbol{y}^1,\boldsymbol{y}^2,\ldots,\boldsymbol{y}^\lambda} P^{(l,m_1,m_2,\ldots,m_{\lambda-1},r)}_{\boldsymbol{y}'^1,\boldsymbol{y}'^2,\ldots,\boldsymbol{y}'^\lambda} &= \delta_{\boldsymbol{y}^1,\boldsymbol{y}'^1}\delta_{\boldsymbol{y}^2,\boldsymbol{y}'^2}\times\cdots\times\delta_{\boldsymbol{y}^\lambda,\boldsymbol{y}'^\lambda} P^{(l,m_1,m_2,\ldots,m_{\lambda-1},r)}_{\boldsymbol{y}^1,\boldsymbol{y}^2,\ldots,\boldsymbol{y}^\lambda} \\ \text{and} \quad \sum_{\boldsymbol{y}^1,\boldsymbol{y}^2,\ldots,\boldsymbol{y}^\lambda} P^{(l,m_1,m_2,\ldots,m_{\lambda-1},r)}_{\boldsymbol{y}^1,\boldsymbol{y}^2,\ldots,\boldsymbol{y}^\lambda} &= \mathbb{1}\ . \end{aligned} \tag{4.11}$$

B. Coarse-grained histories

In order to investigate coarse-grained *evolution* we now construct coarse-grained *histories*. By considering different types of histories constructed from different types of coarse-grained projective partitions we obtain different coarse-grained effective evolutions. Please remember that in this thesis we restrict ourselves to histories constructed from a *fixed* exhaustive set of mutually exclusive propositions, i.e., the projectors within the time-ordered sequences will all be chosen from the same projective partition, for all "times".

Our investigation of the coarse-grained *evolution* of the quantum baker's map starts with looking at properties of sets of coarse-grained histories corresponding to the special cases of 1-scale and 2-scale coarse-grainings as defined in Eqs. (4.6) and (4.7). We would like to compare, in the first instance, the different members within the family

$$\Big\{ \Big\{ P^{(l,m_l,m_r,r)}_{\boldsymbol{y}^1,\boldsymbol{y}^2} \;:\; \boldsymbol{y}^1 \in \{0,1\}^{s_1} \,,\; \boldsymbol{y}^2 \in \{0,1\}^{s_2} \Big\} \;:\; l, r, m_l, m_r, s_1, s_2 \in \{0,1,2,\dots\}$$
$$\text{such that} \quad l + s_1 \le n \,,\; r + s_2 \le N - n \quad \text{and} \quad l + r + s_1 + s_2 + m_l + m_r = N \Big\} \tag{4.12}$$

of coarse-grained descriptions, parameterized by l, r, s_1, s_2, m_l and m_r, with respect to predictability of the evolution. Our results will concern only such members of this family, though, for which s_1 and s_2 are significantly greater than 1. In fact, we will even have to require that $s_1 \ge m_l + m_r$. Furthermore, in order to obtain the classical limit of the quantum baker's map, we will be considering only members with very large value for the parameter l, as $\hbar \to 0$ will correspond to $l \to \infty$. Finally, the results will show that only $m := m_l + m_r$ matters, and the specification "m_l most significant momentum bits and m_r most significant position bits are coarse-grained over" therefore be unnecessary. Note that the local 1-scale coarse-graining (4.1) is included in this family as the special case $m_l + m_r = 0$. Having first examined this family, we will later look at a more general family of coarse-grained histories, namely the family of sets of histories constructed from projective partitions representing arbitrary multi-scale coarse-graining. It is for the sake of comprehensibility that we first consider the special family (4.12) before looking at the more general and, as will turn out, more interesting case of hierarchical coarse-graining. Also, the family (4.12) displays a *formal* analogy with the analysis in Ref. [39] and on that score worth considering. This formal analogy, however, turned out to be misleading.

The histories corresponding to 1-scale and 2-scale coarse-graining (4.6) and (4.7) will be labeled by finite sequences of strings in the first case and pairs of finite sequences of strings in the second case, respectively:

$$\begin{aligned} h_{\vec{\boldsymbol{y}}} &\equiv \left(P^{(l,r)}_{\boldsymbol{y}^1}, P^{(l,r)}_{\boldsymbol{y}^2}, \ldots, P^{(l,r)}_{\boldsymbol{y}^k}\right) \quad , \\ &= \Big(\ \underbrace{\square\square\ldots\square}_{l}\ \boldsymbol{y}^1\ \underbrace{\square\square\ldots\square}_{r}\ , \\ &\qquad \underbrace{\square\square\ldots\square}_{l}\ \boldsymbol{y}^2\ \underbrace{\square\square\ldots\square}_{r}\ , \ \ldots\ , \\ &\qquad \underbrace{\square\square\ldots\square}_{l}\ \boldsymbol{y}^k\ \underbrace{\square\square\ldots\square}_{r}\ \Big)\ , \end{aligned} \tag{4.13}$$

where $\vec{\boldsymbol{y}} = (\boldsymbol{y}^1, \ldots, \boldsymbol{y}^k)$ is a sequence of strings $\boldsymbol{y}^j \in \{0,1\}^{N-l-r}$, $j = 1, \ldots, k$;

$$\begin{aligned} h_{\vec{\boldsymbol{y}}^1, \vec{\boldsymbol{y}}^2} &\equiv \left(P^{(l,m_l,m_r,r)}_{\boldsymbol{y}^{1,1},\,\boldsymbol{y}^{1,2}}, P^{(l,m_l,m_r,r)}_{\boldsymbol{y}^{2,1},\,\boldsymbol{y}^{2,2}}, \ldots, P^{(l,m_l,m_r,r)}_{\boldsymbol{y}^{k,1},\,\boldsymbol{y}^{k,2}}\right) \quad , \\ &= \Big(\ \underbrace{\square\square\ldots\square}_{l}\ \boldsymbol{y}^{1,1}\ \underbrace{\square\square\ldots\square}_{m_l}\ \cdot\ \underbrace{\square\square\ldots\square}_{m_r}\ \boldsymbol{y}^{1,2}\ \underbrace{\square\square\ldots\square}_{r}, \\ &\qquad \underbrace{\square\square\ldots\square}_{l}\ \boldsymbol{y}^{2,1}\ \underbrace{\square\square\ldots\square}_{m_l}\ \cdot\ \underbrace{\square\square\ldots\square}_{m_r}\ \boldsymbol{y}^{2,2}\ \underbrace{\square\square\ldots\square}_{r},\ \ldots\ , \\ &\qquad \underbrace{\square\square\ldots\square}_{l}\ \boldsymbol{y}^{k,1}\ \underbrace{\square\square\ldots\square}_{m_l}\ \cdot\ \underbrace{\square\square\ldots\square}_{m_r}\ \boldsymbol{y}^{k,2}\ \underbrace{\square\square\ldots\square}_{r}\ \Big)\ , \end{aligned} \tag{4.14}$$

where $(\vec{\boldsymbol{y}}^1, \vec{\boldsymbol{y}}^2) = ((\boldsymbol{y}^{1,1}, \ldots, \boldsymbol{y}^{k,1}), (\boldsymbol{y}^{1,2}, \ldots, \boldsymbol{y}^{k,2}))$ is a pair of finite sequences of strings $\boldsymbol{y}^{j,i} \in \{0,1\}^{s_i}$, $j = 1, \ldots, k$, $i = 1, 2$, labeling the history.

To characterize and quantify predictability of the evolution, we use the rate of the entropy production in the course of time. The greater the rate of the entropy production is the more unpredictable is the evolution. *"Time"* is given in terms of the *number k of iterations* of the quantum baker's map. The dependence of entropy on time is provided by considering and comparing the entropy values of sets of histories of different (increasing) length k, i.e., $H[\mathcal{K}[\{P_\mu\}\,;\,k]]$ with $k = 1, 2, 3, \ldots$. Here $\mathcal{K}[\{P_\mu\}\,;\,k]$ denotes the exhaustive set of histories *of length k* constructed from the given (fixed) projective partition $\{P_\mu\}$, as defined in Chap. 3, whereas the entropy of a set of histories $\{h_{\boldsymbol{\alpha}}\}$ is denoted by $H[\{h_{\boldsymbol{\alpha}}\}]$. An entropy of a set of histories can be defined [18, 35, 36] (see Eq. (3.7) of Chap. 3) as soon as probabilities may be assigned to the individual histories of the set in a consistent way. As explained in Chap. 3, the assignment of a probability distribution to a given set of histories is possible in a consistent way, if the set of histories is decoherent. To check decoherence of a set of histories and be able to calculate the probabilities of the individual histories we need to look at the corresponding decoherence functional. In Sec. 4.2.3 we will provide explicit solutions for the decoherence functionals we are concerned with, namely:

$$\mathcal{D}_{B,\,\rho_0}[h_{\vec{\boldsymbol{y}}}, h_{\vec{\boldsymbol{z}}}] = \mathrm{Tr}\ [P^{(l,r)}_{\boldsymbol{y}^k} B P^{(l,r)}_{\boldsymbol{y}^{k-1}} B \cdots P^{(l,r)}_{\boldsymbol{y}^1} B \rho_0 B^\dagger P^{(l,r)}_{\boldsymbol{z}^1} \cdots B^\dagger P^{(l,r)}_{\boldsymbol{z}^{k-1}} B^\dagger P^{(l,r)}_{\boldsymbol{z}^k}]\ , \quad (4.15)$$

and

$$\begin{aligned}&\mathcal{D}_{B,\,\rho_0}[h_{\vec{\boldsymbol{y}}^1,\,\vec{\boldsymbol{y}}^2}, h_{\vec{\boldsymbol{z}}^1,\,\vec{\boldsymbol{z}}^2}] \equiv \\ &= \mathrm{Tr}\ [P^{(l,m_l,m_r,r)}_{\boldsymbol{y}^{k,1},\,\boldsymbol{y}^{k,2}} B P^{(l,m_l,m_r,r)}_{\boldsymbol{y}^{k-1,1},\,\boldsymbol{y}^{k-1,2}} B \cdots P^{(l,m_l,m_r,r)}_{\boldsymbol{y}^{1,1},\,\boldsymbol{y}^{1,2}} B \rho_0 B^\dagger P^{(l,m_l,m_r,r)}_{\boldsymbol{z}^{1,1},\,\boldsymbol{z}^{1,2}} B^\dagger \cdots B^\dagger P^{(l,m_l,m_r,r)}_{\boldsymbol{z}^{k,1},\,\boldsymbol{z}^{k,2}}]\ ,\end{aligned} \quad (4.16)$$

respectively.

Whether the decoherence functional is diagonal or not depends on the initial state ρ_0. In order to check decoherence of a given set of histories and be able to assign probabilities to them we therefore need to specify the initial state from which the histories start. Here we choose a certain class of states as the initial states for the histories, namely, the discrete set of states that are induced by the given set of projectors (which the histories are composed of) via normalization. We therefore assume the initial state ρ_0 to be of the same form as the events in the histories, i.e. to be proportional to one of the projection operators of the set $\{P^{(l,r)}_{\boldsymbol{y}}\}$ or $\{P^{(l,m_l,m_r,r)}_{\boldsymbol{y}^1,\,\boldsymbol{y}^2}\}$, respectively:

$$\rho_0 = \rho^{(l,r)}_{\boldsymbol{x}} \equiv 2^{-(l+r)} P^{(l,r)}_{\boldsymbol{x}} \equiv 2^{-(l+r)}(\underbrace{\Box\Box\ldots\Box}_{l}\ \boldsymbol{x}\ \underbrace{\Box\Box\ldots\Box}_{r})\ , \quad (4.17)$$

or

$$\begin{aligned}\rho_0 &= \rho^{(l,m_l,m_r,r)}_{\boldsymbol{x}^1,\,\boldsymbol{x}^2} \equiv 2^{-(l+m_l+m_r+r)} P^{(l,m_l,m_r,r)}_{\boldsymbol{x}^1,\,\boldsymbol{x}^2} \\ &\equiv 2^{-(l+m_l+m_r+r)}(\underbrace{\Box\Box\ldots\Box}_{l}\ \boldsymbol{x}^1\ \underbrace{\Box\Box\ldots\Box}_{m_l}\ \cdot\ \underbrace{\Box\Box\ldots\Box}_{m_r}\ \boldsymbol{x}^2\ \underbrace{\Box\Box\ldots\Box}_{r})\ .\end{aligned} \quad (4.18)$$

The normalization factor $2^{-(l+r)}$ or $2^{-(l+m_l+m_r+r)}$, respectively, ensures that ρ_0 is a density operator, i.e. $\mathrm{Tr}\ [\rho_0] = 1$.

All calculations in Sec. 4.2.3 will be based on this choice for the initial states, which we regard as the most *natural choice* within our framework of sets of histories constructed from a given, fixed projective partition. *Rough motivations* for this choice are provided in the Appendix 4.4 of the present chapter.

The results derived in Sec. 4.2.3 and summarized in Sec. 4.2.2 display no significant dependence of the coarse-grained behaviour on the non-locality parameter $m = m_l + m_r$. The whole issue turns out to be more interesting if one considers hierarchical coarse-grainings, i.e. coarse-grained histories with coarseness on several different scales of the phase space. So let us now generalize the family of sets of coarse-grained histories from the 1-scale and 2-scale coarse-grained descriptions considered above to the general case of multi-scale coarse-grainings. The corresponding projective partitions have already been introduced in Eqs. (4.9) and (4.10). The generalized family of coarse-grained descriptions is therefore given by the set:

$$\Big\{ \Big\{ P^{(l,m_1,m_2,\dots,m_{\lambda-1},r)}_{\boldsymbol{y}^1,\boldsymbol{y}^2,\dots,\boldsymbol{y}^\lambda} \Big\}_{\boldsymbol{y}^j\in\{0,1\}^{s_j}} \quad : \quad l,r,m_j,s_j \in \{0,1,2,\dots\} \quad , \ \lambda \in \{1,2,3,\dots\}$$
$$\text{such that} \qquad l + r + \sum_{j=1}^{\lambda-1} m_j + \sum_{j=1}^{\lambda} s_j = N \Big\} \ . \tag{4.19}$$

The members of this family are represented by coarse-grained projective partitions displaying coarseness on several different scales in the symbolic representation, an issue which we agreed up on calling multi-scale coarse-graining or hierarchical coarse-graining. The family is parameterized by $l, r, m_1, \dots, m_{\lambda-1}, \ s_1, \dots, s_\lambda$ and λ with the given constraint $l + r + \sum_{j=1}^{\lambda-1} m_j + \sum_{j=1}^{\lambda} s_j = N$. Again, our results will involve only such members of this family, for which $s_1, \dots, s_\lambda$ have values significantly greater than 1, and the value of l is very large (classical limit).

Our generalized type of histories is now labeled by (finite) sequences of finite sequences of binary strings:

$$\Big\{ h_{\vec{\boldsymbol{y}}^1,\vec{\boldsymbol{y}}^2,\dots,\vec{\boldsymbol{y}}^\lambda} \ : \ \vec{\boldsymbol{y}}^{\,i} = (\boldsymbol{y}^{1,i},\dots,\boldsymbol{y}^{k,i}) \quad \text{with} \quad \boldsymbol{y}^{j,i} \in \{0,1\}^{s_i} \ , \ j = 1,\dots,k \ , \ i = 1,\dots,\lambda \Big\} \ . \tag{4.20}$$

They are explicitly defined by time-ordered sequences of (4.10)-type projection operators:

$$h_{\vec{\boldsymbol{y}}^1,\vec{\boldsymbol{y}}^2,\ldots,\vec{\boldsymbol{y}}^\lambda} \equiv \left(P^{(l,m_1,m_2,\ldots,m_{\lambda-1},r)}_{\boldsymbol{y}^{1,1},\boldsymbol{y}^{1,2},\ldots,\boldsymbol{y}^{1,\lambda}}\ ,\ P^{(l,m_1,m_2,\ldots,m_{\lambda-1},r)}_{\boldsymbol{y}^{2,1},\boldsymbol{y}^{2,2},\ldots,\boldsymbol{y}^{2,\lambda}}\ ,\ldots,\ P^{(l,m_1,m_2,\ldots,m_{\lambda-1},r)}_{\boldsymbol{y}^{k,1},\boldsymbol{y}^{k,2},\ldots,\boldsymbol{y}^{k,\lambda}}\right) \quad . \tag{4.21}$$

To examine decoherence of the set of histories (4.20) and calculate its probability distribution we need to look at the decoherence functional

$$\begin{aligned}
&\mathcal{D}_{B,\rho_0}[h_{\vec{\boldsymbol{y}}^1,\vec{\boldsymbol{y}}^2,\ldots,\vec{\boldsymbol{y}}^\lambda}, h_{\vec{\boldsymbol{z}}^1,\vec{\boldsymbol{z}}^2,\ldots,\vec{\boldsymbol{z}}^\lambda}] = \\
&\quad = \mathrm{Tr}\Big[P^{(l,m_1,m_2,\ldots,m_{\lambda-1},r)}_{\boldsymbol{y}^{k,1},\boldsymbol{y}^{k,2},\ldots,\boldsymbol{y}^{k,\lambda}} B P^{(l,m_1,m_2,\ldots,m_{\lambda-1},r)}_{\boldsymbol{y}^{k-1,1},\boldsymbol{y}^{k-1,2},\ldots,\boldsymbol{y}^{k-1,\lambda}} B\cdots P^{(l,m_1,m_2,\ldots,m_{\lambda-1},r)}_{\boldsymbol{y}^{1,1},\boldsymbol{y}^{1,2},\ldots,\boldsymbol{y}^{1,\lambda}} B\rho_0 B^\dagger \times \\
&\qquad \times P^{(l,m_1,m_2,\ldots,m_{\lambda-1},r)}_{\boldsymbol{z}^{1,1},\boldsymbol{z}^{1,2},\ldots,\boldsymbol{z}^{1,\lambda}} B^\dagger\cdots P^{(l,m_1,m_2,\ldots,m_{\lambda-1},r)}_{\boldsymbol{z}^{k-1,1},\boldsymbol{z}^{k-1,2},\ldots,\boldsymbol{z}^{k-1,\lambda}} B^\dagger P^{(l,m_1,m_2,\ldots,m_{\lambda-1},r)}_{\boldsymbol{z}^{k,1},\boldsymbol{z}^{k,2},\ldots,\boldsymbol{z}^{k,\lambda}}\Big]\ ,
\end{aligned} \tag{4.22}$$

an explicit solution of which is provided in Sec. 4.2.3. Again we will choose the initial state to be proportional to one of the projection operators defining our coarse-grained description, i.e. to one of the (4.10)-type projectors:

$$\rho_0 = \rho^{(l,m_1,m_2,\ldots,m_{\lambda-1},r)}_{\boldsymbol{x}^1,\boldsymbol{x}^2,\ldots,\boldsymbol{x}^\lambda} := 2^{-(l+r+m_1+m_2+\cdots+m_{\lambda-1})} P^{(l,m_1,m_2,\ldots,m_{\lambda-1},r)}_{\boldsymbol{x}^1,\boldsymbol{x}^2,\ldots,\boldsymbol{x}^\lambda}\ , \tag{4.23}$$

with the normalization factor ensuring Tr $[\rho_0] = 1$.

4.2.2 Results

This section provides a summary and a short discussion of the main results of this chapter. For their derivations and illustrations see the following Sec. 4.2.3.

We begin with stating the results concerning the family (4.12). First of all we find that in the asymptotic limit $l \to \infty$ all the corresponding members of this family (i.e., all members with very large parameter value l), provided that $m_l + m_r$ is finite, lead to *decoherent* sets of histories, which is the prerequisite for classicality. For finite, but very large l the decoherence functional is approximately diagonal, which means approximate decoherence of histories. For very large l, the diagonal elements of the decoherence functional, $\mathcal{D}_{B,\rho_0}[h_{\vec{\boldsymbol{y}}^1,\vec{\boldsymbol{y}}^2}, h_{\vec{\boldsymbol{y}}^1,\vec{\boldsymbol{y}}^2}]$, may therefore be interpreted as probabilities of the corresponding histories. Furthermore we find that, in the classical limit of very large l, for all members of the corresponding subset within this family, for which

s_1 and s_2 are significantly greater than 1, the probabilities of the individual alternative histories of a set are peaked over histories which display regularities according to the *classical shift property.*

In order to compare the predictability of the evolution the rates of entropy increase of the different sets within the family (4.12) of coarse-grainings are compared. The result for the local coarse-graining (4.1), i.e., for the case $m_l + m_r = 0$, was obtained in [15]. In [15] it was shown that in this case the coarse-grained quantum baker's map exhibits a linear entropy increase at an asymptotic rate given by the Kolmogorov-Sinai entropy [6] of the classical chaotic baker's map, namely 1 bit per iteration step:

$$H[\{h_{\vec{\boldsymbol{y}}}\}] = k + O(\frac{(l+r-k)\log_2(l+r-k)}{2^{l-2(k^2+k)}})\,, \tag{4.24}$$

where the set $\{h_{\vec{\boldsymbol{y}}}\}$ consists of histories of *length* k. In the limit of large l, for any fixed number of iterations, k, the entropy of the coarse-grained quantum baker's map approaches the value of k bits, i.e., 1 bit per iteration.

For nonlocal coarse-grainings $m_l + m_r \neq 0$, the derivation in the next section shows that the entropy increase is 2 bits per iteration step as long as the number of iterations k of the quantum baker's map is smaller than $m_l + m_r$, $k \leq m_l + m_r$, independently of the value of the non-locality parameter $m := m_l + m_r$. As soon as the number of iterations exceeds the value of m, $k > m_l + m_r$, the entropy increase becomes 1 bit per iteration step like in the case of local coarse-graining ($m = 0$). This conclusion can be drawn from the following results:

- Entropy after k iteration steps in case $k \leq m_l + m_r$:

$$H[\{h_{\vec{\boldsymbol{y}}^1,\vec{\boldsymbol{y}}^2}\}] = 2k \ + \ \mathcal{O}(\frac{(l+r-k)\log_2(l+r-k)}{2^{l-2(k^2+(1+m_l+m_r)k)}}) \tag{4.25}$$

- Entropy after k iteration steps in case $k \geq m_l + m_r$:

$$H[\{h_{\vec{\boldsymbol{y}}^1,\vec{\boldsymbol{y}}^2}\}] = k + \ (m_l + m_r) \ + \ \mathcal{O}(\frac{(l+r-k)\log_2(l+r-k)}{2^{l-2(k^2+(1+m_l+m_r)k)}})\,. \tag{4.26}$$

Thus, the greater the non-locality parameter $m = m_l + m_r$ is *the longer* will there be an entropy increase of 2 bits per iteration step. The larger m is the longer does the coarse-grained evolution exhibit more unpredictability as compared to the local coarse-graining ($m = 0$), where the evolution exhibits an entropy increase of 1 bit per iteration step only.

The dependence of the coarse-grained behaviour on the non-locality parameter $m = m_l + m_r$ turns out to be not striking. The behaviour in the *short-time regime*, which we define by $k < m_l + m_r$, is independent of the extent of the non-locality in the above sense. In the short-time regime we always get an entropy increase of 2 bits per iteration step, independently of the value of m, provided that $m > 0$. In the *long-time regime*, defined by $k \gg m_l + m_r$, the entropy increase is always 1 bit per iteration step, and the entire entropy after k iteration steps is only insignificantly dependent on the parameter $m = m_l + m_r$, if $k \gg m$. What is determined by the value of m, though, is the short-time regime itself, i.e., the duration of the regime with a greater entropy increase as compared to the evolution in the case of local coarse-graining ($m = 0$).

The whole issue becomes more interesting in the case of *hierarchical coarse-grainings*, i.e., coarse-grained histories with coarse-graining on several different scales of the phase space. In Sec. 4.2.3.B we will find that *hierarchically* structured families of coarse-grained descriptions display significant differences between their members with respect to predictability of evolution. Again, the results concern only members with very large value for l (classical limit) and s_j-values ($j = 1, 2, \ldots, \lambda$) that are significantly greater than 1. Again we find approximate decoherence for such sets of histories and a probability distribution which is peaked over histories displaying regularities according to the classical shift property. The entropy production in the *short-time regime* which we now define to be given by $k < \min\{m_1, m_2, \ldots, m_{\lambda-1}\}$ is displayed by the following result:

- Entropy after k iteration steps in case $k \leq \min\{m_1, m_2, \ldots, m_{\lambda-1}\}$:

$$H[\{h_{\vec{\boldsymbol{y}}^1, \vec{\boldsymbol{y}}^2, \ldots, \vec{\boldsymbol{y}}^\lambda}\}] = \lambda \cdot k \;+\; \mathcal{O}(\frac{(l + r - k)\log_2(l + r - k)}{2^{l-2(k^2+(1+m_1+m_2+\cdots+m_{\lambda-1})k)}}) \;. \tag{4.27}$$

In the limit of huge l, for any fixed number of iterations $k < \min\{m_1, m_2, \ldots, m_{\lambda-1}\}$, the entropy of the coarse-grained quantum baker's map approaches the value of λk bits, i.e., λ *bits per iteration.* Thus, the short-time behaviour depends extremely on the character of the coarse-graining. The more scales are coarse-grained over in the symbolic representation of the dynamics the more unpredictable is the evolution in the short-time regime. The entropy increase (in bits) per iteration step of the quantum baker's map in the short-time regime is determined by the number of scales which are coarse-grained over in the symbolic representation. The extent of coarse-graining at that scales, i.e. the number of bits that are coarse-grained over at a time on several scales, determines the short-time regime, i.e. the duration of the short-time behaviour.

The behaviour in the long-time regime $k \gg \max\{m_1, m_2, \ldots, m_{\lambda-1}\}$, on the other hand, is independent of the character of the coarse-graining: all members display the same entropy increase, namely approximately just 1 bit per iteration step, as is expressed by the result

- Entropy after k iteration steps in case $k > \max\{m_1, m_2, \ldots, m_{\lambda-1}\}$:

$$H[\{h_{\vec{\boldsymbol{y}}^1,\vec{\boldsymbol{y}}^2,\ldots,\vec{\boldsymbol{y}}^\lambda}\}] = k + \sum_{i=1}^{\lambda-1} m_i \;+\; \mathcal{O}(\frac{(l+r-k)\log_2(l+r-k)}{2^{l-2(k^2+(1+m_1+m_2+\cdots+m_{\lambda-1})k)}})\;. \qquad (4.28)$$

Finally, we would like to note how all the above results for the entropy production in the various coarse-grained descriptions can be understood using the shift property of the coarse-grained evolution of the quantum baker's map, which is explained and illustrated in detail in the next section. For this illustration it is very useful to make use of our diagram notation (4.9). The shift property implies that only such histories are allowed to arise with significant probabilities that satisfy the shift condition: the projectors of the histories have to be related to the initial state via a shift. For instance, if $\rho_0 \propto P^{(l,m_1,m_2,\ldots,m_{\lambda-1},r)}_{\boldsymbol{x}^1,\boldsymbol{x}^2,\ldots,\boldsymbol{x}^\lambda}$ then only such histories can arise with significant probabilities whose first event, represented by the projector $P^{(l,m_1,m_2,\ldots,m_{\lambda-1},r)}_{\boldsymbol{y}^{1,1},\boldsymbol{y}^{1,2},\ldots,\boldsymbol{y}^{1,\lambda}}$, satisfies the shift constraint. Unless $\boldsymbol{y}^{1,1}_{1:(s_1-1)} = \boldsymbol{x}^1_{2:s_1}$ and $\boldsymbol{y}^{1,2}_{1:(s_2-1)} = \boldsymbol{x}^2_{2:s_2}$ and $\ldots$ and $\boldsymbol{y}^{1,\lambda}_{1:(s_\lambda-1)} = \boldsymbol{x}^\lambda_{2:s_\lambda}$ is satisfied by the first event the whole history will have a vanishing probability. On the other hand the last bits $y^{1,1}_{s_1}$, $y^{1,2}_{s_2}$, $\ldots$, $y^{1,\lambda}_{s_\lambda}$ of the strings $\boldsymbol{y}^{1,1}$, $\boldsymbol{y}^{1,2}$, $\ldots$, $\boldsymbol{y}^{1,\lambda}$, which denote the first event of the history, remain undetermined, because the unspecified bits of the empty boxes (see the diagram notation (4.9)) are shifted onto them, which are coarse-grained over. The bits $y^{1,1}_{s_1}$, $y^{1,2}_{s_2}$, $\ldots$, $y^{1,\lambda}_{s_\lambda}$ may therefore be chosen arbitrarily involving a branching into 2^λ possible histories with non-vanishing probabilities.

This branching into 2^λ alternatives repeats with each iteration step of the evolution, as long as $k < \min\{m_1, m_2, \ldots, m_{\lambda-1}\}$, leading to an entropy production of λ bits per iteration step. As soon as the number of iterations k starts to exceed, step by step, the values of $m_1, m_2, \ldots, m_{\lambda-1}$, the entropy production (per iteration step) goes down, step by step, from the value λ to the value 1 in the long-time regime. Consider, for instance, the case in which $k > m_{\lambda-1}$. Only in the first $m_{\lambda-1}$ iteration steps coarse-grained bits (the empty boxes of our diagram notation) are shifted onto the last bits of the strings $\boldsymbol{y}^{j,\lambda-1}$ making them by this means unspecified, i.e. arbitrarily chooseable for the history, and therefore involving a branching into two alternatives meaning an entropy increase of 1 bit. In the subsequent, remaining $k - m_{\lambda-1}$ iteration steps the string $\boldsymbol{x}^\lambda$ of the initial condition enters the scale of the $\boldsymbol{y}^{j,\lambda-1}$-strings, with the consequence that the last bits of the strings $\boldsymbol{y}^{m_{\lambda-1}+1,\lambda-1}, \ldots, \boldsymbol{y}^{k,\lambda-1}$ become determined by the initial condition, which means no branching and therefore no entropy increase.

4.2.3 Derivation and illustration of the results

A. 1-scale and 2-scale coarse-grainings

The decoherence functional for the locally coarse-grained histories (4.15) was calculated in [15]. We briefly review the corresponding result, which is:

$$\mathcal{D}_{B,\rho_{\boldsymbol{x}}^{(l,r)}}[h_{\vec{\boldsymbol{y}}},h_{\vec{\boldsymbol{z}}}] = 2^{-k}\underbrace{\left(\prod_{j=1}^{k}\delta_{\boldsymbol{y}^j}^{\boldsymbol{z}^j}\right)}_{\text{diagonal}}\cdot\underbrace{\left(\delta_{\boldsymbol{y}^1_{1:\gamma-1}}^{\boldsymbol{x}_{2:\gamma}}\prod_{j=1}^{k-1}\delta_{\boldsymbol{y}^{j+1}_{1:\gamma-1}}^{\boldsymbol{y}^j_{2:\gamma}}\right)}_{\text{step-by-step shift}}\cdot\underbrace{\left(\delta_{\boldsymbol{y}^k_{1:\gamma-k}}^{\boldsymbol{x}_{k+1:\gamma}}\right)}_{k\text{th shift}} + O\left(\frac{l+r-k}{2^{l-2(k^2+k)}}\right), \tag{4.29}$$

where $\gamma := |\boldsymbol{x}| = |\boldsymbol{y}^j| = |\boldsymbol{z}^j| = N-(l+r)$. Let us briefly explain the content of this result. The expression in the first parentheses is zero for all off-diagonal elements of the decoherence functional. In the limit of very large l all off-diagonal elements of the decoherence functional vanish, the decoherence condition being therefore established. The diagonal elements of the decoherence functional can therefore be interpreted as probabilities of the corresponding histories (see Ref. [19] for a discussion of approximate decoherence). Asymptotically, only 2^k diagonal elements survive. Moreover, the error terms are exponentially small. We therefore get 2^k histories with asymptotically equal probabilities. The number of such histories doubles after each iteration step resulting in a loss of information at the rate of 1 bit per step. This information loss is quantified by the entropy increase of the set of histories. Since in the limit of large l the set of histories $\{h_{\vec{\boldsymbol{y}}}\}$ is decoherent, the individual alternative histories may be assigned probabilities, which are then given by $p[h_{\vec{\boldsymbol{y}}}] = \mathcal{D}_{B,\rho_{\boldsymbol{x}}^{(l,r)}}[h_{\vec{\boldsymbol{y}}},h_{\vec{\boldsymbol{y}}}]$. Having found the probability distribution we may also define the entropy of the set of all possible alternative histories:

$$\begin{aligned} H[\{h_{\vec{\boldsymbol{y}}}\}] &:= -\sum_{\vec{\boldsymbol{y}}} p[h_{\vec{\boldsymbol{y}}}]\log_2 p[h_{\vec{\boldsymbol{y}}}] \\ &\equiv -\sum_{\vec{\boldsymbol{y}}} \mathcal{D}_{B,\rho_{\boldsymbol{x}}^{(l,r)}}[h_{\vec{\boldsymbol{y}}},h_{\vec{\boldsymbol{y}}}]\log_2\left(\mathcal{D}_{B,\rho_{\boldsymbol{x}}^{(l,r)}}[h_{\vec{\boldsymbol{y}}},h_{\vec{\boldsymbol{y}}}]\right). \end{aligned} \tag{4.30}$$

With (4.29) we obtain:

$$H[\{h_{\vec{\boldsymbol{y}}}\}] = k + O\left(\frac{(l+r-k)\log_2(l+r-k)}{2^{l-2(k^2+k)}}\right). \tag{4.31}$$

In the limit of large l, for any fixed number of iterations, k, the entropy of the coarse-grained quantum baker's map approaches the value of k bits, i.e., 1 bit per iteration.

What kind of histories arise with significant probabilities? This is determined by the expressions within the second and third parentheses of the result (4.29). According to them only histories that satisfy the *step-by-step shift condition* arise with significant probabilities: the projectors of the histories have to be related to the initial state via a shift. For clarity it is very useful to illustrate the issue by means of our diagram notation of the histories introduced above:

$$\begin{array}{c}
\underbrace{\square\square\ldots\square}_{l}\ \ \boldsymbol{x}_1\underline{\boldsymbol{x}_2\ldots\boldsymbol{x}_{\gamma-2}\boldsymbol{x}_{\gamma-1}\boldsymbol{x}_\gamma}\ \ \underbrace{\square\square\ldots\square}_{r}\ , \\
\swarrow \\
\underbrace{\square\square\ldots\square}_{l}\ \ \overline{\boldsymbol{y}^1_1\underline{\boldsymbol{y}^1_2\ldots\boldsymbol{y}^1_{\gamma-2}\boldsymbol{y}^1_{\gamma-1}}}\underline{\ y^1_\gamma}\ \ \underbrace{\square\square\ldots\square}_{r}\ , \\
\swarrow \\
\underbrace{\square\square\ldots\square}_{l}\ \ \overline{\boldsymbol{y}^2_1\underline{\boldsymbol{y}^2_2\ldots\ \boldsymbol{y}^2_{\gamma-2}y^2_{\gamma-1}}}\underline{\ y^2_\gamma}\ \ \underbrace{\square\square\ldots\square}_{r}\ , \\
\swarrow \\
\cdots \\
\swarrow \\
\underbrace{\square\square\ldots\square}_{l}\ \overline{\boldsymbol{y}^k_1\ldots\boldsymbol{y}^k_{\gamma-k}y^k_{\gamma-k+1}\cdots}\,y^k_\gamma\ \underbrace{\square\square\ldots\square}_{r}\ .
\end{array} \tag{4.32}$$

This diagram illustrates symbolically the content of the result (4.29). The first line of this diagram represents the initial condition $\rho^{(l,r)}_{\boldsymbol{x}}$. The subsequent lines correspond to the projectors $P^{(l,r)}_{\boldsymbol{y}^1},\ldots,P^{(l,r)}_{\boldsymbol{y}^k}$ constituting the history $h_{\vec{\boldsymbol{y}}}$. The step-by-step shift condition is depicted by arrows and lines. Underlined substrings are shifted onto those overlined substrings which are indicated by arrows. In order to fulfill the step-by-step shift condition all underlined and overlined substrings that are connected by an arrow have to be equal. In this way it becomes clear which bits of the symbolic specification of a history are completely determined by the initial condition. These bits are indicated by using bold face. The other bits, which are not in bold face, may be chosen arbitrarily. For instance, in the first iteration step the initial condition substring $\boldsymbol{x}_{2:\gamma}\equiv x_2\ldots x_\gamma$ is shifted onto the substring $\boldsymbol{y}^1_{1:\gamma-1}\equiv y^1_1\ldots y^1_{\gamma-1}$. The first

$\gamma - 1$ bits of the string $\boldsymbol{y}^1$ of the first event in the history $h_{\vec{\boldsymbol{y}}}$ are therefore determined by the initial condition. Unless $\boldsymbol{y}^1_{1:\gamma-1} = \boldsymbol{x}_{2:\gamma}$ is satisfied by the first event, the whole history will have a vanishing probability. On the other hand the last bit y^1_γ of the string $\boldsymbol{y}^1$, which denotes the first event of the history, remains undetermined, because the unspecified bit of the empty box is shifted onto it, which is coarse-grained (i.e. summed) over. The bit y^1_γ may therefore be chosen arbitrarily involving a branching into two possible histories with non-vanishing probabilities and therefore an entropy increase of 1 bit. This procedure repeats with each iteration step of the evolution. For the entire history, therefore, there are only k independent bits which can be chosen arbitrarily, given the step-by-step shift constraint.

The calculation of the decoherence functional (4.16) for the non-locally coarse-grained histories can be traced back to using the above result for the local ones. To do so, we may express all the nonlocal projection operators appearing in the decoherence functional as sums over suitable local ones:

$$\begin{aligned}
\rho^{(l,m_l,m_r,r)}_{\boldsymbol{x}^1,\,\boldsymbol{x}^2} &= 2^{-(l+m_l+m_r+r)} \sum_{\boldsymbol{\xi}\in\{0,1\}^{m_l+m_r}} P^{(l,r)}_{\boldsymbol{x}^1\boldsymbol{\xi}\boldsymbol{x}^2} \;, && (4.33)\\
P^{(l,m_l,m_r,r)}_{\boldsymbol{y}^{j,1},\,\boldsymbol{y}^{j,2}} &= \sum_{\boldsymbol{\eta}^j\in\{0,1\}^{m_l+m_r}} P^{(l,r)}_{\boldsymbol{y}^{j,1}\boldsymbol{\eta}^j\boldsymbol{y}^{j,2}} \;, \quad j = 1,2,\ldots,k \;, && (4.34)\\
P^{(l,m_l,m_r,r)}_{\boldsymbol{z}^{j,1},\,\boldsymbol{z}^{j,2}} &= \sum_{\boldsymbol{\zeta}^j\in\{0,1\}^{m_l+m_r}} P^{(l,r)}_{\boldsymbol{z}^{j,1}\boldsymbol{\zeta}^j\boldsymbol{z}^{j,2}} \;, \quad j = 1,2,\ldots,k \;. && (4.35)
\end{aligned}$$

By inserting these expressions into the decoherence functional (4.16) we arrive at:

$$\begin{aligned}
&\mathcal{D}_{B,\,\rho_0}[h_{\vec{\boldsymbol{y}}^1,\,\vec{\boldsymbol{y}}^2}, h_{\vec{\boldsymbol{z}}^1,\,\vec{\boldsymbol{z}}^2}] = \\
&= \sum_{\boldsymbol{\eta}^1\in\{0,1\}^{m_l+m_r}} \cdots \sum_{\boldsymbol{\eta}^k\in\{0,1\}^{m_l+m_r}} \sum_{\boldsymbol{\xi}\in\{0,1\}^{m_l+m_r}} \sum_{\boldsymbol{\zeta}^1\in\{0,1\}^{m_l+m_r}} \cdots \sum_{\boldsymbol{\zeta}^k\in\{0,1\}^{m_l+m_r}} \\
&\quad 2^{-(m_l+m_r)}\mathrm{Tr}\Big[\, P^{(l,r)}_{\boldsymbol{y}^{k,1}\boldsymbol{\eta}^k\boldsymbol{y}^{k,2}} B P^{(l,r)}_{\boldsymbol{y}^{k-1,1}\boldsymbol{\eta}^{k-1}\boldsymbol{y}^{k-1,2}} B \cdots B P^{(l,r)}_{\boldsymbol{y}^{1,1}\boldsymbol{\eta}^1\boldsymbol{y}^{1,2}} \times \\
&\quad \times\; B\Big(\frac{1}{2^{l+r}} P^{(l,r)}_{\boldsymbol{x}^1\boldsymbol{\xi}\boldsymbol{x}^2}\Big) B^\dagger P^{(l,r)}_{\boldsymbol{z}^{1,1}\boldsymbol{\zeta}^1\boldsymbol{z}^{1,2}} B^\dagger \cdots P^{(l,r)}_{\boldsymbol{z}^{k-1,1}\boldsymbol{\zeta}^{k-1}\boldsymbol{z}^{k-1,2}} B^\dagger P^{(l,r)}_{\boldsymbol{z}^{k,1}\boldsymbol{\zeta}^k\boldsymbol{z}^{k,2}}\Big] \;.
\end{aligned} \tag{4.36}$$

Each term of the sum over all possible strings $\boldsymbol{\xi}$, $\{\boldsymbol{\eta}^j\}$ and $\{\boldsymbol{\zeta}^j\}$ is, apart from the factor $2^{-(m_l+m_r)}$, a decoherence functional with respect to histories composed of local projectors. Each such term, therefore, results in an expression of the form (4.29), and we obtain:

$$
\begin{aligned}
&\mathcal{D}_{B,\rho_0}[h_{\vec{\boldsymbol{y}}^1,\vec{\boldsymbol{y}}^2}, h_{\vec{\boldsymbol{z}}^1,\vec{\boldsymbol{z}}^2}] = \\
&= \sum_{\boldsymbol{\eta}^1\in\{0,1\}^{m_l+m_r}} \cdots \sum_{\boldsymbol{\eta}^k\in\{0,1\}^{m_l+m_r}} \sum_{\boldsymbol{\xi}\in\{0,1\}^{m_l+m_r}} \sum_{\boldsymbol{\zeta}^1\in\{0,1\}^{m_l+m_r}} \cdots \sum_{\boldsymbol{\zeta}^k\in\{0,1\}^{m_l+m_r}} \\
&\quad 2^{-(m_l+m_r)} \Bigg\{ \; 2^{-k} \underbrace{\left(\prod_{i=1}^{k} \delta^{\boldsymbol{z}^{i,1}\boldsymbol{\zeta}^i\boldsymbol{z}^{i,2}}_{\boldsymbol{y}^{i,1}\boldsymbol{\eta}^i\boldsymbol{y}^{i,2}}\right)}_{\text{diagonal}} \cdot \underbrace{\left(\delta^{\boldsymbol{x}^1_{2:s_1}\boldsymbol{\xi}\,\boldsymbol{x}^2}_{\boldsymbol{y}^{1,1}\boldsymbol{\eta}^1\boldsymbol{y}^{1,2}_{1:s_2-1}} \prod_{j=1}^{k-1} \delta^{\boldsymbol{y}^{j,1}_{2:s_1}\boldsymbol{\eta}^j\boldsymbol{y}^{j,2}}_{\boldsymbol{y}^{j+1,1}\boldsymbol{\eta}^{j+1}\boldsymbol{y}^{j+1,2}_{1:(s_2-1)}}\right)}_{\text{step−by−step shift}} \times \\
&\quad \times \underbrace{\left(\delta^{(\boldsymbol{x}^1\boldsymbol{\xi}\,\boldsymbol{x}^2)_{k+1:\gamma}}_{(\boldsymbol{y}^{k,1}\boldsymbol{\eta}^k\boldsymbol{y}^{k,2})_{1:\gamma-k}}\right)}_{k\text{th shift}} + \mathcal{O}(\frac{l+r-k}{2^{l-2(k^2+k)}}) \Bigg\} \\
&= \sum_{\boldsymbol{\eta}^1\in\{0,1\}^{m_l+m_r}} \cdots \sum_{\boldsymbol{\eta}^k\in\{0,1\}^{m_l+m_r}} \sum_{\boldsymbol{\xi}\in\{0,1\}^{m_l+m_r}} \Bigg\{ \; 2^{-(m_l+m_r)} \cdot 2^{-k} \cdot \underbrace{\left(\prod_{i=1}^{k} \delta^{\boldsymbol{z}^{i,1}\boldsymbol{\eta}^i\boldsymbol{z}^{i,2}}_{\boldsymbol{y}^{i,1}\boldsymbol{\eta}^i\boldsymbol{y}^{i,2}}\right)}_{\text{diagonal}} \times \\
&\quad \times \underbrace{\left(\delta^{\boldsymbol{x}^1_{2:s_1}\xi_1}_{\boldsymbol{y}^{1,1}} \delta^{\boldsymbol{\xi}_{2:(m_l+m_r)}\boldsymbol{x}^2_1}_{\boldsymbol{\eta}^1} \delta^{\boldsymbol{x}^2_{2:s_2}}_{\boldsymbol{y}^{1,2}_{1:s_2-1}} \cdot \prod_{j=1}^{k-1} \delta^{\boldsymbol{y}^{j,1}_{2:s_1}\eta^j_1}_{\boldsymbol{y}^{j+1,1}} \delta^{\boldsymbol{\eta}^j_{2:(m_l+m_r)}\boldsymbol{y}^{j,2}_1}_{\boldsymbol{\eta}^{j+1}} \delta^{\boldsymbol{y}^{j,2}_{2:s_2}}_{\boldsymbol{y}^{j+1,2}_{1:s_2-1}}\right)}_{\text{step−by−step shift}} \times \\
&\quad \times \underbrace{\left(\delta^{(\boldsymbol{x}^1\boldsymbol{\xi}\,\boldsymbol{x}^2)_{k+1:\gamma}}_{(\boldsymbol{y}^{k,1}\boldsymbol{\eta}^k\boldsymbol{y}^{k,2})_{1:\gamma-k}}\right)}_{k\text{th shift}} \Bigg\} + \left(2^{m_l+m_r}\right)^{2k+1} \cdot \mathcal{O}(\frac{l+r-k}{2^{l-2(k^2+k)}})
\end{aligned}
\tag{4.37}
$$

Here γ denotes the length of the strings $\boldsymbol{y}^{j,1}\boldsymbol{\eta}^j\boldsymbol{y}^{j,2}$ and $\boldsymbol{x}^1\boldsymbol{\xi}\,\boldsymbol{x}^2$, respectively, i.e. $\gamma = |\boldsymbol{x}^1\boldsymbol{\xi}\,\boldsymbol{x}^2| = |\boldsymbol{y}^{j,1}\boldsymbol{\eta}^j\boldsymbol{y}^{j,2}| = s_1 + (m_l + m_r) + s_2$.

First of all the sum over all possible $\boldsymbol{\zeta}^j \in \{0,1\}^{m_l+m_r}$, $j = 1, \ldots, k$, collapses due to the term $\prod_{i=1}^{k} \delta^{\boldsymbol{z}^{i,1}\boldsymbol{\zeta}^i\boldsymbol{z}^{i,2}}_{\boldsymbol{y}^{i,1}\boldsymbol{\eta}^i\boldsymbol{y}^{i,2}}$, apart from contributing a factor $2^{k(m_l+m_r)}$ to the error term. Secondly we note that the step-by-step shift condition causes the whole sum $\sum_{\boldsymbol{\eta}^1}\sum_{\boldsymbol{\eta}^2}\cdots\sum_{\boldsymbol{\eta}^k}$ to collapse, apart from contributing a further factor $2^{k(m_l+m_r)}$ to the bound on the error term, which is furthermore enlarged by a factor $2^{(m_l+m_r)}$ stemming from the sum $\sum_{\boldsymbol{\xi}}$. Let us try to comprehend the collapse of the sums $\sum_{\boldsymbol{\eta}^j}$. For a given fixed string $\boldsymbol{\xi}$ out of the sum $\sum_{\boldsymbol{\xi}}$ all $\boldsymbol{\eta}^1, \boldsymbol{\eta}^2, \ldots, \boldsymbol{\eta}^k$ are through the δ-"functions" determined by the string $\boldsymbol{\xi}$ and the given fixed string $\boldsymbol{x}^2$ of the initial condition. The first shift leads to a determination of $\boldsymbol{\eta}^1$: according to $\delta^{\boldsymbol{\xi}_{2:(m_l+m_r)}\boldsymbol{x}^2_1}_{\boldsymbol{\eta}^1}$ the sum over all possible $\boldsymbol{\eta}^1 \in \{0,1\}^{m_l+m_r}$ collapses and only the string $\boldsymbol{\eta}^1 = \boldsymbol{\xi}_{2:(m_l+m_r)}\boldsymbol{x}^2_1$ survives. The second shift determines $\boldsymbol{\eta}^2$, since according to $\delta_{\boldsymbol{\eta}^2}^{\boldsymbol{\eta}^1_{2:(m_l+m_r)}\boldsymbol{y}_1^{1,2}}$ the sum over all possible $\boldsymbol{\eta}^2 \in \{0,1\}^{m_l+m_r}$ collapses and only the string $\boldsymbol{\eta}^2 = \boldsymbol{\eta}^1_{2:(m_l+m_r)}\boldsymbol{y}_1^{1,2} = \boldsymbol{\xi}_{3:(m_l+m_r)}\boldsymbol{x}_1^2\boldsymbol{x}_2^2$ does lead to a non-vanishing contribution to the decoherence functional. It is easy to see that due to the step-by-step shift condition all the sums $\sum_{\boldsymbol{\eta}^j}$, $j = 1, \ldots, k$, collapse and only the strings

$$\boldsymbol{\eta}^j \stackrel{!}{=} \boldsymbol{\xi}_{(j+1):(m_l+m_r)}\boldsymbol{x}^2_{1:j} \tag{4.38}$$

out of these sums survive leading together to a non-vanishing contribution to the decoherence functional. In fact the step-by-step shift condition can also be expressed in the following way:

$$\prod_{j=1}^{k} \delta^{(\boldsymbol{x}^1\boldsymbol{\xi}\,\boldsymbol{x}^2)_{j+1:\gamma}}_{(\boldsymbol{y}^{j,1}\boldsymbol{\eta}^j\boldsymbol{y}^{j,2})_{1:\gamma-j}} \quad , \tag{4.39}$$

meaning that only such strings $\boldsymbol{\eta}^j$ out of the corresponding sums $\sum_{\boldsymbol{\eta}^j}$, $j = 1, \ldots, k$, lead to a non-vanishing contribution to the decoherence functional which are determined by $\boldsymbol{\xi}$ and $\boldsymbol{x}^2$ according to (4.38). Next we note that as a consequence of the step-by-step shift condition also the sum over all possible $\boldsymbol{\xi} \in \{0,1\}^{m_l+m_r}$ collapses. It collapses only *partially* in case $k < m_l + m_r$ and it collapses *completely* in case $k \geq m_l + m_r$. Let us first consider the case $k < m_l + m_r$. After the first shift the first bit of $\boldsymbol{\xi}$ is determined by the last bit of the string $\boldsymbol{y}^{1,1}$ of the given history, i.e. $\xi_1 = y^{1,1}_{s_1}$, according to the term $\delta^{\boldsymbol{x}^1_{2:s_1}\xi_1}_{\boldsymbol{y}^{1,1}}$. The second shift leads to $\delta^{\boldsymbol{y}^{1,1}_{2:s_1}\eta^1_1}_{\boldsymbol{y}^{2,1}}$, so that $\eta^1_1 = y^{2,1}_{s_1}$. But we have $\eta^1_1 = \xi_2$ due to the first shift, so we arrive at a determination of ξ_2, namely $\xi_2 = y^{2,1}_{s_1}$. In this way the sum over all possible $\boldsymbol{\xi} \in \{0,1\}^{m_l+m_r}$ collapses to a sum over all possible $\boldsymbol{\xi}_{(k+1):(m_l+m_r)} \in \{0,1\}^{m_l+m_r-k}$,

$$\sum_{\boldsymbol{\xi}\in\{0,1\}^{m_l+m_r}} \longrightarrow \sum_{\boldsymbol{\xi}_{(k+1):(m_l+m_r)}\in\{0,1\}^{m_l+m_r-k}} \quad , \tag{4.40}$$

since the first k bits $\xi_1, \ldots, \xi_k$ out of the sum $\sum_{\boldsymbol{\xi}}$ have to fulfill the step-by-step shift condition and are therefore determined by $\xi_j = y^{j,1}_{s_1}$. That the first k bits of the string $\boldsymbol{\xi}$ out of the sum $\sum_{\boldsymbol{\xi}}$ are determined by the given history $h_{\vec{\boldsymbol{y}}^1,\vec{\boldsymbol{y}}^2}$ and therefore the sum over the first k bits of $\boldsymbol{\xi} = \xi_1\xi_2\ldots\xi_{m_l+m_r}$ collapses can also be seen by looking at the k-th shift factor which in fact appears as a redundant factor in the result: according to $\delta^{(\boldsymbol{x}^1\boldsymbol{\xi}\,\boldsymbol{x}^2)_{k+1:\gamma}}_{(\boldsymbol{y}^{k,1}\boldsymbol{\eta}^k\boldsymbol{y}^{k,2})_{1:\gamma-k}}$ only such strings $\boldsymbol{\xi}$ out of the sum $\sum_{\boldsymbol{\xi}}$ lead to a non-vanishing contribution to the decoherence functional for which $\boldsymbol{\xi}_{1:k} = \boldsymbol{y}^{k,1}_{(s_1-k+1):s_1}$ holds. The remaining m_l+m_r-k bits of $\boldsymbol{\xi} = \xi_1\xi_2\ldots\xi_{m_l+m_r}$ remain undetermined and are still summed over. There are $2^{m_l+m_r-k}$ possible different substrings $\boldsymbol{\xi}_{k+1:m_l+m_r} \in \{0,1\}^{m_l+m_r-k}$ in this remaining sum leading to a non-vanishing contribution to the decoherence functional. Since the contributions of all these strings are equal, as can be seen by looking at the result, we may replace the remaining sum over all possible $\boldsymbol{\xi}_{k+1:m_l+m_r}$ by the factor $2^{m_l+m_r-k}$. Furthermore, all the δ-terms containing bits of the unspecified strings $\boldsymbol{\xi}$ and $\boldsymbol{\eta}^j$, $j=1,\ldots,k$, which are summed over, may now be replaced by 1 after having been exploited for the determination of that strings $\boldsymbol{\xi}$ and $\boldsymbol{\eta}^j$ out of the sums $\sum_{\boldsymbol{\xi}}$ and $\sum_{\boldsymbol{\eta}^1}\sum_{\boldsymbol{\eta}^2}\cdots\sum_{\boldsymbol{\eta}^k}$ which lead to a non-vanishing contribution to the value of the decoherence functional. In case $k < m_l + m_r$ we therefore arrive at the following result:

- Decoherence functional in case $k < m_l + m_r$:

$$
\begin{aligned}
\mathcal{D}_{B,\rho_0}[h_{\vec{\boldsymbol{y}}^1,\vec{\boldsymbol{y}}^2}, h_{\vec{\boldsymbol{z}}^1,\vec{\boldsymbol{z}}^2}] \; = \; & \underbrace{2^{m_l+m_r-k}\cdot 2^{-(m_l+m_r)}\cdot 2^{-k}}_{=2^{-2k}} \cdot \underbrace{\left(\prod_{i=1}^{k}\delta^{\boldsymbol{z}^{i,1}}_{\boldsymbol{y}^{i,1}}\delta^{\boldsymbol{z}^{i,2}}_{\boldsymbol{y}^{i,2}}\right)}_{\text{diagonal}} \times \\
& \times \underbrace{\left(\delta^{\boldsymbol{x}^1_{2:s_1}}_{\boldsymbol{y}^{1,1}_{1:s_1-1}}\delta^{\boldsymbol{x}^2_{2:s_2}}_{\boldsymbol{y}^{1,2}_{1:s_2-1}}\cdot\prod_{j=1}^{k-1}\delta^{\boldsymbol{y}^{j,1}_{2:s_1}}_{\boldsymbol{y}^{j+1,1}_{1:s_1-1}}\delta^{\boldsymbol{y}^{j,2}_{2:s_2}}_{\boldsymbol{y}^{j+1,2}_{1:s_2-1}}\right)}_{\text{step-by-step shift}} \times \\
& \times \underbrace{\left(\delta^{\boldsymbol{x}^1_{k+1:s_1}}_{\boldsymbol{y}^{k,1}_{1:s_1-k}}\delta^{\boldsymbol{x}^2_{k+1:s_2}}_{\boldsymbol{y}^{k,2}_{1:s_2-k}}\right)}_{k\text{th shift}} + \mathcal{O}\Big(\frac{l+r-k}{2^{l-2(k^2+(1+m_l+m_r)k)}}\Big)\,.
\end{aligned}
\tag{4.41}
$$

Let us now consider the case $k \geq m_l + m_r$. As already mentioned in this case the whole sum $\sum_{\boldsymbol{\xi}}$ collapses to a single string $\boldsymbol{\xi} \in \{0,1\}^{m_l+m_r}$ satisfying the step-by-step shift condition. This can be seen, again, by looking at the k-th shift condition given by the factor $\delta^{(\boldsymbol{x}^1\boldsymbol{\xi}\,\boldsymbol{x}^2)_{k+1:\gamma}}_{(\boldsymbol{y}^{k,1}\boldsymbol{\eta}^k\boldsymbol{y}^{k,2})_{1:\gamma-k}}$; according to this factor each string $\boldsymbol{\xi}$ out of the sum $\sum_{\boldsymbol{\xi}}$ is shifted onto (m_l+m_r) bits of the string $\boldsymbol{y}^{k,1}$, but since $\boldsymbol{y}^{k,1}$ is a *fixed* string specifying the last event of the *given* history, only the string $\boldsymbol{\xi} = \boldsymbol{y}^{k,1}_{(s_1-k+1):(s_1-k+m_l+m_r)}$ out of the sum $\sum_{\boldsymbol{\xi}}$ survives. Here, of course, we assumed that $s_1 \geq k \geq m_l + m_r$! In case $k \geq m_l + m_r$ we therefore receive:

- Decoherence functional in case $k \geq m_l + m_r$:

$$\begin{aligned}\mathcal{D}_{B,\rho_0}[h_{\vec{\boldsymbol{y}}^1,\vec{\boldsymbol{y}}^2}, h_{\vec{\boldsymbol{z}}^1,\vec{\boldsymbol{z}}^2}] \;=\;& 2^{-(m_l+m_r)} \cdot 2^{-k} \cdot \underbrace{\left(\prod_{i=1}^{k} \delta_{\boldsymbol{y}^{i,1}}^{\boldsymbol{z}^{i,1}} \delta_{\boldsymbol{y}^{i,2}}^{\boldsymbol{z}^{i,2}}\right)}_{\text{diagonal}} \times \\ & \times \underbrace{\left(\delta_{\boldsymbol{y}^{1,1}_{1:s_1-1}}^{\boldsymbol{x}^1_{2:s_1}} \delta_{\boldsymbol{y}^{1,2}_{1:s_2-1}}^{\boldsymbol{x}^2_{2:s_2}} \cdot \prod_{j=1}^{k-1} \delta_{\boldsymbol{y}^{j+1,1}_{1:s_1-1}}^{\boldsymbol{y}^{j,1}_{2:s_1}} \delta_{\boldsymbol{y}^{j+1,2}_{1:s_2-1}}^{\boldsymbol{y}^{j,2}_{2:s_2}}\right)}_{\text{step-by-step shift}} \times \\ & \times \underbrace{\left(\delta_{\boldsymbol{y}^{k,1}_{1:s_1-k}}^{\boldsymbol{x}^1_{k+1:s_1}} \delta_{\boldsymbol{y}^{k,1}_{s_1-k+(m_l+m_r)+1\,:\,s_1}}^{\boldsymbol{x}^2_{1:\,k-(m_l+m_r)}} \delta_{\boldsymbol{y}^{k,2}_{1:s_2-k}}^{\boldsymbol{x}^2_{k+1:s_2}}\right)}_{k\text{th shift}} \\ & + \mathcal{O}\left(\frac{l+r-k}{2^{l-2(k^2+(1+m_l+m_r)k)}}\right) . \end{aligned} \tag{4.42}$$

Let us now discuss the obtained results (4.41) and (4.42) for the decoherence functional (4.16). First of all we get approximate decoherence: for very large l the decoherence functional is approximately diagonal. In the asymptotic limit $l \to \infty$ our set of histories $\{h_{\vec{\boldsymbol{y}}^1,\vec{\boldsymbol{y}}^2}\}$ becomes decoherent. The diagonal elements of the functional, $\mathcal{D}_{B,\rho_0}[h_{\vec{\boldsymbol{y}}^1,\vec{\boldsymbol{y}}^2}, h_{\vec{\boldsymbol{y}}^1,\vec{\boldsymbol{y}}^2}]$, may therefore be interpreted as probabilities of the corresponding histories, i.e., $p(h_{\vec{\boldsymbol{y}}^1,\vec{\boldsymbol{y}}^2},) = \mathcal{D}_{B,\rho_0}[h_{\vec{\boldsymbol{y}}^1,\vec{\boldsymbol{y}}^2}, h_{\vec{\boldsymbol{y}}^1,\vec{\boldsymbol{y}}^2}]$. Again there is no single dominant history. Several different histories arise with significant probabilities . In case $k < m_l+m_r$ we get 2^{2k} different histories with asymptotically equal probabilities (given by 2^{-2k}). The number of histories with asymptotically nonzero probabilities becomes four times larger after each iteration step of the quantum baker's map resulting in a *loss of information of* 2 *bits per step*. The entropy increase is therefore 2 *bits per iteration step*, which can also be seen by calculating the entropy of the approximately decoherent set of histories $\{h_{\vec{\boldsymbol{y}}^1,\vec{\boldsymbol{y}}^2}\}$:

- Entropy after k iteration steps in case $k \le m_l + m_r$:

$$
\begin{aligned}
H[\{h_{\vec{\boldsymbol{y}}^1,\vec{\boldsymbol{y}}^2}\}] &= -\sum_{\vec{\boldsymbol{y}}^1,\vec{\boldsymbol{y}}^2} p[h_{\vec{\boldsymbol{y}}^1,\vec{\boldsymbol{y}}^2}]\log_2 p[h_{\vec{\boldsymbol{y}}^1,\vec{\boldsymbol{y}}^2}] \\
&= 2k \;+\; \mathcal{O}\Big(\frac{(l+r-k)\log_2(l+r-k)}{2^{l-2(k^2+(1+m_l+m_r)k)}}\Big)\,. \qquad (4.43)
\end{aligned}
$$

Again, only such histories are allowed to arise with significant probabilities that satisfy the shift condition: the projectors of the histories have to be related to the initial state via a shift. Let us illustrate this issue once again by means of our diagram notation:

$$
\begin{array}{c}
\underbrace{\Box\ldots\Box}_{l}\ \boldsymbol{x}^1_1\underline{\boldsymbol{x}^1_2\ldots\boldsymbol{x}^1_{s_1-2}\boldsymbol{x}^1_{s_1-1}\boldsymbol{x}^1_{s_1}}\ \underbrace{\Box\Box\ldots\Box}_{m_l+m_r}\ \boldsymbol{x}^2_1\underline{\boldsymbol{x}^2_2\ldots\boldsymbol{x}^2_{s_2-2}\boldsymbol{x}^2_{s_2-1}\boldsymbol{x}^2_{s_2}}\ \underbrace{\Box\ldots\Box}_{r} \\
\swarrow \qquad\qquad\qquad\qquad\qquad \swarrow \\
\underbrace{\Box\ldots\Box}_{l}\ \overline{\boldsymbol{y}^{1,1}_1\underline{\boldsymbol{y}^{1,1}_2\ldots\boldsymbol{y}^{1,1}_{s_1-2}\boldsymbol{y}^{1,1}_{s_1-1}}}\underline{\,y^{1,1}_{s_1}}\ \underbrace{\Box\Box\ldots\Box}_{m_l+m_r}\ \overline{\boldsymbol{y}^{1,2}_1\underline{\boldsymbol{y}^{1,2}_2\ldots\boldsymbol{y}^{1,2}_{s_2-2}\boldsymbol{y}^{1,2}_{s_2-1}}}\underline{\,y^{1,2}_{s_2}}\ \underbrace{\Box\ldots\Box}_{r} \\
\swarrow \qquad\qquad\qquad\qquad\qquad \swarrow \\
\underbrace{\Box\ldots\Box}_{l}\ \overline{\boldsymbol{y}^{2,1}_1\underline{\boldsymbol{y}^{2,1}_2\ldots\boldsymbol{y}^{2,1}_{s_1-2}y^{2,1}_{s_1-1}}}\underline{\,y^{2,1}_{s_1}}\ \underbrace{\Box\Box\ldots\Box}_{m_l+m_r}\ \overline{\boldsymbol{y}^{2,2}_1\underline{\boldsymbol{y}^{2,2}_2\ldots\boldsymbol{y}^{2,2}_{s_2-2}y^{2,2}_{s_2-1}}}\underline{\,y^{2,2}_{s_2}}\ \underbrace{\Box\ldots\Box}_{r} \\
\swarrow \qquad\qquad\qquad\qquad\qquad \swarrow \\
\cdots \qquad\qquad\qquad\qquad\qquad \cdots \\
\swarrow \qquad\qquad\qquad\qquad\qquad \swarrow \\
\underbrace{\Box\ldots\Box}_{l}\ \overline{\boldsymbol{y}^{k,1}_1\ldots\boldsymbol{y}^{k,1}_{s_1-k}y^{k,1}_{s_1-k+1}\cdots} y^{k,1}_{s_1}\ \underbrace{\Box\Box\ldots\Box}_{m_l+m_r}\ \overline{\boldsymbol{y}^{k,2}_1\ldots\boldsymbol{y}^{k,2}_{s_2-k}y^{k,2}_{s_2-k+1}\cdots} y^{k,2}_{s_2}\ \underbrace{\Box\ldots\Box}_{r}
\end{array}
\qquad (4.44)
$$

This diagram illustrates symbolically the content of the result (4.41). Again, the first line of this diagram represents the initial condition $\rho_0 = \rho^{(l,m_l,m_r,r)}_{\boldsymbol{x}^1,\boldsymbol{x}^2}$. The subsequent lines correspond to the projectors $P^{(l,m_l,m_r,r)}_{\boldsymbol{y}^{1,1},\boldsymbol{y}^{1,2}},\ldots,P^{(l,m_l,m_r,r)}_{\boldsymbol{y}^{k,1},\boldsymbol{y}^{k,2}}$ representing the subsequent propositions of the history $h_{\vec{\boldsymbol{y}}^1,\vec{\boldsymbol{y}}^2}$. The coarse-grained islands in the middle of each line, with m_l+m_r empty boxes each, subsequently represent the sums $\sum_{\boldsymbol{\xi}}$, $\sum_{\boldsymbol{\eta}^1}, \sum_{\boldsymbol{\eta}^2}, \ldots \sum_{\boldsymbol{\eta}^k}$ in our calculation. Again, the step-by-step shift condition is depicted by arrows and lines. Underlined substrings are shifted onto those overlined substrings which are indicated by arrows. In order to fulfill the step-by-step shift condition all underlined and overlined substrings that are connected by an arrow must be equal. In this way we immediately see which bits of the symbolic specification of a history are completely determined by the initial condition. In the diagram these bits are indicated by using bold face. The remaining bits, which are not in bold face, may be chosen arbitrarily. For instance, in the first iteration step the initial condition substrings $\boldsymbol{x}^1_{2:s_1} \equiv x^1_2 \ldots x^1_{s_1-2} x^1_{s_1-1} x^1_{s_1}$ and $\boldsymbol{x}^2_{2:s_2} \equiv x^2_2 \ldots x^2_{s_2-2} x^2_{s_2-1} x^2_{s_2}$ are shifted onto the substrings $\boldsymbol{y}^{1,1}_{1:(s_1-1)} \equiv y^{1,1}_1 y^{1,1}_2 \ldots y^{1,1}_{s_1-2} y^{1,1}_{s_1-1}$ and $\boldsymbol{y}^{1,2}_{1:(s_2-1)} \equiv y^{2,2}_1 y^{2,2}_2 \ldots y^{2,2}_{s_2-2} y^{2,2}_{s_2-1}$, respectively. The first $(s_1 - 1)$ bits of the string $\boldsymbol{y}^{1,1}$ and the first $(s_2 - 1)$ bits of the string $\boldsymbol{y}^{1,2}$ of the first event in the history are therefore determined by the initial condition. Unless $\boldsymbol{y}^{1,1}_{1:(s_1-1)} = \boldsymbol{x}^1_{2:s_1}$ and $\boldsymbol{y}^{1,2}_{1:(s_2-1)} = \boldsymbol{x}^2_{2:s_2}$ is satisfied by the first event the whole history will have a vanishing probability. On the other hand the last bits $y^{1,1}_{s_1}$ and $y^{1,2}_{s_2}$ of the strings $\boldsymbol{y}^{1,1}$ and $\boldsymbol{y}^{1,2}$, which denote the first event of the history, remain undetermined, because the unspecified bits of the empty boxes are shifted onto them, which are coarse-grained (i.e. summed) over. The bits $y^{1,1}_{s_1}$ and $y^{1,2}_{s_2}$ may therefore be chosen arbitrarily involving a branching into four possible histories with non-vanishing probabilities. This procedure repeats with each iteration step of the evolution. The second step leads to a determination of the first $(s_1 - 1)$ bits of the string $\boldsymbol{y}^{2,1}$ and the first $(s_2 - 1)$ bits of the string $\boldsymbol{y}^{2,2}$ symbolizing the second event of the history, whereas, again, the last bits of these strings remain unspecified and may be chosen arbitrarily implicating a branching into further four alternatives with non-vanishing probabilities. And so on. It becomes clear from the above discussion which histories arise with significant probabilities during the evolution and why the number of alternative equiprobable histories is quadruplicated after each iteration step. After k iteration steps — we still assume $k < m_l + m_r$ — there are therefore $2k$ independent bits which can be chosen arbitrarily, given the step-by-step shift constraint. As a result, 2^{2k} alternative, equiprobable histories arise with significant probability after k iteration steps.

Our result for $k > m_l + m_r$, Eq. (4.42), may be interpreted in the following way. As long as the number of iterations k is smaller than $m = m_l + m_r$ the number of histories with asymptotically non-vanishing probabilities becomes four times larger after each iteration step of the quantum baker's map resulting in an entropy increase of 2 bits per iteration step. As soon as the number of iterations becomes greater than $m = m_l + m_r$, the entropy increase becomes 1 bit per iteration step. This is what is expressed by the result $2^{-(m_l+m_r)} \cdot 2^{-k} = 2^{-2(m_l+m_r)} \cdot 2^{-(k-(m_l+m_r))}$ for the probability of the histories which are allowed to occur. The first $m_l + m_r$ iteration steps lead to an entropy increase of 2 bits per step involving $2^{2(m_l+m_r)}$ asymptotically equiprobable histories. The remaining $k - (m_l + m_r)$ iteration steps produce an entropy increase of 1 bit per step only, with the number of histories with significant probabilities being doubled at each step implicating a branching factor $2^{k-(m_l+m_r)}$. The entire number of histories arising with significant probabilities after k iteration steps therefore becomes $2^{2(m_l+m_r)} \cdot 2^{k-(m_l+m_r)} = 2^{(m_l+m_r)} \cdot 2^k$, the histories being asymptotically equiprobable. Again the issue becomes clearer when using our diagram picture. The "size" of the middle coarse-grained islands is now only $m_l + m_r < k$. So only in the first $m_l + m_r$ iteration steps coarse-grained bits are shifted onto the last bits of the strings $\boldsymbol{y}^{j,1}$, making them by this means unspecified, i.e. arbitrarily chooseable for the history. In the subsequent, remaining $k - (m_l + m_r)$ iteration steps the string $\boldsymbol{x}^2$ of the initial condition enters the scale of the $\boldsymbol{y}^{j,1}$-strings, with the consequence that the last bits of the strings $\boldsymbol{y}^{m_l+m_r+1,1}, \ldots, \boldsymbol{y}^{k,1}$ become determined by the initial condition. At the end, after the k-th iteration step, only $m_l + m_r$ bits of the string $\boldsymbol{y}^{k,1}$ may be chosen arbitrarily, the first $s_1 - k$ bits and the last $k - (m_l + m_r)$ bits of it being determined by the initial condition. On the other hand only the first $s_2 - k$ bits of the string $\boldsymbol{y}^{k,2}$ become determined by the initial condition, whereas all the last k bits of it remain arbitrarily chooseable for the history, provided that $k < r$. This explains the result $2^{(m_l+m_r)} \cdot 2^k$ for the number of alternative histories satisfying the shift constraint. For the entropy of the approximately decoherent set of histories $\{h_{\vec{\boldsymbol{y}}^1, \vec{\boldsymbol{y}}^2}\}$ we get the result:

- Entropy after k iteration steps in case $k \geq m_l + m_r$:

$$\begin{aligned} H[\{h_{\vec{\boldsymbol{y}}^1, \vec{\boldsymbol{y}}^2}\}] &= -\sum_{\vec{\boldsymbol{y}}^1, \vec{\boldsymbol{y}}^2} p[h_{\vec{\boldsymbol{y}}^1, \vec{\boldsymbol{y}}^2}] \log_2 p[h_{\vec{\boldsymbol{y}}^1, \vec{\boldsymbol{y}}^2}] \\ &= k + \ (m_l + m_r) \ + \ \mathcal{O}\Big(\frac{(l+r-k)\log_2(l+r-k)}{2^{l-2(k^2+(1+m_l+m_r)k)}}\Big) \ . \end{aligned} \tag{4.45}$$

B. Hierarchical (multi-scale) coarse-grainings

We will see in the following that by introducing more and more scales that are coarse-grained over in the symbolic representation of the dynamics the short-time behaviour of the coarse-grained evolution of the quantum baker's map will exhibit a growing entropy increase per iteration step, i.e. more unpredictability.

So let us now look at the generalized type of histories (4.20). The evaluation of the corresponding decoherence functional (4.22) is done in a similar way as for the case $\lambda = 2$. We first state the result for the short-time regime which we now define to be given by $k < \min\{m_1, m_2, \ldots, m_{\lambda-1}\}$:

- Decoherence functional in case $k < \min\{m_1, m_2, \ldots, m_{\lambda-1}\}$:

$$\mathcal{D}_{B,\rho_0}[h_{\vec{\boldsymbol{y}}^1,\vec{\boldsymbol{y}}^2,\ldots,\vec{\boldsymbol{y}}^\lambda}\,,\,h_{\vec{\boldsymbol{z}}^1,\vec{\boldsymbol{z}}^2,\ldots,\vec{\boldsymbol{z}}^\lambda}] = 2^{-\lambda k}\cdot\underbrace{\left(\prod_{j=1}^{k}\prod_{i=1}^{\lambda}\delta_{\boldsymbol{y}^{j,i}}^{\boldsymbol{z}^{j,i}}\right)}_{\text{diagonal}}\cdot\underbrace{\left(\prod_{i=1}^{\lambda}\delta_{\boldsymbol{y}^{1,i}_{1:s_i-1}}^{\boldsymbol{x}^i_{2:s_i}}\right)}_{\text{first shift}}\times$$
$$\times\underbrace{\left(\prod_{j=1}^{k-1}\prod_{i=1}^{\lambda}\delta_{\boldsymbol{y}^{j+1,i}_{1:s_i-1}}^{\boldsymbol{y}^{j,i}_{2:s_i}}\right)}_{\text{step-by-step shift}}\cdot\underbrace{\left(\prod_{i=1}^{\lambda}\delta_{\boldsymbol{y}^{k,i}_{1:s_i-k}}^{\boldsymbol{x}^i_{k+1:s_i}}\right)}_{k\text{-th shift}}+$$
$$+\,\mathcal{O}\Big(\frac{l+r-k}{2^{l-2(k^2+(1+m_1+m_2+\cdots+m_{\lambda-1})k)}}\Big)\,. \tag{4.46}$$

In the limit of large l the off-diagonal elements of the decoherence functional vanish and the set of histories becomes decoherent. The diagonal elements of the functional may therefore be interpreted as probabilities. The coarse-grained evolution is again governed by shift constraints. Only such histories are allowed to arise with significant probabilities that satisfy the shift condition, which has been illustrated in detail for the case $\lambda = 2$, see above. Here we are mainly interested in the rate of the entropy increase. The result (4.46) shows that in the short-time regime, i.e. as long as $k < \min\{m_1, m_2, \ldots, m_{\lambda-1}\}$, the coarse-grained evolution exhibits an entropy increase of λ *bits per iteration step*, provided that l is very large (classical limit). This is quantitatively expressed by the entropy of the approximately decoherent set of histories:

- Entropy after k iteration steps in case $k < \min\{m_1, m_2, \ldots, m_{\lambda-1}\}$:

$$
\begin{aligned}
H[\{h_{\vec{\boldsymbol{y}}^1,\vec{\boldsymbol{y}}^2,\ldots,\vec{\boldsymbol{y}}^\lambda}\}] &= -\sum_{\vec{\boldsymbol{y}}^1,\vec{\boldsymbol{y}}^2,\ldots,\vec{\boldsymbol{y}}^\lambda} p[h_{\vec{\boldsymbol{y}}^1,\vec{\boldsymbol{y}}^2,\ldots,\vec{\boldsymbol{y}}^\lambda}] \log_2 p[h_{\vec{\boldsymbol{y}}^1,\vec{\boldsymbol{y}}^2,\ldots,\vec{\boldsymbol{y}}^\lambda}] \\
&= \lambda \cdot k \; + \; \mathcal{O}\left(\frac{(l+r-k)\log_2(l+r-k)}{2^{l-2(k^2+(1+m_1+m_2+\cdots+m_{\lambda-1})k)}}\right) , \qquad (4.47)
\end{aligned}
$$

where we used $p[h_{\vec{\boldsymbol{y}}^1,\vec{\boldsymbol{y}}^2,\ldots,\vec{\boldsymbol{y}}^\lambda}] = \mathcal{D}_{B,\rho_0}[h_{\vec{\boldsymbol{y}}^1,\vec{\boldsymbol{y}}^2,\ldots,\vec{\boldsymbol{y}}^\lambda} \, , \, h_{\vec{\boldsymbol{y}}^1,\vec{\boldsymbol{y}}^2,\ldots,\vec{\boldsymbol{y}}^\lambda}]$.

As far as the long-time regime is concerned, which we define to be given by $k \gg \max\{m_1, m_2, \ldots, m_{\lambda-1}\}$, our analysis yields the following results for the decoherence functional and the value for the entropy:

- Decoherence functional in case $k > \max\{m_1, m_2, \ldots, m_{\lambda-1}\}$:

$$
\begin{aligned}
&\mathcal{D}_{B,\rho_0}[h_{\vec{\boldsymbol{y}}^1,\vec{\boldsymbol{y}}^2,\ldots,\vec{\boldsymbol{y}}^\lambda} \, , \, h_{\vec{\boldsymbol{z}}^1,\vec{\boldsymbol{z}}^2,\ldots,\vec{\boldsymbol{z}}^\lambda}] = \\
&= 2^{-k-(m_1+\cdots+m_{\lambda-1})} \cdot \underbrace{\left(\prod_{j=1}^{k}\prod_{i=1}^{\lambda} \delta_{\boldsymbol{y}^{j,i}}^{\boldsymbol{z}^{j,i}}\right)}_{\text{diagonal}} \cdot \underbrace{\left(\prod_{i=1}^{\lambda} \delta_{\boldsymbol{y}^{1,i}_{1:s_i-1}}^{\boldsymbol{x}^i_{2:s_i}}\right)}_{\text{first shift}} \times \\
&\quad \times \underbrace{\left(\prod_{j=1}^{k-1}\prod_{i=1}^{\lambda} \delta_{\boldsymbol{y}^{j+1,i}_{1:s_i-1}}^{\boldsymbol{y}^{j,i}_{2:s_i}}\right)}_{\text{step-by-step shift}} \cdot \underbrace{\left(\prod_{i=1}^{\lambda} \delta_{\boldsymbol{y}^{k,i}_{1:s_i-k}}^{\boldsymbol{x}^i_{k+1:s_i}}\right)}_{k\text{-th shift}} \\
&\quad + \mathcal{O}\left(\frac{l+r-k}{2^{l-2(k^2+(1+m_1+m_2+\cdots+m_{\lambda-1})k)}}\right) \qquad (4.48)
\end{aligned}
$$

- Entropy after k iteration steps in case $k > \max\{m_1, m_2, \ldots, m_{\lambda-1}\}$:

$$
H[\{h_{\vec{\boldsymbol{y}}^1,\vec{\boldsymbol{y}}^2,\ldots,\vec{\boldsymbol{y}}^\lambda}\}] = k + \sum_{i=1}^{\lambda-1} m_i \; + \; \mathcal{O}\left(\frac{(l+r-k)\log_2(l+r-k)}{2^{l-2(k^2+(1+m_1+m_2+\cdots+m_{\lambda-1})k)}}\right) . \qquad (4.49)
$$

The interpretation of these results is analogous to the special case $\lambda = 2$ of the last section. In the long-time regime $k \gg \max\{m_1, m_2, \ldots, m_{\lambda-1}\}$ the entropy production is always just 1 bit per iteration step, independently of the values of the parameters $m_1, \ldots, m_{\lambda-1}$, which determine the "border" between the regimes. It becomes also clear what happens in the "intermediate regime" at this border. As soon as the number of iterations, k, exceeds, one after another, step by step, the (in general different) values of $m_1, m_2, \ldots, m_{\lambda-1}$, the entropy production (per iteration step) decreases, step by step, from the value λ to the value 1.

4.3 Summary and conclusion

Let us summarize the conclusions of this chapter. We have investigated the issue of how classical predictability of the coarse-grained evolution of the quantum baker's map depends on the character of coarse-graining. Our analysis was motivated by Brun and Hartle's work in [39], which examines the same question for the one-dimensional quantum harmonic chain. As opposed to their system of consideration the quantum dynamical system considered here is a non-linear system displaying chaos. We shall regard our analysis as complementary to Brun and Hartle's investigations.

In our analysis we have compared the members of a family of different coarse-grained descriptions for the quantum baker's map with respect to predictability of its coarse-grained evolution. The family of coarse-grainings we have considered, is parameterized by the *number of scales* at which information is discarded in the symbolic representation on the one hand, and the *extent of coarse-graining* on any particular scale on the other hand. We have employed the decoherent histories formalism to represent coarse-grained evolution, the predictability of which we have characterized and quantified by the entropy production in the course of time. We have found that it is the number of scales at which information is discarded rather than the extent of coarse-graining on that scales what is directly related to the amount of entropy production during the coarse-grained evolution of the quantum baker's map. The short-time entropy production has been shown to be determined by the number of scales that are coarse-grained over. On the other hand, the short-time regime, i.e., the duration of the short-time behaviour, is determined by the extent of coarse-graining on that scales. In summary, *hierarchical* coarse-grainings display a significantly more unpredictable evolution than 1-scale coarse-grainings with the same degree of prior knowledge.

4.4 Appendix — Comments on the choice of the initial states

In our analysis of the coarse-grained evolution of the quantum baker's map we have made a special choice for the initial states of the histories. We have chosen the initial states to be proportional to one of the projectors of the given projective partition defining the coarse-grained description, cf. Eqs. (4.17), (4.18) and (4.23). In this appendix I shall provide some *rough motivations* for this special choice.

Firstly, the corresponding assumption is reasonable if one takes the point of view that *a "state"* of a physical system is the *result of a certain "preparation" procedure*, cf. [40] (p. 92). A projective partition $\{P_{\boldsymbol{y}}^{(l,r)}\}$ or $\{P_{\boldsymbol{y}^1,\boldsymbol{y}^2}^{(l,m_l,m_r,r)}\}$, or $\{P_{\boldsymbol{y}^1,\boldsymbol{y}^2,\ldots,\boldsymbol{y}^\lambda}^{(l,m_1,m_2,\ldots,m_{\lambda-1},r)}\}$, of the Hilbert space can be regarded as defining a projective measurement on the system. Only states that are *block-diagonal* with respect to $\{P_{\boldsymbol{y}}^{(l,r)}\}$ or $\{P_{\boldsymbol{y}^1,\boldsymbol{y}^2}^{(l,m_l,m_r,r)}\}$, or $\{P_{\boldsymbol{y}^1,\boldsymbol{y}^2,\ldots,\boldsymbol{y}^\lambda}^{(l,m_1,m_2,\ldots,m_{\lambda-1},r)}\}$, respectively, can be prepared by such measurements. The density operators (4.17), (4.18) and (4.23) represent the most coarse-grained states that can be prepared in a *selective* projective measurement defined by $\{P_{\boldsymbol{y}}^{(l,r)}\}$ or $\{P_{\boldsymbol{y}^1,\boldsymbol{y}^2}^{(l,m_l,m_r,r)}\}$, or $\{P_{\boldsymbol{y}^1,\boldsymbol{y}^2,\ldots,\boldsymbol{y}^\lambda}^{(l,m_1,m_2,\ldots,m_{\lambda-1},r)}\}$, respectively. The choice of these coarse-grained states as initial states can then be motivated by the *principle of uniform local probability* from statistical mechanics [41, 42], which is the basic assumption for the *classical H-theorem*, i.e. the law of increase of entropy in classical statistical mechanics. This principle states that for the purpose of calculating the probabilities of future events we may always take the initial probability density to be *uniform* within a sufficiently small cell in phase space. It is this empirical principle which introduces an asymmetry in time, involving a distinction between the "future" and the "past" [41, 42]. It is the coarse-grained entropy for the coarse-grained probability density which obeys the classical H-theorem. According to the principle of uniform local probability the absence of a detailed microstructure in the initial probability density is assumed and by this means any entropy-decreasing behaviour of the system avoided. The initial density matrices in our analysis correspond to a quantum version of the principle of uniform local probability together with the notion of a *selective* preparation procedure in a projective measurement defined by the projectors representing the sequential events (propositions) of the histories.

An alternative, very rough, motivation for our special choice of the initial states is provided by a result which will be derived in the next chapter, Sec. 5.4 (see also Ref. [2]). In Sec. 5.4 (and Ref. [2]) we show that, for histories constructed from a fixed projective partition $\{P_\mu\}$ of a finite dimensional Hilbert space, the requirement of decoherence of such histories for arbitrary history lengths and for all initial states that are naturally induced by the projectors $\{P_\mu\}$ via normalization, i.e. $\rho_0 = P_\mu/\mathrm{Tr}\,[P_\mu]$, *implies* the decoherence of such histories *for arbitrary initial states.* I.e., if decoherence is established for arbitrary history lengths and all initial states of the form $\rho_0 = P_\mu/\mathrm{Tr}\,[P_\mu]$, then any set of histories constructed from $\{P_\mu\}$ is decoherent for *all possible* initial states. The crucial prerequisite in the derivation of this result is the *finite dimension* of the Hilbert space. In the present chapter, however, we have to let the dimension of the Hilbert space go to infinity, so as to compute the classical limit of the quantum baker's map. Moreover, in computing the classical limit of a chaotic map, the limit $\hbar = 1/2\pi D \to 0$ has always to be taken *before* the limit $k \to \infty$ for the number of iterations of the map [43], i.e. the history length. In this respect the result of Sec. 5.4 is not applicable to the analysis of the present chapter. Nevertheless, we still regard this result concerning the relationship between decoherence of histories and initial states as a reasonable motivation for our choice (4.17) and (4.18), respectively.

A further motivation is worth mentioning. Our choice for the initial states emerges in a natural way in the absence of any information about the state of the system. If interpreting quantum states as "states of knowledge" — in the fashion of Bayesianism — a completely mixed state, $\rho = \mathbb{1}/\mathrm{Tr}\,[\mathbb{1}]$, is commonly used as a *prior*, to represent complete lack of any knowledge with regard to the system. Choosing such a completely mixed state as the initial state for the histories effectively involves our class of initial states. For, inserting $\rho_0 = \mathbb{1}/\mathrm{Tr}\,[\mathbb{1}]$ into the expression $\mathrm{Tr}\left[C_{\boldsymbol{\alpha}}\,\rho_0\,C^\dagger_{\boldsymbol{\beta}}\right]$ leads to:

$$
\begin{aligned}
\mathrm{Tr}\left[C_{\boldsymbol{\alpha}}\left(\frac{\mathbb{1}}{\mathrm{Tr}\,[\mathbb{1}]}\right)C^\dagger_{\boldsymbol{\beta}}\right] = & \\
= & \ \frac{1}{\mathrm{Tr}\,[\mathbb{1}]}\mathrm{Tr}\Big[P_{\alpha_k}UP_{\alpha_{k-1}}U\dots P_{\alpha_1}\underbrace{U\,\mathbb{1}\,U^\dagger}_{=\mathbb{1}}P_{\beta_1}\dots U^\dagger P_{\beta_{k-1}}U^\dagger P_{\beta_k}\Big] \\
\propto & \ \delta_{\alpha_1\beta_1}\mathrm{Tr}\left[P_{\alpha_k}UP_{\alpha_{k-1}}U\dots P_{\alpha_2}U\left(\frac{P_{\alpha_1}}{\mathrm{Tr}\,[P_{\alpha_1}]}\right)U^\dagger P_{\beta_2}\dots U^\dagger P_{\beta_{k-1}}U^\dagger P_{\beta_k}\right] \\
= & \ \delta_{\alpha_1\beta_1}\mathrm{Tr}\left[C_{\boldsymbol{\alpha}_{2:k}}\left(\frac{P_{\alpha_1}}{\mathrm{Tr}\,[P_{\alpha_1}]}\right)C^\dagger_{\boldsymbol{\beta}_{2:k}}\right]\ , \qquad (4.50)
\end{aligned}
$$

which is, apart from the factor $\delta_{\alpha_1\beta_1}$, the value of the decoherence functional for the pair of histories $h_{\boldsymbol{\alpha}_{2:k}}$ and $h_{\boldsymbol{\beta}_{2:k}}$, both starting from the initial state $\tilde{\rho}_0 = P_{\alpha_1}/\mathrm{Tr}\,[P_{\alpha_1}]$.

Chapter 5

Decoherence properties of arbitrarily long histories

5.1 Introduction

In the last chapter we demonstrated how our decoherent histories framework, as introduced in Chap. 3, can be used to analyze dynamical features of a unitary quantum map. Having examined carefully a special but very interesting quantum map, the quantum version of the classical baker's transformation, within our framework, we would like to proceed with the investigation of properties of arbitrary unitary quantum maps. In this chapter we focus our attention on *decoherence properties*.

The investigation of properties of the quantum baker's map within the decoherent histories framework would be considerably facilitated, if simpler decoherence conditions were available. In the last chapter we considered only families of quite special coarse grainings, namely such which were natural with regard to the symbolic representation of the quantum baker's map. Choosing such families made analytical studies of the dynamical properties feasible. Checking decoherence of histories for other types of coarse-grainings, however, other than that in view of the symbolic representation of the quantum baker's map naturally suggesting coarse grainings, turned out to be very difficult [1]. Research on more general types of coarse-grained descriptions for the quantum baker's map requires *simpler decoherence conditions*, simpler than that given in terms of properties of the decoherence functional, cf. Eq. (3.6).

The need for *simpler decoherence conditions* has been the main motivation for the research presented in this chapter. It is a rather technical issue, but of great relevance to the investigation of dynamical features of quantum maps within the histories framework of quantum mechanics — within which only decoherent sets of histories have predictive content. In general, it is very difficult to decide whether a given set of histories is decoherent. As the length of the histories increases, checking the decoherence conditions (3.6) soon becomes extremely cumbersome. This is especially true when the system dynamics is difficult to simulate as, e.g., in the case of chaotic quantum maps, for which typically only the first iteration is known in closed analytical form [14]. The quantum baker's map is one example for such a map. Establishing decoherence directly, by computing the off-diagonal elements of the decoherence functional (cf. Eq. (3.6)), would require enormous computational resources in the case of large history lengths [2]. It would therefore be of great practical importance to have a simple decoherence criterion that *uses only a single iteration* of a given unitary quantum map.

In the present chapter we study *decoherence properties* of *arbitrarily long* histories constructed from a *fixed projective partition* of a *finite dimensional* Hilbert space. A number of interesting results will have been obtained within this framework. The presentation of these results and their derivations closely follow our published work in [1, 2, 3]. In particular, simple necessary decoherence conditions are derived and the dependence of decoherence on the initial state is investigated. Moreover, a first step towards generalizations of these results to the case of *approximate decoherence* is accomplished.

Unfortunately, however, the results obtained here are not applicable to the research

[1] These investigations and the associated difficulties have not explicitly been discussed in Chap. 4

[2] That we were able to establish, analytically, approximate decoherence for the coarse-grained evolution of the quantum baker's map is due to a very suitable choice of the coarse-grained description! Checking decoherence for other types of coarse-grained histories proved extremely cumbersome.

on the classical limit of the quantum baker's map, or any other chaotic quantum map [3]. For the following reason. The results of the present chapter are valid for situations, in which the Hilbert space dimension $d := \dim(\mathcal{H})$, being arbitrary but finite, is kept *fixed*, and the histories are extended arbitrarily far into the future. Our results are therefore based on the crucial assumption that we deal with a Hilbert space of a *fixed* finite dimension and allow the length of the histories to go to infinity, $k \to \infty$. Thus, if one wants to make the dimension of the Hilbert space arbitrarily large, our results can be valid only if the limit $k \to \infty$ for the history length is taken before the limit $d \to \infty$. On the other hand, when computing the classical limit of the quantum baker's map, or any other chaotic quantum map, the limit $\hbar = 1/(2\pi d) \to 0$, or equivalently $d \to \infty$, has always to be taken *before* the limit $k \to \infty$ for the number of iterations of the map [43], i.e. the history length.

The results derived in this chapter involve decoherence properties of *arbitrarily long histories*. To state it more precisely: given a projective partition $\{P_\mu\}$ of a finite dimensional Hilbert space we address the issue of *under which conditions the sets of histories* $\mathcal{K}[\{P_\mu\}\,;\,k\,]$ *be decoherent for all* $k \in \mathbb{N}$. What is necessary and what is sufficient for this to be true? There exists a trivially sufficient condition: $\mathcal{K}[\{P_\mu\}\,;\,k\,]$ will be decoherent for all $k \in \mathbb{N}$ and all initial states that are classical w.r.t. $\{P_\mu\}$ if the given unitary quantum map U preserves the classicality of all such classical states w.r.t. $\{P_\mu\}$. Preservation of classicality of states guarantees decoherence for *arbitrary lengths*. It turns out, however, that this property of the unitary map (w.r.t. $\{P_\mu\}$) is, in general, *not necessary* for decoherence of $\mathcal{K}[\{P_\mu\}\,;\,k\,]$ for all $k \in \mathbb{N}$. It then becomes natural to ask the question of *what is necessary* for this decoherence issue. Answers will be provided by the results of this chapter.

Why are we interested in decoherence for arbitrary history lengths? This requirement is physically motivated. Decoherence, i.e. consistency of sets of histories, is

[3]The research presented in this chapter originated from studies on the coarse-grained evolution of the quantum baker's map within the decoherent histories framework. The necessity of simpler decoherence conditions issued from these studies. In search of such simpler decoherence conditions, however, results have been obtained, which, unfortunately, are too weak to be applicable to the investigation of the classical limit of chaotic quantum maps within the histories formalism.

a prerequisite for classical behaviour within the decoherent histories framework of quantum mechanics. The requirement, or rather assumption, that decoherence be established for arbitrary history lengths, reflects the belief that the observed phenomenon of decoherence is persistent: with a fixed level of coarse-graining typical systems usually become even more classical with time [4]. It is worth noting, however, that there are problems with pursuing this belief within the consistent histories framework [34, 44, 45] [5]. We do not want to expand on them here, though. In Sec. 5.3.2 we demonstrate, by means of a simple example, that there are unitary maps that lead to completely decoherent sets of histories for all classical initial states and all history lengths up to a certain maximal length K, but induce non-decoherent sets as soon as the length of the histories exceeds the number K. Such maps only pretend "classicality" for a limited period of time, but sooner or later they lead to a non-consistent set of histories. A further motivation for being interested in decoherence being established for arbitrary history lengths is provided by our concern to introduce a description of the evolution which resembles the method of classical symbolic dynamics, where coarse-grained histories are extended infinitely far into the past and into the future so as to obtain doubly-infinitely long symbolic sequences (cf. Sec. 2.1).

Considering *very long* histories can be problematic from the following (practical) point of view. What most people have worked on within the decoherent histories programme are histories of finite length which obey the decoherence principle only *approximately* [18, 19]. Approximate decoherence of histories was briefly introduced in the background chapter, Sec. 2.3, and will be expanded on in Sec. 5.5 at the end of the present chapter, when we generalize our results to the case of approximately consistent histories. The usual assumption is that if the total number of histories is not too large, there is always a closely related set of histories "nearby" for which decoherence is obeyed exactly. In fact, using naive but very plausible counting arguments, Dowker and Kent demonstrated in [34] that, in the neighborhood of generic

[4]No-one would expect the Solar system to look less classical in the future than it does already.

[5]Let me quote Dowker and Kent in [34]: "... present-day quasiclassicality in a consistent set does not imply persisting quasiclassicality in that set, and in fact quasiclassicality does not persist in generic consistent future extensions; this is clearly a nontrivial problem".

approximately consistent sets of histories, an exactly consistent set can always be found. This assumption cannot be sustained any longer if considering very long histories. Considering very long histories involves a huge total number of alternative histories, i.e. a huge dimension of the histories Hilbert space [6]. It is a well known fact that Hilbert spaces with very large dimensions exhibit exponentially-large sets of "almost orthogonal" states. Roughly speaking, this means that, in a Hilbert space with a huge dimension "almost all" vectors are "almost orthogonal". Having learned about this fact, we must realize that, in the case of *very long* histories, the above assumption involving the existence of a "nearby" exactly consistent set of histories in the neighborhood, will fail in a very dramatic way.

Another objection is worth mentioning, an objection to considering arbitrarily long histories in connection with finite dimensional Hilbert spaces. The issue has already been discussed above. In the case of finite dimensional Hilbert spaces the dynamics must be quasi-periodic in the long run. So if we want the unitary map U to represent some quantum chaotic map, then obviously only the short-term regime can display chaotic dynamical features.

We begin our analysis, in the following Sec. 5.2, with proving a very powerful Lemma, which all the subsequent proofs of the results on the decoherence properties mainly will be based on. This technical result (cf. [1]) being used here as an auxiliary tool is remarkable on its own. It involves a quantum *"uniform recurrence" phenomenon* for unitaries acting on finite dimensional Hilbert spaces. Its relevance to the research on decoherence properties of histories becomes clear by the following rough idea: it can be used to show that histories starting from the same initial state will eventually come together again at a later time and thus form a closed loop meaning interference unless some "decoherence properties" are satisfied. This is the basic idea for how we will proceed in deriving our results.

[6]The histories Hilbert space is the vector space whose elements are various histories obtained by allowing for all possible projective partitions of the system Hilbert space $\mathcal{H}$ at different times. The histories Hilbert space for all possible histories of length k is in a way representable as the tensor product $\mathcal{H}^{\otimes k}$, where $\mathcal{H}$ is the Hilbert space of the system in question, and the factors within the tensor product correspond to different times $j = 1, 2, \ldots, k$.

In Section 5.3 we make progress towards finding simpler decoherence conditions for a set of histories. Simple *necessary* decoherence conditions for sets of arbitrarily long histories are provided — *necessary* conditions that employ only a *single iteration* of a given unitary quantum map. The results obtained will raise some issues which a lot have been debated within the decoherent histories community as well as the decoherence programme. We briefly address the issue of "classicality" and some related issues referring to other existing work.

In Sec. 5.4 we then present a mathematically precise demonstration of the surprising result that decoherence of histories that extend infinitely far into the future is in a sense independent of the initial state. More precisely, if decoherence is present for arbitrary history lengths and all initial states from the smallest natural set of states that is associated with our framework, namely the discrete set $\mathcal{S}_{\{P_\mu\}}$, then any set of histories constructed from $\{P_\mu\}$ is decoherent for all possible initial states. Even though obtained within our slightly simplified framework, this quite surprising result is of relevance to the decoherence programme (see, e.g., [46] and references therein). It is especially an interesting result in relation to one of the main issues in quantum cosmology [18, 47]: the issue of how the classical features of our world evolve from the initial quantum state of the Universe.

In the last part of this chapter, Sec. 5.5, *approximate decoherence* of histories is introduced and the corresponding implications examined. There we make a first step towards generalization of the results that will have been obtained for exact decoherence within our framework to the case of a approximate decoherence. The corresponding result can also be found in Ref. [3].

5.2 A quantum recurrence theorem for finite dimensional Hilbert spaces

This section provides a technical result [1], which all our derivations in the present chapter mainly will be based on. We state this auxiliary result, which is very interesting on its own, in the form of a Lemma. We prove the Lemma and discuss its meaning and consequences. An additional corollary (together with its proof) completes the main issues of the Lemma.

Lemma:

Let $\mathcal{H}$ be a finite dimensional Hilbert space, and let U be a unitary map on $\mathcal{H}$. Then $\forall \epsilon > 0 \;\; \exists q \in \mathbb{N}$ such that $\| U^q - \mathbb{1}_{\mathcal{H}} \| < \epsilon$, where $\| \cdot \|$ denotes the conventional operator norm, $\|A\| = \sup\{\| Av \| : \; v \in \mathcal{H}, \; \|v\| = 1\}$ for any operator A on $\mathcal{H}$.

Proof of the Lemma:

Since our Hilbert space is finite dimensional, U has a discrete eigenvalue spectrum. All eigenvalues of a unitary operator have modulus 1. The spectral decomposition of U can therefore be written in the form

$$U = \sum_{j=1}^{d} e^{2\pi i \, \xi_j} \, |\Omega_j\rangle\langle\Omega_j| \; , \tag{5.1}$$

where $d := \dim(\mathcal{H})$, $\xi_1, \ldots, \xi_d$ are real numbers, and $|\Omega_j\rangle$ are the eigenvectors of U. The Lemma is trivially true if $\xi_1, \ldots, \xi_d$ are all rational. In this case we immediately get $U^q = \mathbb{1}_{\mathcal{H}}$, if q is a common denominator of $\xi_1, \ldots, \xi_d$. For arbitrary $\xi_1, \ldots, \xi_d$, we make use of a number-theoretical result, known as *Dirichlet's theorem on simultaneous diophantine approximation* [48]. We wish to get a simultaneous approximation of $\xi_1, \ldots, \xi_d$ by fractions

$$\frac{p_1}{q}, \frac{p_2}{q}, \ldots, \frac{p_d}{q} \tag{5.2}$$

with a common denominator q. Furthermore we wish to have the ability to choose the common denominator q in such a way that $\max\{|q\xi_1 - p_1|, \ldots, |q\xi_d - p_d|\}$ becomes arbitrarily small. According to Dirichlet's theorem this is possible: *If $\xi_1, \ldots, \xi_d$ are any real numbers such that at least one of them is irrational, then the system of inequalities*

$$\left| \xi_j - \frac{p_j}{q} \right| < \frac{1}{q^{1+\frac{1}{d}}} \quad \textit{with} \quad q, p_j \in \mathbb{N} \quad (j = 1, 2, \ldots, d) \tag{5.3}$$

has infinitely many solutions. In particular, $\max\{|q\xi_1 - p_1|, \ldots, |q\xi_d - p_d|\} < q^{-\frac{1}{d}}$ *holds for infinitely many integers* $q \in \mathbb{N}$. As a consequence, given any $\epsilon > 0$, we can always find an integer $q \in \mathbb{N}$ so that, for every $j \in \{1, 2, \ldots, d\}$, the product $q\xi_j$ differs from an integer by less than ϵ.

To prove the Lemma, let any $\epsilon > 0$ be given. Define $\epsilon' := \frac{\epsilon}{d(e^{2\pi}-1)}$. According to Dirichlet's Theorem there always exists a $q = q(\epsilon') \in \mathbb{N}$ such that, for every j, $q\xi_j$ differs from an integer by less than ϵ'. It follows that

$$U^q = \sum_{j=1}^{d} e^{2\pi i\, q\, \xi_j} \,|\Omega_j\rangle\langle\Omega_j| = \sum_{j=1}^{d} e^{2\pi i\, \epsilon_j} \,|\Omega_j\rangle\langle\Omega_j| \tag{5.4}$$

with some very small numbers ϵ_j satisfying $|\epsilon_j| < \epsilon'$ for all j. Hence

$$\begin{aligned} \| \, U^q - \mathbb{1}_{\mathcal{H}} \, \| &= \| \sum_{j=1}^{d} (e^{2\pi i \epsilon_j} - 1)|\Omega_j\rangle\langle\Omega_j| \, \| \\ &\leq \sum_{j=1}^{d}\sum_{\nu=1}^{\infty} \frac{(2\pi)^\nu}{\nu!} |\epsilon_j|^\nu \underbrace{\| \; |\Omega_j\rangle\langle\Omega_j| \; \|}_{=1} \\ &< \sum_{j=1}^{d}\sum_{\nu=1}^{\infty} \frac{(2\pi)^\nu}{\nu!} \epsilon'^\nu < \sum_{j=1}^{d}\sum_{\nu=1}^{\infty} \frac{(2\pi)^\nu}{\nu!} \epsilon' \\ &= d \cdot \epsilon' \cdot (e^{2\pi} - 1) = \epsilon \, . \end{aligned} \tag{5.5}$$

This proves the Lemma. □

Remark:

The essential basic ingredient in the proof of the Lemma is Dirichlet's theorem on simultaneous diophantine approximation. This number-theoretical result states that for any *finite* set of real numbers $\xi_1, \ldots, \xi_d \in \mathbb{R}$ with at least one of them being irrational there exist *infinitely many* $q \in \mathbb{N}$ such that, for every $j \in \{1, 2, \ldots, d\}$, the product $q\xi_j$ differs from an integer by less than $q^{-\frac{1}{d}}$. It therefore follows from Dirichlet's theorem with regard to the above Lemma that U^q is arbitrarily close to the identity operator $\mathbb{1}_{\mathcal{H}}$ not just for one $q \in \mathbb{N}$, but for *infinitely many* integers $q \in \mathbb{N}$. Hence the statement of the Lemma may be reformulated as an even stronger proposition: for any $\epsilon > 0$ there exist *infinitely many* $q \in \mathbb{N}$ such that $\| \, U^q - \mathbb{1}_{\mathcal{H}} \, \| < \epsilon$. Thus, any unitary dynamics in finite dimensional Hilbert spaces induces an evolution which is infinitely often *recurrent*. Furthermore, it is also important to note the issue that the Lemma involves a *uniform recurrence*, meaning that *all states* $\rho \in \mathcal{S}(\mathcal{H})$ return arbitrarily close to themselves *at the same "time"*. Let us formulate the precise meaning of this consequence as a corollary.

Corollary (uniform recurrence):

Let $\mathcal{H}$ be a finite-dimensional Hilbert space, and let $U : \mathcal{H} \to \mathcal{H}$ be a unitary map on $\mathcal{H}$. Then for any $\epsilon > 0$ there exist infinitely many integers $q \in \mathbb{N}$ such that for any density operator ρ_0 on $\mathcal{H}$ we have $\delta_E(U^q \rho_0 U^{\dagger\, q}, \rho_0) < \epsilon$, where $\delta_E(\cdot,\cdot)$ denotes the standard Euclidean distance measure[7]*, which to any pair of density operators ρ' and ρ'' on $\mathcal{H}$ assigns the distance $\delta_E(\rho', \rho'') := \| \rho' - \rho'' \|_2 \equiv \sqrt{\mathrm{Tr}\, |\rho' - \rho''|^2}$.*

Proof: Let any, arbitrarily small $\epsilon > 0$ be given. Define $\epsilon' := \epsilon/3$. According to the stronger version of the Lemma (stated in the above remark) we can always find infinitely many positive integers $q \in \mathbb{N}$ such that $U^q = \mathbb{1}_{\mathcal{H}} + \hat{\mathcal{O}}(\epsilon')$, where $\hat{\mathcal{O}}(\epsilon')$ is some operator with norm bounded by ϵ': $\|\hat{\mathcal{O}}(\epsilon')\| < \epsilon'$. It then follows, *for any density operator* ρ_0 on $\mathcal{H}$, that

$$\begin{aligned}
\delta_E^2(U^q \rho_0 U^{\dagger\, q}, \rho_0) &= \| U^q \rho_0 U^{\dagger\, q} - \rho_0 \|_2^2 \equiv \mathrm{Tr}\left[\left| U^q \rho_0 U^{\dagger\, q} - \rho_0 \right|^2 \right] \\
&= \mathrm{Tr}\left[\left| \left(\mathbb{1}_{\mathcal{H}} + \hat{\mathcal{O}}(\epsilon') \right) \rho_0 \left(\mathbb{1}_{\mathcal{H}} + \hat{\mathcal{O}}^\dagger(\epsilon') \right) - \rho_0 \right|^2 \right] \\
&= \mathrm{Tr}\left[\left| \rho_0 \hat{\mathcal{O}}^\dagger(\epsilon') + \hat{\mathcal{O}}(\epsilon')\rho_0 + \hat{\mathcal{O}}(\epsilon')\rho_0\hat{\mathcal{O}}^\dagger(\epsilon') \right|^2 \right] \\
&= \mathrm{Tr}\left[\left(\rho_0 \hat{\mathcal{O}}^\dagger(\epsilon') + \hat{\mathcal{O}}(\epsilon')\rho_0 + \hat{\mathcal{O}}(\epsilon')\rho_0\hat{\mathcal{O}}^\dagger(\epsilon') \right)^2 \right] =
\end{aligned}$$

$$\begin{aligned}
= \; & \mathrm{Tr}\left[\left(\rho_0 \hat{\mathcal{O}}^\dagger(\epsilon') \right)^2 \right] + \mathrm{Tr}\left[\rho_0^2 |\hat{\mathcal{O}}(\epsilon')|^2 \right] + \mathrm{Tr}\left[\rho_0 |\hat{\mathcal{O}}(\epsilon')|^2 \rho_0 \hat{\mathcal{O}}^\dagger(\epsilon') \right] + \\
& + \mathrm{Tr}\left[\rho_0^2 |\hat{\mathcal{O}}(\epsilon')|^2 \right] + \mathrm{Tr}\left[\left(\hat{\mathcal{O}}(\epsilon')\rho_0 \right)^2 \right] + \mathrm{Tr}\left[\rho_0 \hat{\mathcal{O}}(\epsilon')\rho_0 |\hat{\mathcal{O}}(\epsilon')|^2 \right] + \\
& + \mathrm{Tr}\left[\rho_0 \hat{\mathcal{O}}^\dagger(\epsilon')\rho_0 |\hat{\mathcal{O}}(\epsilon')|^2 \right] + \mathrm{Tr}\left[\hat{\mathcal{O}}(\epsilon')\rho_0 |\hat{\mathcal{O}}(\epsilon')|^2 \rho_0 \right] + \mathrm{Tr}\left[\left(\rho_0 |\hat{\mathcal{O}}(\epsilon')|^2 \right)^2 \right] \\
\leq \; & 2 \left| \mathrm{Tr}\left[\rho_0^2 |\hat{\mathcal{O}}(\epsilon')|^2 \right] \right| + \left| \mathrm{Tr}\left[\left(\rho_0 \hat{\mathcal{O}}^\dagger(\epsilon') \right)^2 \right] \right| + \\
& + 2 \left| \mathrm{Tr}\left[\rho_0 |\hat{\mathcal{O}}(\epsilon')|^2 \rho_0 \hat{\mathcal{O}}^\dagger(\epsilon') \right] \right| + \left| \mathrm{Tr}\left[\left(\hat{\mathcal{O}}(\epsilon')\rho_0 \right)^2 \right] \right| + \\
& + 2 \left| \mathrm{Tr}\left[\hat{\mathcal{O}}(\epsilon')\rho_0 |\hat{\mathcal{O}}(\epsilon')|^2 \rho_0 \right] \right| + \left| \mathrm{Tr}\left[\left(\rho_0 |\hat{\mathcal{O}}(\epsilon')|^2 \right)^2 \right] \right| . \qquad (5.6)
\end{aligned}$$

[7]The Euclidean distance measure can be viewed as being induced by the Hilbert-Schmidt operator norm, which is defined by $\|A\|_2 := \sqrt{\mathrm{Tr}[A^\dagger A]}$ for any operator A on $\mathcal{H}$.

Utilizing the inequality $|\,\mathrm{Tr}[BT]\,| \leq \|B\|\ \mathrm{Tr}\sqrt{T^\dagger T}$ for bounded operators $B : \mathcal{H} \to \mathcal{H}$ and operators $T : \mathcal{H} \to \mathcal{H}$ with finite trace norm $\|T\|_1 := \mathrm{Tr}\sqrt{T^\dagger T}$, see Ref. [57], we deduce:

$$
\begin{aligned}
\delta_E^2(U^q\rho_0 U^{\dagger\, q}, \rho_0) \;\leq\; & 2\ \|\rho_0\hat{\mathcal{O}}^\dagger(\epsilon')\hat{\mathcal{O}}(\epsilon')\|\ \mathrm{Tr}\,[\rho_0] + \|\hat{\mathcal{O}}^\dagger(\epsilon')\rho_0\hat{\mathcal{O}}^\dagger(\epsilon')\|\ \mathrm{Tr}\,[\rho_0] + \\
& +2\ \|\hat{\mathcal{O}}^\dagger(\epsilon')\rho_0|\hat{\mathcal{O}}(\epsilon')|^2\|\ \mathrm{Tr}\,[\rho_0] + \|\hat{\mathcal{O}}(\epsilon')\rho_0\hat{\mathcal{O}}(\epsilon')\|\ \mathrm{Tr}\,[\rho_0] + \\
& +2\ \|\hat{\mathcal{O}}(\epsilon')\rho_0|\hat{\mathcal{O}}(\epsilon')|^2\|\ \mathrm{Tr}\,[\rho_0] + \|\ |\hat{\mathcal{O}}(\epsilon')|^2\rho_0|\hat{\mathcal{O}}(\epsilon')|^2\|\ \mathrm{Tr}\,[\rho_0]\ .
\end{aligned}
$$

Using the fact that $\|B^\dagger\|=\|B\|$ for any bounded operator B and it's adjoint $B^\dagger$ [58], we have $\|\hat{\mathcal{O}}^\dagger(\epsilon')\|=\|\hat{\mathcal{O}}(\epsilon')\|< \epsilon'$. Utilizing the submultiplicativity property of operator norms, see e.g. Ref. [58], and the fact that $\mathrm{Tr}\,[\rho_0] = 1$ and $\|\ \rho_0\ \|\leq 1$, we can therefore infer:

$$
\begin{aligned}
\delta_E^2(U^q\rho_0 U^{\dagger\, q}, \rho_0) \;\leq\; & 2\ \|\rho_0\|\ \|\hat{\mathcal{O}}^\dagger(\epsilon')\|\ \|\hat{\mathcal{O}}(\epsilon')\| + \|\hat{\mathcal{O}}^\dagger(\epsilon')\|\ \|\rho_0\|\ \|\hat{\mathcal{O}}^\dagger(\epsilon')\| + \\
& +2\ \|\hat{\mathcal{O}}^\dagger(\epsilon')\|\ \|\rho_0\|\ \|\hat{\mathcal{O}}^\dagger(\epsilon')\|\ \|\hat{\mathcal{O}}(\epsilon')\| + \\
& +\ \|\hat{\mathcal{O}}(\epsilon')\|\ \|\rho_0\|\ \|\hat{\mathcal{O}}(\epsilon')\| + \\
& +2\ \|\hat{\mathcal{O}}(\epsilon')\|\ \|\rho_0\|\ \|\hat{\mathcal{O}}^\dagger(\epsilon')\|\ \|\hat{\mathcal{O}}(\epsilon')\| + \\
& +\ \|\hat{\mathcal{O}}^\dagger(\epsilon')\|\ \|\hat{\mathcal{O}}(\epsilon')\|\ \|\rho_0\|\ \|\hat{\mathcal{O}}^\dagger(\epsilon')\|\ \|\hat{\mathcal{O}}(\epsilon')\| \\
<\; & 2\epsilon'^2 + \epsilon'^2 + 2\epsilon'^3 + \epsilon'^2 + 2\epsilon'^3 + \epsilon'^4 \\
<\; & 9\epsilon'^2 = \epsilon^2\ , \qquad (5.7)
\end{aligned}
$$

where we assumed $\epsilon' \ll 1$. Thus we have $\delta_E(U^q\rho_0 U^{\dagger\, q}, \rho_0) < \epsilon$ for infinitely many positive integers $q \in \mathbb{N}$ and for any density operator ρ_0 on $\mathcal{H}$. Since $\epsilon > 0$ was arbitrary, this proves the corollary. □

5.3 Simple necessary decoherence conditions for a set of histories

5.3.1 Introduction

One of our main concerns within the research presented in this chapter is to derive simpler criteria for checking decoherence of a set of histories — simpler as compared to the consistency conditions in terms of properties of the decoherence functional, given by Eq. (3.6). In particular, we are especially interested in deriving a *single-iteration decoherence condition*, i.e., a criterion for decoherence of a set of histories that employs only a single iteration of a given unitary quantum map.

In this section we make progress towards finding a simple characterization of the set of unitary quantum maps that, given a classical initial state [8], lead to decoherent histories of arbitrary length. The choice of classical states as the initial states from which the histories are to start was motivated in Chap. 3 by the fact, that only classical states $\rho \in \mathcal{S}^{\rm cl}_{\{P_\mu\}}$ can be prepared by the projective measurement [9] which one *may* associate with a given projective partition $\{P_\mu\}$ representing the exhaustive set of mutually exclusive events the histories are composed of. As for the requirement of "decoherence for arbitrary history lengths", see the preliminary notes in the

[8] "Classical" in the sense as defined in Chapter 3, i.e., classical with respect to a given projective partition $\{P_\mu\}$ of a Hilbert space $\mathcal{H}$.

[9] It should be stressed, however, once again, that I do not interpret the events of the histories as *"dynamical events"* or *outcomes of a measurement* that can be associated with the given projective partition $\{P_\mu\}$ the set of histories is constructed of. I am aware of the fact, that the decoherent histories framework is being regarded (cf., e.g., [18, 19, 20]) as a formulation of quantum mechanics which, at a fundamental level, does not need the concept of measurement theory. In contrast to the orthodox Copenhagen interpretation it makes no reference to external observers and classical measurement apparatuses, although, of course, the physical process of measurements can be described from within this formalism. It is only the choice of the initial states which I *motivate* — and not truly justify — by the concept of measurements. I regard this as reasonable if one takes the point of view that a *"state"* of a physical system is the *result of a certain "preparation" procedure*, cf. [40] (p. 92). The above choice of the initial states is furthermore supported by the formal similarities between the decoherent histories formalism and the framework of classical symbolic dynamics.

introduction of this chapter.

Provided that these two additional assumptions are posed, i.e., decoherence *for all classical initial states* and *for arbitrary history lengths*, we manage to obtain interesting results (cf. also [1, 2]) involving *necessary* single-iteration decoherence conditions. One of them (see Sec. 5.3.2) applies only to the case of fine-grained histories, but it can be shown *trivially* to be a sufficient condition for decoherence as well, even in the most general case. The other one (see Sec. 5.3.3) is applicable to the general case of arbitrary coarse-grainings, but it fails to be a sufficient condition for decoherence in this general case. Our analysis below therefore concentrates on proving single-iteration criteria which pose nontrivial *necessary conditions* for a set of histories to be decoherent.

In Sec. 5.3.2 we prove that fine-grained histories of arbitrary length decohere for all classical initial states *if and only if* the unitary evolution preserves classicality of states (using our natural formal definition of classicality of Chap. 3). We give a counterexample showing that this equivalence does not hold for coarse-grained histories. In particular, decoherence of coarse-grained histories does not, in general, imply that the unitary evolution preserves classicality of states. In Sec. 5.3.3 we then provide a necessary single-iteration decoherence condition that applies to arbitrary coarse-grainings and is equivalent to the preservation of classicality of states in the fine-grained case. We discuss the obtained results in Sec. 5.3.4.

5.3.2 A simple necessary decoherence condition for a set of fine-grained histories

Imagine a unitary evolution that transforms every classical state into a classical state. If the initial state is classical, this evolution trivially leads to decoherent histories. One can easily see that in this case the decoherence functional is diagonal for histories of any length. It is not immediately clear, however, whether any unitary that leads to the desired decoherence effect must preserve classicality of states. In what follows we show that this is the case *only for fine-grained histories*.

Theorem 1:

Let a fine-grained projective partition $\{P_\mu\}$ of a finite dimensional Hilbert space $\mathcal{H}$ and a unitary map U on $\mathcal{H}$ be given. The decoherence conditions are then satisfied for all classical initial states and arbitrarily long histories if and only if U preserves classicality of states, i.e.,

$$\forall\,\rho \in \mathcal{S}^{\mathrm{cl}}_{\{P_\mu\}}\ \forall\,k \in \mathbb{N}\ \ \forall\,h_{\boldsymbol{\alpha}}, h_{\boldsymbol{\beta}} \in \mathcal{K}[\{P_\mu\}\,;\,k\,]:\ \mathcal{D}_{U,\rho}\,[h_{\boldsymbol{\alpha}}, h_{\boldsymbol{\beta}}] \propto \delta_{\boldsymbol{\alpha\beta}} \tag{5.8}$$

if and only if

$$\forall\,\rho \in \mathcal{S}^{\mathrm{cl}}_{\{P_\mu\}}\ :\ \ U\rho U^\dagger\ \in \mathcal{S}^{\mathrm{cl}}_{\{P_\mu\}}\ . \tag{5.9}$$

Theorem 2:

For coarse-grained partitions, the classicality condition (5.9) of Theorem 1 is in general not a necessary condition. More precisely, there exists a coarse-grained projective partition and a unitary map such that the classicality condition (5.9) is not satisfied but the decoherence condition (5.8) is valid.

Thus, decoherence for arbitrarily long histories and classical initial states is a sufficient condition for U to preserve classicality of states in the fine-grained case, but not in the coarse-grained case. In general, decoherence does not imply that the unitary evolution preserves classicality.

In our theorems, the decoherence condition is formulated for any $k \in \mathbb{N}$, i.e., arbitrary history lengths, corresponding to an arbitrary number of iteration steps of the unitary map U. This is a very strong condition. It can be relaxed if the Hilbert space is two-dimensional. In this case, decoherence of all histories of length $k = 2$ for all classical initial states is equivalent to the condition that the unitary evolution preserves classicality of states.

In general, however, it is not sufficient to restrict attention to histories of a fixed finite length. This is made precise in the following example. For a given $K \in \mathbb{N}$ consider a Hilbert space $\mathcal{H}$ with dimension $d = 2K$. Let $\{P_\mu = |\mu\rangle\langle\mu|\ :\ \mu = 0, 1, \ldots, d-1\,\}$ be a fine-grained partition of $\mathcal{H}$, where the kets $|\mu\rangle$ form an orthonormal basis of $\mathcal{H}$.

Define a unitary map $U : \mathcal{H} \to \mathcal{H}$ by

$$\begin{aligned}
|0\rangle &\rightarrow U|0\rangle = \frac{1}{\sqrt{2}}\left(|2\rangle + |3\rangle\right) \\
|1\rangle &\rightarrow U|1\rangle = \frac{1}{\sqrt{2}}\left(|2\rangle - |3\rangle\right) \\
|\nu\rangle &\rightarrow U|\nu\rangle = |\nu + 2\rangle \quad \text{for} \quad \nu = 2, 3, \ldots, (d-3) \\
|d-2\rangle &\rightarrow U|d-2\rangle = |0\rangle \\
|d-1\rangle &\rightarrow U|d-1\rangle = |1\rangle \, .
\end{aligned} \tag{5.10}$$

The map U does not preserve classicality w.r.t. $\{P_\mu\}$. For $k > K$ and, e.g., the classical initial state $\rho = |0\rangle\langle 0| \in \mathcal{S}^{\mathrm{cl}}_{\{P_\mu\}}$, the set of histories $\mathcal{K}[\{P_\mu\}\,;\,k\,]$ does not decohere. One can easily show, however, that $\mathcal{K}[\{P_\mu\}\,;\,k\,]$ decoheres for all $\rho \in \mathcal{S}^{\mathrm{cl}}_{\{P_\mu\}}$ and all $k \leq K$. We have thus found, for any $K \in \mathbb{N}$, an example in which U does not preserve classicality, but the decoherence condition is satisfied for all classical initial states and all histories up to length K.

Proof of Theorem 1:

The classicality condition (5.9) implies the decoherence condition (5.8) trivially. We will prove the converse by contradiction, i.e., we will assume that the classicality condition (5.9) is not satisfied, and then show that this assumption contradicts the decoherence condition (5.8).

Assume condition (5.9) is not satisfied. This means there exists a classical state $\rho \in \mathcal{S}^{\mathrm{cl}}_{\{P_\mu\}}$ such that $U\rho U^\dagger \notin \mathcal{S}^{\mathrm{cl}}_{\{P_\mu\}}$. Since the partition $\{P_\mu\}$ is fine-grained, it consists of one-dimensional projectors, $P_\mu = |\mu\rangle\langle\mu|$, where the vectors $|\mu\rangle$ form an orthonormal basis of $\mathcal{H}$. The state ρ can be written as $\rho = \sum_\mu p_\mu |\mu\rangle\langle\mu|$, where $p_\mu \geq 0$ and $\sum_\mu p_\mu = 1$. The assumption $U\rho U^\dagger \notin \mathcal{S}^{\mathrm{cl}}_{\{P_\mu\}}$ implies that for at least one term in the decomposition $\rho = \sum_\mu p_\mu |\mu\rangle\langle\mu|$ classicality is not preserved. If it were not so, $U\rho U^\dagger$ would be classical. Hence there exists μ_0 such that $p_{\mu_0} \neq 0$ and $(U|\mu_0\rangle\langle\mu_0|U^\dagger) \notin \mathcal{S}^{\mathrm{cl}}_{\{P_\mu\}}$. This means there exist μ',μ'', $\mu' \neq \mu''$, such that

$$\begin{aligned}
\langle\mu'|U|\mu_0\rangle &\equiv c_{\mu'} \neq 0 \, , \\
\langle\mu''|U|\mu_0\rangle &\equiv c_{\mu''} \neq 0 \, .
\end{aligned} \tag{5.11}$$

Now we derive a *necessary condition* for decoherence and then show that the above assumption contradicts it. Written out, the decoherence condition (5.8) is

$$\mathrm{Tr}\left[P_{\alpha_k} U P_{\alpha_{k-1}} U \dots P_{\alpha_1} U\, \rho_0\, U^\dagger P_{\beta_1} \dots P_{\beta_{k-1}} U^\dagger P_{\beta_k}\right] \propto \prod_{j=1}^{k} \delta_{\alpha_j \beta_j} \tag{5.12}$$

for all $k \in \mathbb{N}$, all initial states $\rho_0 \in \mathcal{S}^{\mathrm{cl}}_{\{P_\mu\}}$, and arbitrary histories $h_{\boldsymbol{\alpha}}$, $h_{\boldsymbol{\beta}}$. By summing over $\alpha_2, \dots, \alpha_{k-1}$ and $\beta_2, \dots, \beta_{k-1}$, and using $\sum_\mu P_\mu = \mathbb{1}_{\mathcal{H}}$, we obtain

$$\mathrm{Tr}\left[P_{\alpha_k} U^{k-1} P_{\alpha_1} U\, \rho_0\, U^\dagger P_{\beta_1} (U^\dagger)^{k-1} P_{\beta_k}\right] \propto \delta_{\alpha_k \beta_k} \delta_{\alpha_1 \beta_1} \tag{5.13}$$

for all $k \in \mathbb{N}$, any $\rho_0 \in \mathcal{S}^{\mathrm{cl}}_{\{P_\mu\}}$, and arbitrary α_1, β_1, α_k, β_k.

To derive a contradiction we let our histories start with the initial state $\rho_0 = P_{\mu_0} \equiv |\mu_0\rangle\langle\mu_0|$. Furthermore we choose $\alpha_1 = \mu'$, $\beta_1 = \mu''$, and $\alpha_k = \beta_k = \mu_0$. Since $\mu' \neq \mu''$, condition (5.13) becomes

$$\mathrm{Tr}\left[P_{\mu_0} U^{k-1} P_{\mu'} U\, \rho_0\, U^\dagger P_{\mu''} (U^\dagger)^{k-1} P_{\mu_0}\right] = 0 \tag{5.14}$$

for all $k \in \mathbb{N}$. On the other hand, since $\rho_0 = |\mu_0\rangle\langle\mu_0|$, and using Eqs. (5.11), we get for the left hand side of Eq. (5.14):

$$\mathrm{Tr}\left[P_{\mu_0} U^{k-1} P_{\mu'} U\, \rho_0\, U^\dagger P_{\mu''} (U^\dagger)^{k-1} P_{\mu_0}\right] = \underbrace{c_{\mu'} c^*_{\mu''}}_{\neq 0} \langle\, \mu'' \,|(U^\dagger)^{k-1} P_{\mu_0} U^{k-1}|\, \mu' \,\rangle\ . \tag{5.15}$$

We now make use of our Lemma on uniform recurrence from Sec. 5.2. According to the Lemma, for any given, arbitrarily small $\epsilon > 0$ we can always find a $q \in \mathbb{N}$ such that $U^q = \mathbb{1}_{\mathcal{H}} + \hat{\mathcal{O}}(\epsilon)$, where $\hat{\mathcal{O}}(\epsilon)$ is some operator with norm bounded by ϵ: $\|\hat{\mathcal{O}}(\epsilon)\| < \epsilon$. Using the submultiplicativity property of operator norms, we have

$$\|\, U^{-1} \hat{\mathcal{O}}(\epsilon) \,\| \leq \|\, U^{-1} \,\| \times \|\, \hat{\mathcal{O}}(\epsilon) \,\| = \|\, \hat{\mathcal{O}}(\epsilon) \,\| \tag{5.16}$$

and hence $U^{q-1} = U^{-1} + \hat{\mathcal{O}}'(\epsilon)$, where $\|\, \hat{\mathcal{O}}'(\epsilon) \,\| < \epsilon$. Choosing $k = q$ in Eq. (5.15),

$$\begin{aligned}
&\mathrm{Tr}\left[P_{\mu_0} U^{q-1} P_{\mu'} U\, \rho_0\, U^\dagger P_{\mu''} (U^\dagger)^{q-1} P_{\mu_0}\right] \\
&\quad= c_{\mu'} c^*_{\mu''} \langle\, \mu'' \,|(U^\dagger)^{q-1} P_{\mu_0} U^{q-1}|\, \mu' \,\rangle \\
&\quad= c_{\mu'} c^*_{\mu''} \langle\, \mu'' \,|\big(U + \hat{\mathcal{O}}'^{\dagger}(\epsilon)\big)\, |\mu_0\rangle\langle\mu_0|\, \big(U^\dagger + \hat{\mathcal{O}}'(\epsilon)\big)\, |\, \mu' \,\rangle \\
&\quad= \underbrace{c_{\mu'} c^*_{\mu''}}_{\neq 0}\ \underbrace{\langle\, \mu'' \,|U\, |\mu_0\rangle}_{=c_{\mu''}}\ \underbrace{\langle\mu_0|\, U^\dagger\, |\, \mu' \,\rangle}_{=c^*_{\mu'}} + O(\epsilon) \\
&\quad= \underbrace{|c_{\mu'} c_{\mu''}|^2}_{\neq 0} + O(\epsilon)\ ,
\end{aligned} \tag{5.17}$$

where $O(\epsilon) \to 0$ as $\epsilon \to 0$. This contradicts condition (5.14), which is a necessary consequence of our decoherence condition (5.8), and thus proves the theorem. $\square$

Proof of Theorem 2:

We prove theorem 2 by constructing a coarse-grained partition and a unitary map with the required properties. Let $\mathcal{H}$ be a 4-dimensional Hilbert space. We can write $\mathcal{H} = \mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{E}$, and think of it as the Hilbert space of two qubits, regarding one of them as the system $\mathcal{S}$, the other one as the environment $\mathcal{E}$. Let $\{|0\rangle, |1\rangle\}$ and $\{|e_0\rangle, |e_1\rangle\}$ be orthonormal bases of $\mathcal{H}_\mathcal{S}$ and $\mathcal{H}_\mathcal{E}$, respectively. The states $|\mu, e_\lambda\rangle := |\mu\rangle \otimes |e_\lambda\rangle$, where $\mu, \lambda \in \{0, 1\}$, form an orthonormal basis of $\mathcal{H}$.

We now define a coarse-grained projective partition, $\{P_0, P_1\}$, by

$$\begin{aligned} P_\mu &= |\mu\rangle\langle\mu| \otimes \mathbb{1}_{\mathcal{H}_\mathcal{E}} \\ &= |\mu, e_0\rangle\langle\mu, e_0| + |\mu, e_1\rangle\langle\mu, e_1| \ , \end{aligned} \tag{5.18}$$

and a unitary map $U : \mathcal{H} \to \mathcal{H}$ by

$$\begin{aligned} U|0\,, e_0\rangle &= |0\,, e_1\rangle \\ U|0\,, e_1\rangle &= |1\,, e_0\rangle \\ U|1\,, e_0\rangle &= |1\,, e_1\rangle \\ U|1\,, e_1\rangle &= |0\,, e_0\rangle \ . \end{aligned} \tag{5.19}$$

The map U is a permutation of the basis states. A more compact definition of U is

$$U|\mu, e_\lambda\rangle = \sum_{\nu=0}^{1} \delta_{\nu\lambda} |\mu + \nu, e_{1+\nu}\rangle \ , \tag{5.20}$$

where $\mu, \lambda \in \{0, 1\}$ and addition is understood modulo 2. The map U does not preserve classicality w.r.t. $\{P_0, P_1\}$. This can be seen by considering the pure classical state $\rho = |\psi\rangle\langle\psi| \in \mathcal{S}^{\mathrm{cl}}_{\{P_\mu\}}$, where $|\psi\rangle = \frac{1}{\sqrt{2}}(|0, e_0\rangle + |0, e_1\rangle) \in \mathrm{supp}(P_0)$. Since $U|\psi\rangle = \frac{1}{\sqrt{2}}(|0, e_1\rangle + |1, e_0\rangle)$ is a superposition of states that belong to $\mathrm{supp}(P_0)$ and $\mathrm{supp}(P_1)$, respectively, we have $U\rho U^\dagger \notin \mathcal{S}^{\mathrm{cl}}_{\{P_\mu\}}$. This shows that, with our choice of unitary map and partition, the classicality condition (5.9) is not satisfied.

It remains to be shown that the decoherence condition (5.8) is satisfied for this choice of unitary map and partition. The most general classical state w.r.t. $\{P_0, P_1\}$

is given by $\rho = p_0\rho_0 + p_1\rho_1$, where $p_0 + p_1 = 1$ and ρ_0, ρ_1 are any density matrices satisfying $\mathrm{supp}(\rho_\mu) \subseteq \mathrm{supp}(P_\mu)$ for $\mu = 0, 1$. Let $\rho_\mu = \sum_{j=0}^{1} r_\mu^j |\omega_\mu^j\rangle\langle\omega_\mu^j|$ be their spectral decompositions. In terms of the basis vectors $|\mu, e_\lambda\rangle$ the eigenvectors can be written as $|\omega_\mu^j\rangle = \sum_{\lambda=0}^{1} c_{\mu,\lambda}^j |\mu\,, e_\lambda\rangle$. Putting everything together, we find that every $\rho \in \mathcal{S}_{\{P_\mu\}}^{\mathrm{cl}}$ can be written in the form

$$\rho = \sum_{\mu=0}^{1}\sum_{j=0}^{1}\sum_{\lambda,\lambda'=0}^{1} p_\mu r_\mu^j c_{\mu,\lambda}^j c_{\mu,\lambda'}^{j\,*} |\mu\,, e_\lambda\rangle\langle\mu\,, e_{\lambda'}|\;. \tag{5.21}$$

Substituting this into the expression

$$\mathcal{R}_\rho(k) := P_{\alpha_k} U P_{\alpha_{k-1}} U \ldots P_{\alpha_1} U\, \rho\, U^\dagger P_{\beta_1} \ldots P_{\beta_{k-1}} U^\dagger P_{\beta_k} \tag{5.22}$$

and using the *principle of induction*, one can show that, for $k \geq 2$ and any $\rho \in \mathcal{S}_{\{P_\mu\}}^{\mathrm{cl}}$,

$$\mathcal{R}_\rho(k) \propto \delta_{\alpha_1+\alpha_{k-1}+\alpha_k,\beta_1+\beta_{k-1}+\beta_k} |\alpha_k, e_{\alpha_{k-1}+\alpha_k+1}\rangle\langle\beta_k, e_{\beta_{k-1}+\beta_k+1}|\;, \tag{5.23}$$

where again addition is understood modulo 2. This can be shown to be equivalent to

$$\mathcal{R}_\rho(k) \propto \left(\prod_{j=1}^{\lfloor\frac{k-1}{2}\rfloor} \delta_{\alpha_1+\alpha_{2j+1},\beta_1+\beta_{2j+1}}\right)\left(\prod_{j=1}^{\lfloor\frac{k}{2}\rfloor} \delta_{\alpha_{2j},\beta_{2j}}\right) |\alpha_k, e_{\alpha_{k-1}+\alpha_k+1}\rangle\langle\beta_k, e_{\beta_{k-1}+\beta_k+1}|\;. \tag{5.24}$$

Taking the trace on both sides gives

$$\mathrm{Tr}[\mathcal{R}_\rho(k)] \propto \left(\prod_{j=1}^{\lfloor\frac{k-1}{2}\rfloor} \delta_{\alpha_1+\alpha_{2j+1},\beta_1+\beta_{2j+1}}\right)\left(\prod_{j=1}^{\lfloor\frac{k}{2}\rfloor} \delta_{\alpha_{2j},\beta_{2j}}\right) \langle\beta_k, e_{\beta_{k-1}+\beta_k+1}|\alpha_k, e_{\alpha_{k-1}+\alpha_k+1}\rangle\;. \tag{5.25}$$

The scalar product on the right hand side is equal to $\delta_{\alpha_k,\beta_k}\delta_{\alpha_{k-1},\beta_{k-1}}$. Using

$$\delta_{\alpha_k,\beta_k}\delta_{\alpha_{k-1},\beta_{k-1}} \prod_{j=1}^{\lfloor\frac{k-1}{2}\rfloor} \delta_{\alpha_1+\alpha_{2j+1},\beta_1+\beta_{2j+1}} = \prod_{j=1}^{\lfloor\frac{k-1}{2}\rfloor} \delta_{\alpha_{2j+1},\beta_{2j+1}}\;, \tag{5.26}$$

and the fact that $\mathrm{Tr}[\mathcal{R}_\rho(k=1)] \propto \delta_{\alpha_1,\beta_1}$, we finally obtain

$$\mathrm{Tr}[\mathcal{R}_\rho(k)] \propto \prod_{j=1}^{k} \delta_{\alpha_j,\beta_j} \tag{5.27}$$

for all $k \in \mathbb{N}$ and all $\rho \in \mathcal{S}_{\{P_\mu\}}^{\mathrm{cl}}$. The decoherence condition (5.8) is thus satisfied, which completes the proof of Theorem 2. □

Our results of this subsection can be summarized as follows. We have analyzed the relationship between the condition that a unitary map is classicality-preserving on the one hand, and the decoherence condition for all classical initial states and arbitrarily long histories on the other hand. We have shown that for fine-grained histories, these two conditions are equivalent, but that decoherence of coarse-grained histories does not, in general, imply that the unitary evolution preserves classicality of states.

In the *fine-grained* case, our general goal is thus achieved. Namely, we have found a *single-iteration necessary and sufficient criterion* for decoherence: preservation of classicality of states by the evolution is both necessary and sufficient for the decoherence of arbitrarily long histories starting from any classical initial state.

5.3.3 A simple necessary decoherence condition for a set of coarse-grained histories

We now provide a *necessary* single-iteration decoherence condition that applies to arbitrary coarse-grainings and is equivalent to (5.9) in the fine-grained case. It is the content of the following theorem.

Theorem 3:

Let a projective partition $\{P_\mu\}$ of a finite dimensional Hilbert space $\mathcal{H}$ and a unitary map U on $\mathcal{H}$ be given. The medium decoherence condition is then satisfied for all classical initial states and arbitrarily long histories, i.e.,

$$\forall\, \rho \in \mathcal{S}^{\mathrm{cl}}_{\{P_\mu\}}\; \forall\, k \in \mathbb{N}\;\; \forall\, h_{\boldsymbol{\alpha}}, h_{\boldsymbol{\beta}} \in \mathcal{K}[\{P_\mu\}\,;\,k] :\; \mathcal{D}_{U,\rho}\,[h_{\boldsymbol{\alpha}}, h_{\boldsymbol{\beta}}] \propto \delta_{\boldsymbol{\alpha\beta}}\,, \tag{5.28}$$

only if the following necessary condition is fulfilled:

$$\forall\, P_{\mu'}, P_{\mu''} \in \{P_\mu\} :\;\; \left[U P_{\mu'} U^\dagger\,,\, P_{\mu''}\right] = 0\,. \tag{5.29}$$

Proof:

Theorem 3 will turn out as an immediate simple corollary of Theorem 4 of Section 5.4. □

Theorem 3 thus provides a *single-iteration necessary decoherence condition* for sets of *arbitrarily coarse-grained histories*[10] of arbitrary length starting from any classical initial state. This new condition can be regarded as a generalization of the single-iteration necessary decoherence condition that was derived for fine-grained histories in Sec. 5.3.2.

5.3.4 Discussion

Let us conclude with a summary and a discussion of our results. In search of simpler criteria for checking decoherence of histories we found a simple, necessary and sufficient, decoherence condition for sets of fine-grained histories of arbitrary length. In Section 5.3.2 it was shown that in the case of *fine-grained* partitions sets of histories of arbitrary length decohere for all classical initial states *if and only if* the unitary dynamics *preserves the classicality of states*, i.e. *if and only if*

$$\forall \rho \in \mathcal{S}^{\mathrm{cl}}_{\{P_\mu\}} \;:\; U\rho U^\dagger \in \mathcal{S}^{\mathrm{cl}}_{\{P_\mu\}}. \tag{5.30}$$

It is a *single-iteration* criterion: to verify that it holds for a particular unitary quantum map U, only a single iteration of the map has to be taken into account, which can be much easier[11] than establishing decoherence directly by computing the off-diagonal elements of the decoherence functional. This is especially useful for studying chaotic quantum maps, for which typically only the first iteration is known in a closed analytical form [14]. Unfortunately, condition (5.30) fails to be necessary [12] in the coarse-grained case. For general coarse-grainings, the *commutativity condition*

$$\forall\, P_{\mu'}, P_{\mu''} \in \{P_\mu\} \;:\; \left[U P_{\mu'} U^\dagger \,,\, P_{\mu''} \right] = 0 \tag{5.31}$$

[10]Of course we mean the most general case *within our slightly restricted framework* of Chapter 3

[11]Checking the preservation of classicality of states, condition (5.30), involves the use of only one iteration of the unitary map, as opposed to the decoherence condition (3.6), where checking decoherence of histories of length k needs $O(k^2)$ equations, each using k applications of the unitary map U.

[12]Condition (5.30) is *trivially* a sufficient single-iteration decoherence condition both in the fine-grained as well as in the most general coarse-grained case.

proved to be a *necessary single-iteration condition* for a set of histories to be decoherent for all classical initial states and arbitrary history lengths. Since this condition is equivalent to (5.30) in the fine-grained case, we thus got a generalization. Unfortunately, however, condition (5.31) is, in general, *not* a sufficient condition for decoherence of histories, as the following simple example shows. Let $\mathcal{H}$ be a 4-dimensional Hilbert space. We can write $\mathcal{H} = \mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{E}}$, and think of it as the Hilbert space of two qubits, one of them being regarded as the system $\mathcal{S}$, the other one as the environment $\mathcal{E}$. Let $\{|0\rangle, |1\rangle\}$ and $\{|e_0\rangle, |e_1\rangle\}$ be orthonormal bases of $\mathcal{H}_{\mathcal{S}}$ and $\mathcal{H}_{\mathcal{E}}$, respectively, the states $|\mu, e_\lambda\rangle := |\mu\rangle \otimes |e_\lambda\rangle$, where $\mu, \lambda \in \{0, 1\}$, thus forming an orthonormal basis of $\mathcal{H}$. A coarse-grained projective partition, $\{P_0, P_1\}$, shall be given by

$$\begin{aligned} P_\mu &= |\mu\rangle\langle\mu| \otimes \mathbb{1}_{\mathcal{H}_{\mathcal{E}}} \qquad (\mu \in \{0,1\}) \\ &= |\mu, e_0\rangle\langle\mu, e_0| + |\mu, e_1\rangle\langle\mu, e_1| \, , \end{aligned} \tag{5.32}$$

and a unitary map $U : \mathcal{H} \to \mathcal{H}$ by

$$\begin{aligned} U|0\,, e_0\rangle &= \frac{1}{\sqrt{2}}\,(|0\,, e_0\rangle + |1\,, e_1\rangle) \\ U|0\,, e_1\rangle &= \frac{1}{\sqrt{2}}\,(|0\,, e_0\rangle - |1\,, e_1\rangle) \\ U|1\,, e_0\rangle &= \frac{1}{\sqrt{2}}\,(|0\,, e_1\rangle + |1\,, e_0\rangle) \\ U|1\,, e_1\rangle &= \frac{1}{\sqrt{2}}\,(|0\,, e_1\rangle - |1\,, e_0\rangle) \ . \end{aligned} \tag{5.33}$$

As can be checked easily, the projective partition and the unitary map of this example satisfy the commutativity condition (5.31). But the corresponding sets of histories, $\mathcal{K}[\{P_\mu\}\,;\,k\,] \equiv \{P_\mu\}^k$, are not decoherent for all $\rho \in \mathcal{S}^{\mathrm{cl}}_{\{P_\mu\}}$ and all $k \in \mathbb{N}$. For instance, in the case $k = 3$, the two histories $h_{001} = (P_0, P_0, P_1)$ and $h_{101} = (P_1, P_0, P_1)$, if both starting from the classical initial state $\rho_0 = |0\,, e_0\rangle\langle 0\,, e_0|$, do interfere, since $\mathcal{D}_{U,\rho_0}[h_{001}, h_{101}] = 1/8$, as can be checked in a few lines. We have thus shown that the commutativity condition (5.31) is not sufficient for decoherence of histories. Whether there exists, for the most general case of arbitrary coarse-grainings, a single-iteration decoherence criterion which is both sufficient and necessary at the same time, remains an open question. Such a condition would certainly be weaker than the very strong (and almost trivial) sufficient condition (5.30), and stronger than the weak necessary commutativity condition (5.31).[13]

A few further notes regarding the above results and their proofs should be said. Our proofs, including that of Theorem 4 of Sec. 5.4, are based on the Lemma on uniform recurrence from Sec. 5.2. The task has been to show — by making use of the above Lemma on uniform recurrence — that, unless the stated necessary property of the unitary map with respect to the given projective partition is fulfilled, there exists an initial state $\tilde{\rho}_0$ from the set $\mathcal{S}^{\rm cl}_{\{P_\mu\}}$ (or $\mathcal{S}_{\{P_\mu\}}$ in the proof of Theorem 4 of the next section 5.4, respectively), such that two alternative histories *both starting from* $\tilde{\rho}_0$ will eventually *come back together again* at a later time and thus *form a closed loop*. This means interference and therefore violation of the particular decoherence condition. As pointed out by the referee of Ref. [1], the result stated in Theorem 1 of Sec. 5.3.2 could possibly also be obtained using different methods and techniques, namely the theory of classical Markov processes together with Birkhoff's theorem on doubly stochastic matrices. Let me *sketch* this promising alternative way of proving Theorem 1. Again, since the partition $\{P_\mu\}$ is fine-grained, it consists of one-dimensional projectors, $P_\mu = |\mu\rangle\langle\mu|$, with the vectors $|\mu\rangle$ forming an orthonormal basis $\{|\mu\rangle\}_\mu$ of $\mathcal{H}$. The assumption that the unitary quantum map U does not preserve classicality of states means that it has the property of mapping, with finite amplitudes $c_{\mu'} \neq 0$ *and* $c_{\mu''} \neq 0$, some basis state $|\mu_0\rangle \in \{|\mu\rangle\}_\mu$ onto two *different* basis kets $|\mu'\rangle, |\mu''\rangle \in \{|\mu\rangle\}_\mu$, i.e.,

$$U|\mu_0\rangle = c_{\mu'}|\mu'\rangle + c_{\mu''}|\mu''\rangle \; + \; \cdots \tag{5.34}$$

with $c_{\mu'} \neq 0$ *and* $c_{\mu''} \neq 0$ *and* $\mu' \neq \mu''$ for some $\mu', \mu'' \in \{\mu\}$. The dots indicate terms with $\mu \neq \mu', \mu''$, which can occur or may vanish. And the task is again to show that the two alternative paths emerging from this branching will come back together again at some later time, thus involving interference. This will be so if and only if

[13] At this point another "commutativity condition" is worth mentioning, which, however, is *trivially equivalent to the strong classicality condition* (5.30), namely:

$$\forall \rho \in \mathcal{S}^{\rm cl}_{\{P_\mu\}}\; \forall P_{\mu'} \in \{P_\mu\} : \;\; \left[U\rho U^\dagger \, , \, P_{\mu'}\right] = 0 \, .$$

This very strong condition mustn't be confused with our *much weaker* necessary commutativity condition (5.31).

the corresponding classical Markov process with transition probabilities equal to the absolute squares of the matrix elements of U (of the matrix representation with respect to the orthonormal basis $\{|\mu\rangle\}_\mu$) has the same property: given distinct paths emerging from a single state, there is a finite probability that they will come back together at some later time. This classical result can be shown using Birkhoff's theorem on doubly stochastic matrices. The unitarity of U implies that the corresponding Markov transition matrix is doubly stochastic. Birkhoff's theorem states that such a doubly stochastic matrix is a convex combination of *permutations*. Then, since there is a finite probability to jump from μ_0 to μ' and from μ_0 to μ'', the Markov transition matrix contains a contribution from a permutation π' that maps μ_0 to μ' and another permutation π'' that maps μ_0 to μ''. Using the cycle representation of permutations [14] it becomes clear that, both π' as well as π'', will eventually bring μ_0 back to μ_0. And if q is the least common multiple of the number of steps required in these two cases, it then follows that, both branches emerging from the two alternatives $\mu_0 \to \mu'$ and $\mu_0 \to \mu''$ in the course of the Markov process contain, among many other possible paths, also a path, which returns back to μ_0 after the q^{th} step. The two alternative transitions $\mu_0 \to \mu'$ and $\mu_0 \to \mu''$ therefore lead, with finite probabilities, to two distinct classical paths in the course of the Markov process, such that both of them return back to μ_0 after q steps, thus involving the existence of a closed loop and therefore a violation of consistency in the corresponding quantum process.

This alternative way of proving Theorem 1 is certainly very illustrative. Our method, on the other hand, is self-contained, and it is obviously more straightforward, as it directly refers to the decoherence functional. And as such it is immediately applicable to the consideration of generalizations. Basically the same method is used for the derivation of a simpler decoherence criterion in the case of general coarse-grainings. Furthermore, it is especially more appropriate for the investigation of similar results

[14] Any permutation π is, as is generally known, a product of cycles, no two of which contain a common numeral (see, e.g., Ref. [22]). For example, the 5-term cycle $(1\,3\,7\,2\,4)$ is a permutation which maps 1 into 3, 3 into 7, 7 into 2, 2 into 4 and 4 into 1 again, and the permutation $\pi = \binom{1\,2\,3\,4\,5\,6\,7\,8\,9}{3\,4\,7\,1\,9\,8\,2\,6\,5}$ is, in terms of its 3 cycles, representable as the product $(1\,3\,7\,2\,4)\,(5\,9)\,(6\,8)$. The reduction of a permutation into a product of its cycles is unique.

for *approximate decoherence* of histories (see Sec. 5.5), which is defined in terms of bounds on the non-diagonal elements of the decoherence functional.

In retrospect to our investigations, the result stated in Theorem 2 is not surprising. Indeed, the classicality condition (5.9) of Theorem 1 is too strong to be a necessary condition for decoherence of arbitrarily coarse-grained histories. In fact, the classicality condition (5.9) implies a *deterministic evolution*, since it does not allow any stochastic branching: the set of projectors $\{UP_\mu U^\dagger\}_\mu$ has to be identical to the given projective partition $\{P_\mu\}_\mu$ itself, i.e., the unitary map U transforms every $P_{\mu'} \in \{P_\mu\}_\mu$ either into another partition element $P_{\mu''} \in \{P_\mu\}_\mu$, $UP_{\mu'}U^\dagger = P_{\mu''}$ (with $\mu'' \neq \mu'$), or it leaves it alone, $UP_{\mu'}U^\dagger = P_{\mu'}$. In other words, *the unitary map U acts on $\{P_\mu\}_\mu$ as a permutation*, if also taking into account the well known fact that every unitary map is a bijective (*injective* as well as *surjective*) map. That the classicality condition (5.9) prevents histories from branching can be made clear as follows. The existence of branching would mean that there exist $P_{\tilde\mu}, P_{\mu'}, P_{\mu''} \in \{P_\mu\}_\mu$, with $P_{\mu'} \neq P_{\mu''}$, such that $\mathrm{supp}(UP_{\tilde\mu}U^\dagger) \cap \mathrm{supp}(P_{\mu'}) = A \neq \emptyset$ and $\mathrm{supp}(UP_{\tilde\mu}U^\dagger) \cap \mathrm{supp}(P_{\mu''}) = B \neq \emptyset$. We show that this is in contradiction to classicality condition (5.9). Indeed, this means that there exist two pure classical states $|\psi_1\rangle \in \mathrm{supp}(P_{\tilde\mu})$ and $|\psi_2\rangle \in \mathrm{supp}(P_{\tilde\mu})$, such that $U|\psi_1\rangle \in A \subseteq \mathrm{supp}(P_{\mu'})$ and $U|\psi_2\rangle \in B \subseteq \mathrm{supp}(P_{\mu''})$. Thus, an arbitrary superposition of $|\psi_1\rangle$ and $|\psi_2\rangle$, for instance $|\psi\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle) \in \mathrm{supp}(P_{\tilde\mu})$, is transformed by the unitary U into a *superposition* of states that belong to $\mathrm{supp}(P_{\mu'})$ and $\mathrm{supp}(\mathrm{P}_{\mu''})$, respectively. Such a superposition of states *from different "blocks"* certainly does not represent a (pure) classical state w.r.t. $\{P_\mu\}_\mu$. Since for $\rho := |\psi\rangle\langle\psi|$ we thus have $\rho \in \mathcal{S}^{\mathrm{cl}}_{\{P_\mu\}}$, but $(U\rho U^\dagger) \notin \mathcal{S}^{\mathrm{cl}}_{\{P_\mu\}}$, we obtain a contradiction to the classicality condition (5.9). Hence, the statement of Theorem 2 involves a result which one would expect anyway if taking into account the above conclusion. For, if the classicality condition (5.9) had also been a necessary condition for decoherence in the most general case of arbitrarily coarse-grained histories, only deterministic processes would have been describable by decoherent sets of histories within our framework. This cannot be true in general! And it is indeed not true, as displayed by Theorem 2. In the special case of completely fine-grained histories within our framework, however, decoherence of histories is possible if, and only if, the set of histories is deterministic. Deterministic sets of histories are of course trivially decoherent. Determinism is trivially sufficient for decoherence, even in the most general case! Theorem 1 shows that *in the fine-grained case determinism is also necessary for decoherence* within our framework. In the fine-grained case decoherence is possible *only if* the corresponding set of histories is deterministic. Decoherence of fine-grained histories (for all history lengths and all classical initial states) *implies determinism.* For any pure classical initial state only one history is ever realized, occurring with certainty, i.e. probability equal to 1. No branching occurs. There is therefore no entropy increase. The entropy remains constant being given by the initial entropy $S = -\sum_\mu p_\mu \log(p_\mu)$ of the initial density matrix $\rho_0 = \sum_\mu p_\mu P_\mu$, in case of a mixed classical initial state.

In this context another result concerning properties of decoherent histories is worth mentioning, which is related to the above issue regarding determinism. It was proven, independently by Diosi in [49] and by Dowker and Kent in [34, 44], that for *finite-dimensional* Hilbert spaces there is an upper bound on the number of decohering histories with non-vanishing probabilities. Diosi proved that, in the case of *finite-dimensional* Hilbert spaces of dimension $d < \infty$, there can be at most d histories with non-vanishing probabilities in a decoherent set, if the initial state is pure. If the initial state from which the histories start is mixed, represented by a density operator ρ of rank $r := \text{rank}(\rho) > 1$, then the maximal number of non-zero probability histories in a decoherent set is given by rd. Hence, there can be no more than d^2 decoherent histories *with non-vanishing probabilities.* In other words, if the Hilbert space is of *finite* dimension, there is a strict upper bound on the number of probabilistic events. Once this number is reached, branching stops, and the evolution continues completely deterministically after that. Diosi's and Dowker and Kent's results do not say, however, how long it takes to use up all of the "allowed" branchings. In general, therefore, we do not know, when the quantum stochasticity is going to be exhausted and the very last probabilistic event will happen. The very last branching could in principle occur very far in the future, i.e. — to use the language of our framework — at a time corresponding to a very large value of the iteration number k of the unitary map, with k being finite but going to infinity, after a huge period of time with no branchings.

Another observation concerning the statement of Theorem 2 is worth mentioning. The result stated as Theorem 2 is interesting from the following point of view. One would expect that any *"classical evolution"*, whatever this may mean, always transforms classical states into classical ones. Theorem 2, however, insists that, generally, decoherence of histories is not sufficient for "classical evolution". Decoherence, therefore, is not enough to guarantee a feature that one would expect for a "classical behaviour". In spite of our very simplified framework and the restriction to considering evolutions that are induced by unitary maps it is tempting to relate this simple observation to a well known fact among experts on the decoherent histories formulation of quantum mechanics, namely, that consistency of histories alone does not suffice to define classical behaviour [18, 20]. Decoherence is one, but by no means the only, aspect of classicality. In general, one can have infinitely many sets of consistent histories in the formal sense [34, 44, 45], but which do not describe classical behaviour. Decoherence of histories is only a *precondition* for classical behaviour. Some authors even claim — Adrian Kent being the most advocated critic — that "classicality", i.e. the appearance of classical behaviour in a world governed at a fundamental level by quantum-dynamical laws, cannot be derived from within the decoherent (consistent) histories formalism of quantum mechanics at all. See References [34, 44, 45] for a very critical review of the issue of classicality within the decoherent (consistent) histories formulation of quantum mechanics.

The above reference to the classicality issue within the consistent (decoherent) histories formulation of quantum mechanics is of course very vague.[15] The above observation regarding "classical evolution" is suggestive of the classicality issue, but by

[15] Please let me note, that I do not claim our Theorem 2 to be of any relevant contribution to the research on the important issue of "classicality" within the decoherence programme (see [46] and references therein; see also [18, 20]), the ultimate goal of which is to explain the appearance of a classical world within quantum theory. Our simple result of Theorem 2 does only indicate a possible connection with the complex phenomenon of "classicality" in quantum theory; it is suggestive of the classicality issue, but by no means of any convincing relevance.

no means of any convincing relevance. In fact it is to some extent misleading to view this observation in connection with the classicality issue. Here we are considering a closed quantum dynamical system whose evolution is given by a *unitary* quantum map. From Theorem 2 we then draw the not unexpected conclusion that decoherence of histories does not imply a typically classical feature for this unitary map, namely the preservation of classicality of states. It is of course absolutely not clear, however, what, in the present context, is meant by "classical states" of a *closed* quantum system. The notion of "classical states" would make sense, if bringing an environment into the picture. Then it would be suggesting to associate our notion of classical states, as defined in Chapter 3, with *Zurek's (classical) "pointer states"* [50, 51, 52, 53], which arise as "preferred set of states" in a process called *"environment-induced superselection"* [51]. But this would mean considering an open quantum system. The problem of formulating a decoherence functional for an open quantum system interacting with some environment was investigated by Paz and Zurek in [54]. In their work Paz and Zurek make contact between the two main approaches addressing the classicality issue: the decoherent histories formalism and the environment-induced decoherence programme. In their analysis they assume that the total, closed, system for which the decoherence functional is formulated can be conveniently separated into a subsystem called "system" and another subsystem called "environment",[16] and examine the issue under what conditions a decoherence functional can be constructed *entirely* from the

[16]Such a fixed system-environment split is intuitively accessible and correctly models many mechanisms of decoherence, but it is not general. Rather, *coarse-graining* is the universal concept within the histories formalism of quantum mechanics, which, *when possible*, determines a system-environment split. It is possible, if and only if — for a given coarse-grained set of alternatives — the Hilbert space $\mathcal{H}$ of the given closed system can be written as a tensor product $\mathcal{H}_\mathcal{S} \otimes \mathcal{H}_\mathcal{E}$ where $\mathcal{H}_\mathcal{S}$ refers to quantities followed by the coarse-graining and $\mathcal{H}_\mathcal{E}$ refers to the quantities that are ignored. Such a tensor product factorization is called a system-environment split. Different coarse-grainings lead to different possible notions of "system" and "environment", that division is usually not unique, and for some kinds of coarse-graining no system-environment split is possible at all. Even when a system-environment split is possible at one time, a *different* system-environment split could be needed at another moment of time. A fixed system-environment split is therefore neither general nor necessary (cf. Appendix of Ref. [39]).

point of view of the "*reduced theory*", i.e., in terms of the *reduced density operator* of the "system" and projectors acting on the "system" *alone*, after the environment has been "traced out". They find that such a construction is possible if the correlations between system and environment evolve in a Markovian way. If the correlations dynamically established between the system and the environment do not affect the future evolution of the reduced density matrix of the system, the later will satisfy a master equation which is local in time. Under this condition a decoherence functional can be formulated entirely in terms of elements of the "reduced theory " for the open system. Provided this can be done, Paz and Zurek show how to construct sets of perfectly consistent (decoherent) histories. If the projectors $P^j_{\mu_j}$ defining the histories of length k, are chosen, for all $j = 1, \ldots, k$, such that they project on the instantaneous eigenstates of the *"path-projected"* reduced density matrix $\rho^{\mathrm{red}}(t_j)$,[17] at the particular time t_j, one obviously gets a diagonal decoherence functional and thus an exactly decoherent set of histories. The instantaneous eigenbasis of the path-projected reduced density matrix is given by the "Schmidt basis", the corresponding histories being therefore called "Schmidt histories" by Paz and Zurek. In the Markovian regime such Schmidt histories always decohere. These Schmidt histories are, in general, highly *branch dependent*, though, since the eigenstates of the path-projected reduced density matrix at time t_j depend on the choice of earlier alternatives. The set of projectors defining the Schmidt events in the "next time" thus depend on which projectors were applied in the past. For instance, one can get completely different Schmidt histories by choosing a different time-sequence (this may already be the case if one adds or omits an event at a single instant of time). Schmidt histories are thus in general highly unstable, and because of that they do not describe classical behaviour, in spite of being decoherent. Paz and Zurek investigated the conditions under which Schmidt histories are stable. They become stable if the eigenbasis of the path-projected reduced density matrix is *time-independent*, because in that case the commutation properties remain unaltered

[17]The *"path-projected"* reduced density matrix is the density matrix (cf. [54]) which is obtained from the initial density matrix by consecutively applying global unitary evolution operators, projectors on the system alone and the operation $\mathrm{Tr}_{\mathcal{E}}[\,\cdot\,]$ ("tracing out the environment") — using the Schrödinger picture. See Ref. [54] for details.

under the addition or omission of projectors. This, in turn, will be so, if there exists a stable *pointer basis*, selected by the environment as a robust preferred set of states for the system, with respect to which the reduced density matrix is approximately diagonal. However, this can only be achieved if we wait long enough between the intermediate times for which the history events are specified. In other words, the difference $\Delta t = t_j - t_{j-1}$ has to be larger than the typical decoherence time scale of the dynamics.

There are a few points worth noting here with regard to our results of this section in relation to Paz and Zurek's work [54] just discussed. The main reason for mentioning their work here is to address the issue of the *classicality concept* within the consistent histories framework, which we were prodded to by the simple observation inferred from Theorem 2. Another reason concerns the actual topic of this section, namely, finding simpler consistency conditions for a set of histories. Let me start with the second point. In their work Paz and Zurek provide a *sufficient* consistency condition for a set of histories of an *open* quantum system: consistency is always easily achieved by means of the Schmidt histories. In a way, the abstract "classicality" condition (5.30) together with the special structure of the histories within our simplified framework lead to the very same situation. We can, of course, define a "path-projected" density operator as well, and our unitary evolution, if having the property of preserving the classicality of states, generates path-projected density operators which are classical w.r.t. the partition $\{P_\mu\}_\mu$ the histories are constructed of. Thus, the projectors $P^j_{\mu_j}$ representing the events at times t_j automatically project on the eigenbasis of the path-projected density operators $\rho(t_j)$, since they are all chosen from the same partition $\{P_\mu\}_\mu$, with respect to which $\rho(t_j)$ is block-diagonal for all $j = 1, 2, \ldots$. Note, however, that, while Paz and Zurek are concerned with a set of histories for an *open* quantum system, we are considering histories for *closed* quantum systems. Paz and Zurek's sufficient criterion for constructing consistent sets of histories is, if applied within our framework, trivially a sufficient condition for consistency of sets of histories. We showed that in the fine-grained case it is also necessary within our framework.

Let us now, once again, expand on the "classicality" issue, which Paz and Zurek provide a good understanding for. In their conclusions they point out, that consistency of histories, being a primary, necessary criterion for classicality, is not sufficient to define quasiclassical domain: sets of Schmidt histories being always consistent are in general highly non-classical as explained above. Paz and Zurek demonstrate how classical Schmidt histories arise: "A classical history is a chain of events recorded by the environment". *Predictability* is the main ingredient: histories become classical if they are predictable in the sense that they are stable under the addition or omission of intermediate times. This, again, will be the case if Schmidt histories are constructed with pointer states, which means that we need to require the separation between the time slices to be larger than the typical decoherence time. In that case, stability of Schmidt histories is established and the projectors representing the history events are determined to be the ones associated with the pointer basis selected by the environment — *determined by the environment and not by a subjective choice of an observer.* In this way Schmidt histories become good candidates for describing quasi-classical domain [54]. The quantum operations $\mathcal{E}_{t_j,t_{j+1}}[\,\cdot\,]$ which transform the *path-projected* reduced density operators $\rho^{\mathrm{red}}(t_j)$ at times t_j into those at the "next" times t_{j+1},

$$\rho^{\mathrm{red}}(t_{j+1}) = \mathcal{E}_{t_j,t_{j+1}}\left[\rho^{\mathrm{red}}(t_j)\right] \quad , \tag{5.35}$$

within such *classical Schmidt histories*, are of course *not unitary* (i.e. cannot be written in terms of *unitary* Kraus operators). This is because the dynamics of *open* quantum systems are not unitary. Therefore, the "classical evolution" which appears in such *classical Schmidt histories* is *not a unitary evolution.* The above observation we inferred from Theorem 2, on the other hand, concerns only unitary evolutions, induced by some unitary quantum map U, for *closed* systems. The use of the terms "classical states" and "classical evolution" within our framework, which involves *closed* quantum systems, is very uncommon, but can be motivated — additionally to the measurement-theoretically motivated arguments given in Chap. 3 — in the following way. The main idea to justify the use of these terms within our framework consists in first having an *open* quantum system *"prepared"* in an (improper) mixture of (classical) pointer states by its environment, then completely shielding that system from further interactions with all of the surrounding environment and letting it evolve alone according to its own unitary dynamics, after it has been isolated. In this situation an initially open system is forced by the environment-induced decoherence process [46, 50, 51, 52, 53, 55] into a state which is block-diagonal w.r.t. the eigen-spaces of the pointer observable, selected by the environment as the stable observable for the system. This will in general be a mixed state. Or that is to say it will be an *improper mixture* of pointer states, which means that there is still entanglement with the environment — which will have been established during the decoherence process up to time $t = 0$ — even if we isolate the system from further interactions with the surrounding environment after that decoherence process. After isolating that system at time $t = 0$ we can ask questions with regard to histories of that system, which will now be *closed* [18]. In this way we get a situation as described by our framework: the initial state from which the histories start is given by a classical state, i.e. a state which is block-diagonal with respect to the eigen-spaces of the (stable) pointer observable. And we can ask the question what kind of unitaries U are required between the intermediate times t_{j+1} and t_j of the histories in order to obtain decoherent sets of histories. And we can say that such a unitary induces a "classical evolution" if it transforms every mixture of pointer states into a mixture of pointer states [19]. Theorem 2 tells us that such a "classical evolution" is not necessary for obtaining a consistent set of histories.

Finally, another notion regarding the "classicality" concept is worth mentioning here. Our observation that decoherence of histories is not sufficient for "classical evolution" might also be relevant to the study of "classicality" in quantum information processing [31]. "Classicality" is understood in this context as that part of a quantum algorithm that can be substituted by classical information processing without appreciably slowing down the computation. The basic idea of this study is the belief that

[18]It will be isolated in terms of interactions, but the entanglement with the environment, which has been established during the decoherence process up to time $t = 0$, will in general persist.

[19]The pointer states which constitute the preferred set of states when the system is open, are, of course, not going to be the favoured states any longer after the system has been isolated from all the interactions with its surrounding environment. We thus define "classical states" for a given system to be that class of states, which are block-diagonal w.r.t. the eigen spaces of the observable, which *would* be selected as the stable "pointer observable" by the environment-induced decoherence process, *if* the system in question *were* made interact with its surrounding environment.

not all the information processed by a quantum computer is required to be quantum for the success of the quantum algorithm. That classical simulation of quantum systems might become efficient when decoherence is taken into account, was noticed by Aharonov and Ben-Or in [56]. In [31] Poulin uses the consistent histories formalism to investigate "classicality" of quantum algorithms, in particular the issue of how and when one can substitute quantum information by classical information without appreciably affecting the speed of the computation. In his analysis parts of the system, some qubits of the quantum computer, are "forced" to "classical states", in a way that does not affect the result and speed of the quantum computation. The quantum information conveyed by these qubits can thus be replaced by classical information, until a new transformation makes them quantum again. Poulin uses the term "classical states" in connection with consistent histories in the same fashion as we do above, relating their notion to Zurek's (classical) pointer states.

5.4 Initial states and decoherence of histories

5.4.1 Introduction

Whether the decoherence condition (3.6) is fulfilled or not depends on the initial state, the unitary dynamics of the system and the propositions from which the histories are constructed. The *dependence on the initial state* is connected to one of the central questions of the decoherence programme (see, e.g., [46] and references therein), especially the programme of decoherence in quantum cosmology [18, 47]: the question of how the classical features of our world emerge from the initial quantum state of the Universe. This question provides the main motivation for the research presented in this section.

Here we investigate this question within our slightly restricted and simplified framework, i.e., for the special class of histories that are constructed from a *fixed* exhaustive set of mutually exclusive propositions, $\{P_\mu\}$. The issue of how the choice of the initial state affects decoherence of such histories is the central content of the following theorem (Theorem 4 of Section 5.4.2). We show that decoherence of arbitrarily long histories for all initial states that are induced by the projectors $\{P_\mu\}$ via normalization implies the decoherence of such histories for arbitrary initial states. Thus, establishing decoherence for arbitrary history lengths and all initial states from the set $\mathcal{S}_{\{P_\mu\}}$ involves the decoherence of such histories for all $\rho \in \mathcal{S}$. It is relevant to note that, unlike the set $\mathcal{S}$ of all possible states, the set $\mathcal{S}_{\{P_\mu\}}$ is discrete and may contain as few as just two elements (in the case of "yes-no" propositions).

In addition, with Theorem 4 we make up for the proof of Theorem 3 of the last section. Theorem 3 of Section 5.3.3, which concerns single-iteration necessary decoherence criterions for sets of arbitrarily coarse-grained histories, is a straightforward corollary of the following Theorem 4.

5.4.2 Result

Theorem 4:

Let a projective partition $\{P_\mu\}$ of a finite dimensional Hilbert space $\mathcal{H}$ and a unitary map U on $\mathcal{H}$ be given. Then the following three statements are equivalent:

$$(a) \quad \forall\, \rho \in \mathcal{S}_{\{P_\mu\}}\, \forall\, k \in \mathbb{N}\;\; \forall\, h_{\boldsymbol{\alpha}}, h_{\boldsymbol{\beta}} \in \mathcal{K}[\{P_\mu\}\,;\,k\,] :\; \mathcal{D}_{U,\rho}\,[h_{\boldsymbol{\alpha}}, h_{\boldsymbol{\beta}}] \propto \delta_{\boldsymbol{\alpha\beta}}$$

$$(b) \quad \forall\, P_{\mu'}, P_{\mu''} \in \{P_\mu\}\;\; \forall\, n \in \mathbb{N} :\;\; \left[U^n P_{\mu'} (U^\dagger)^n\,,\, P_{\mu''}\right] = 0$$

$$(c) \quad \forall\, \rho \in \mathcal{S}\;\; \forall\, k \in \mathbb{N}\;\; \forall\, h_{\boldsymbol{\alpha}}, h_{\boldsymbol{\beta}} \in \mathcal{K}[\{P_\mu\}\,;\,k\,] :\; \mathcal{D}_{U,\rho}\,[h_{\boldsymbol{\alpha}}, h_{\boldsymbol{\beta}}] \propto \delta_{\boldsymbol{\alpha\beta}}\,.$$

Proof:

We will prove the theorem by showing that (a) implies (b), (b) implies (c), and (c) implies (a). The last implication, (c)⇒(a), is trivial, and the second implication, (b)⇒(c), can be easily shown using the notation of Eq. (3.5). It remains to prove the implication (a)⇒(b).

The proof is constructed as follows. We first show that the proposition

$$\begin{aligned} &\forall\, \rho \in \mathcal{S}_{\{P_\mu\}}\;\; \forall\, n \in \mathbb{N}\;\; \forall\, \mu_0, \mu', \mu'' \quad \text{with} \quad \mu' \neq \mu'' : \\ &\mathrm{Tr}\left[P_{\mu''} (U^n P_{\mu_0} U^{\dagger\, n}) P_{\mu'} (U^n \rho U^{\dagger\, n}) P_{\mu''}\right] = 0 \end{aligned} \tag{5.36}$$

is a necessary consequence of the decoherence condition (a) and then conclude that this proposition implies the commutativity condition (b) of the theorem.

The first part of the proof will be accomplished by contradiction, i.e. we will assume that (5.36) is not satisfied, and then show that this assumption contradicts the decoherence condition (a) of the theorem.

Assume condition (5.36) is not satisfied. This means there exist a partition state $\tilde{\rho} \in \mathcal{S}_{\{P_\mu\}}$, an integer $\tilde{n} \in \mathbb{N}$, and partition-element labels μ_0, μ', μ'', with $\mu' \neq \mu''$, such that

$$\mathrm{Tr}\left[P_{\mu''}(U^{\tilde{n}}P_{\mu_0}U^{\dagger\,\tilde{n}})P_{\mu'}(U^{\tilde{n}}\tilde{\rho}\,U^{\dagger\,\tilde{n}})P_{\mu''}\right] = c \neq 0\ . \tag{5.37}$$

This, as we will see, is in contradiction to decoherence condition (a). Written out, the decoherence condition (a) is

$$\mathrm{Tr}\left[P_{\alpha_k}UP_{\alpha_{k-1}}U\ldots P_{\alpha_1}U\,\rho_0\,U^\dagger P_{\beta_1}\ldots P_{\beta_{k-1}}U^\dagger P_{\beta_k}\right] \propto \prod_{j=1}^{k}\delta_{\alpha_j\beta_j} \tag{5.38}$$

for all $k \in \mathbb{N}$, all initial states $\rho_0 \in \mathcal{S}_{\{P_\mu\}}$, and arbitrary histories $h_{\boldsymbol{\alpha}}$, $h_{\boldsymbol{\beta}}$. Since the length k of the histories is arbitrary, we may choose $k = q\tilde{n}$ with arbitrary $q \in \mathbb{N}$. By summing over $\alpha_1, \ldots, \alpha_{\tilde{n}-1}, \alpha_{\tilde{n}+1}, \ldots, \alpha_{q\tilde{n}-1}$ and $\beta_1, \ldots, \beta_{\tilde{n}-1}, \beta_{\tilde{n}+1}, \ldots, \beta_{q\tilde{n}-1}$, and using $\sum_\mu P_\mu = \mathbb{1}_{\mathcal{H}}$, we obtain

$$\mathrm{Tr}\left[P_{\alpha_{q\tilde{n}}}(U^{q-1})^{\tilde{n}}P_{\alpha_{\tilde{n}}}U^{\tilde{n}}\,\rho_0\,U^{\dagger\,\tilde{n}}P_{\beta_{\tilde{n}}}(U^{\dagger\,q-1})^{\tilde{n}}P_{\beta_{q\tilde{n}}}\right] \propto \delta_{\alpha_{q\tilde{n}}\beta_{q\tilde{n}}}\delta_{\alpha_{\tilde{n}}\beta_{\tilde{n}}} \tag{5.39}$$

for all $q \in \mathbb{N}$, any $\rho_0 \in \mathcal{S}_{\{P_\mu\}}$, and arbitrary $\alpha_{\tilde{n}}$, $\beta_{\tilde{n}}$, $\alpha_{q\tilde{n}}$, $\beta_{q\tilde{n}}$.

In order to proceed we now make use of our *Lemma on uniform recurrence* from Section 5.2. According to this Lemma, for any given arbitrarily small $\epsilon > 0$ we can always find a $q \in \mathbb{N}$ such that $U^q = \mathbb{1}_{\mathcal{H}} + \hat{\mathcal{O}}(\epsilon)$, where $\hat{\mathcal{O}}(\epsilon)$ is some operator whose norm is of order ϵ: $\|\hat{\mathcal{O}}(\epsilon)\| < \epsilon$. Using the submultiplicativity property of operator norms, we have

$$\|U^{-1}\hat{\mathcal{O}}(\epsilon)\| \leq \|U^{-1}\| \times \|\hat{\mathcal{O}}(\epsilon)\| = \|\hat{\mathcal{O}}(\epsilon)\| \tag{5.40}$$

and hence $U^{q-1} = U^{-1} + \hat{\mathcal{O}}'(\epsilon)$, where $\|\hat{\mathcal{O}}'(\epsilon)\| < \epsilon$.

Now we are in a position to derive a contradiction. We let our histories start with the initial state $\rho_0 = \tilde{\rho}$. Furthermore we choose $\alpha_{\tilde{n}} = \mu'$, $\beta_{\tilde{n}} = \mu''$, and $\alpha_{q\tilde{n}} = \beta_{q\tilde{n}} = \mu_0$.

Since $\mu' \neq \mu''$, condition (5.39) becomes

$$\forall q \in \mathbb{N} : \ \mathrm{Tr}\left[P_{\mu_0}(U^{q-1})^{\tilde{n}} P_{\mu'} U^{\tilde{n}}\, \tilde{\rho}\, U^{\dagger\,\tilde{n}} P_{\mu''} (U^{\dagger\,q-1})^{\tilde{n}} P_{\mu_0}\right] = 0 \ . \tag{5.41}$$

Choosing q such that $\| U^q - \mathbb{1}_{\mathcal{H}} \| < \epsilon$ for a given *arbitrarily small* $\epsilon > 0$, we get a situation where the expressions $(U^{q-1})^{\tilde{n}}$ and $(U^{\dagger\,q-1})^{\tilde{n}}$ in Eq. (5.41) can be replaced by $(U^\dagger + \hat{\mathcal{O}}'(\epsilon))^{\tilde{n}}$ and $(U + \hat{\mathcal{O}'}^\dagger(\epsilon))^{\tilde{n}}$, respectively. In the following it will be convenient to use the definition

$$A_{r_1,r_2,\ldots,r_{\tilde{n}}} := \prod_{i=1}^{\tilde{n}} \left(U^{\dagger\, r_i} (\hat{\mathcal{O}}'(\epsilon))^{1-r_i} \right) \ , \tag{5.42}$$

where the operators inside the product are written out from left to right in the order of increasing index i. Using this definition we have:

$$(U^\dagger + \hat{\mathcal{O}}'(\epsilon))^{\tilde{n}} \ = \sum_{r_1,\ldots,r_{\tilde{n}} \in \{0,1\}} A_{r_1,\ldots,r_{\tilde{n}}} \ , \tag{5.43}$$

$$(U + \hat{\mathcal{O}'}^\dagger(\epsilon))^{\tilde{n}} \ = \sum_{r_1,\ldots,r_{\tilde{n}} \in \{0,1\}} A^\dagger_{r_1,\ldots,r_{\tilde{n}}} \ . \tag{5.44}$$

This yields for the left hand side of Eq. (5.41):

$$\begin{aligned} &\mathrm{Tr}\left[P_{\mu_0}(U^{q-1})^{\tilde{n}} P_{\mu'} U^{\tilde{n}}\, \tilde{\rho}\, U^{\dagger\,\tilde{n}} P_{\mu''} (U^{\dagger\,q-1})^{\tilde{n}} P_{\mu_0}\right] = \\ &= \ \mathrm{Tr}\left[P_{\mu_0}\left(\sum_{r_1,\ldots,r_{\tilde{n}} \in \{0,1\}} A_{r_1,\ldots,r_{\tilde{n}}} \right) P_{\mu'} U^{\tilde{n}}\, \tilde{\rho}\, U^{\dagger\,\tilde{n}} P_{\mu''} \left(\sum_{s_1,\ldots,s_{\tilde{n}} \in \{0,1\}} A^\dagger_{s_1,\ldots,s_{\tilde{n}}} \right) P_{\mu_0}\right] \\ &= \ \sum_{r_1,\ldots,r_{\tilde{n}} \in \{0,1\}} \ \sum_{s_1,\ldots,s_{\tilde{n}} \in \{0,1\}} \mathrm{Tr}\left[P_{\mu_0} A_{r_1,\ldots,r_{\tilde{n}}} P_{\mu'} U^{\tilde{n}}\, \tilde{\rho}\, U^{\dagger\,\tilde{n}} P_{\mu''} A^\dagger_{s_1,\ldots,s_{\tilde{n}}} P_{\mu_0} \right] . \end{aligned} \tag{5.45}$$

According to (5.41) the left hand side of this equation must be zero. Hence we have:

$$\begin{aligned} &\mathrm{Tr}\left[P_{\mu_0}(U^\dagger)^{\tilde{n}} P_{\mu'} U^{\tilde{n}}\, \tilde{\rho}\, U^{\dagger\,\tilde{n}} P_{\mu''} U^{\tilde{n}} P_{\mu_0}\right] = \\ &= \ - \sum_{\substack{r_1,\ldots,r_{\tilde{n}} \in \{0,1\} \\ r_1+\cdots+r_{\tilde{n}} < \tilde{n}}} \ \sum_{\substack{s_1,\ldots,s_{\tilde{n}} \in \{0,1\} \\ s_1+\cdots+s_{\tilde{n}} < \tilde{n}}} \mathrm{Tr}\left[P_{\mu_0} A_{r_1,\ldots,r_{\tilde{n}}} P_{\mu'} U^{\tilde{n}}\, \tilde{\rho}\, U^{\dagger\,\tilde{n}} P_{\mu''} A^\dagger_{s_1,\ldots,s_{\tilde{n}}} P_{\mu_0} \right] . \end{aligned} \tag{5.46}$$

Using the cyclic permutation-invariance property of the trace and the triangle inequality, we obtain

$$\left|\text{Tr}\left[P_{\mu''}(U^{\tilde{n}}P_{\mu_0}U^{\dagger\,\tilde{n}})P_{\mu'}(U^{\tilde{n}}\,\tilde{\rho}\,U^{\dagger\,\tilde{n}})P_{\mu''}\right]\right| \leq$$
$$\leq \sum_{\substack{r_1,\ldots,r_{\tilde{n}}\in\{0,1\}\\ r_1+\cdots+r_{\tilde{n}}<\tilde{n}}} \sum_{\substack{s_1,\ldots,s_{\tilde{n}}\in\{0,1\}\\ s_1+\cdots+s_{\tilde{n}}<\tilde{n}}} \left|\text{Tr}\left[A^{\dagger}_{s_1,\ldots,s_{\tilde{n}}}P_{\mu_0}A_{r_1,\ldots,r_{\tilde{n}}}P_{\mu'}U^{\tilde{n}}\,\tilde{\rho}\,U^{\dagger\,\tilde{n}}P_{\mu''}\right]\right| . \tag{5.47}$$

Utilizing the inequality $|\,\text{Tr}[BT]\,| \leq \|\,B\,\|\;\text{Tr}\sqrt{T^{\dagger}T}$ for bounded operators $B:\mathcal{H}\to\mathcal{H}$ and operators $T:\mathcal{H}\to\mathcal{H}$ with finite trace norm $\|\,T\,\|_1 := \text{Tr}\sqrt{T^{\dagger}T}$, see Ref. [57], we deduce from Eq. (5.47):

$$\left|\text{Tr}\left[P_{\mu''}(U^{\tilde{n}}P_{\mu_0}U^{\dagger\,\tilde{n}})P_{\mu'}(U^{\tilde{n}}\,\tilde{\rho}\,U^{\dagger\,\tilde{n}})P_{\mu''}\right]\right| \leq$$
$$\leq \sum_{\substack{r_1,\ldots,r_{\tilde{n}}\in\{0,1\}\\ r_1+\cdots+r_{\tilde{n}}<\tilde{n}}} \sum_{\substack{s_1,\ldots,s_{\tilde{n}}\in\{0,1\}\\ s_1+\cdots+s_{\tilde{n}}<\tilde{n}}} \|\,B^{s_1,\ldots,s_{\tilde{n}}}_{r_1,\ldots,r_{\tilde{n}}}\,\|\;\text{Tr}\sqrt{T^{\dagger}T}\;, \tag{5.48}$$

where we defined

$$B^{s_1,\ldots,s_{\tilde{n}}}_{r_1,\ldots,r_{\tilde{n}}} := A^{\dagger}_{s_1,\ldots,s_{\tilde{n}}}P_{\mu_0}A_{r_1,\ldots,r_{\tilde{n}}}\;, \tag{5.49}$$
$$T := P_{\mu'}U^{\tilde{n}}\,\tilde{\rho}\,U^{\dagger\,\tilde{n}}P_{\mu''}\;. \tag{5.50}$$

Using the fact that $\|B^{\dagger}\|=\|B\|$ for any bounded operator B and it's adjoint $B^{\dagger}$ [58], we have $\|\,\hat{\mathcal{O}}'^{\dagger}(\epsilon)\,\|=\|\,\hat{\mathcal{O}}'(\epsilon)\,\|<\epsilon$. Utilizing the submultiplicativity property of operator norms we deduce that the norms of the operators $B^{s_1,\ldots,s_{\tilde{n}}}_{r_1,\ldots,r_{\tilde{n}}}$ are all bounded from above by ϵ, except in the case where all $s_1,\ldots,s_{\tilde{n}}$ and all $r_1,\ldots,r_{\tilde{n}}$ are equal 1, which is excluded from the sum on the right-hand side of Eq. (5.48). Indeed we have:

$$\begin{aligned}
\|\,B^{s_1,\ldots,s_{\tilde{n}}}_{r_1,\ldots,r_{\tilde{n}}}\,\| &\leq \left(\prod_{i=1}^{\tilde{n}}\|\,U\,\|^{s_i}\,\|\,\hat{\mathcal{O}}'^{\dagger}(\epsilon)\,\|^{1-s_i}\right)\times \\
&\quad\times\|\,P_{\mu_0}\,\|\left(\prod_{i=1}^{\tilde{n}}\|\,U^{\dagger}\,\|^{r_i}\,\|\,\hat{\mathcal{O}}'(\epsilon)\,\|^{1-r_i}\right) \\
&\leq \left(\prod_{i=1}^{\tilde{n}}\epsilon^{1-s_i}\right)\left(\prod_{j=1}^{\tilde{n}}\epsilon^{1-r_j}\right) \\
&\leq \epsilon^2<\epsilon\;,\quad \text{if}\quad s_1+\cdots+s_{\tilde{n}}<\tilde{n}\;,\;r_1+\cdots+r_{\tilde{n}}<\tilde{n}\;,
\end{aligned} \tag{5.51}$$

where we used $\|P_{\mu_0}\|=\|U\|=\|U^\dagger\|=1$ and $\epsilon \ll 1$. With the definition $M := \mathrm{Tr}\sqrt{T^\dagger T}$ we finally conclude from Eq. (5.48):

$$\left|\mathrm{Tr}\left[P_{\mu''}(U^{\tilde{n}} P_{\mu_0} U^{\dagger\,\tilde{n}}) P_{\mu'} (U^{\tilde{n}}\,\tilde{\rho}\, U^{\dagger\,\tilde{n}}) P_{\mu''}\right]\right| < 2^{2\tilde{n}} M\epsilon \,. \tag{5.52}$$

Since c, $\tilde{n}$ and M are fixed constants, we can always arrange $2^{2\tilde{n}}M\epsilon < |c|$ by choosing a sufficiently small $\epsilon > 0$. This contradicts the assumption (5.37) and thus proves our proposition (5.36).

We are now in a position to derive the commutativity condition (b) of the theorem. It is a straightforward consequence of proposition (5.36) we have just proven. Taking condition (5.36) and choosing in it the state $\rho \in \mathcal{S}_{\{P_\mu\}}$ to be proportional to the projector sandwiched between U^n and $U^{\dagger\,n}$ within the first bracket,

$$\rho = \frac{P_{\mu_0}}{\mathrm{Tr}\left[P_{\mu_0}\right]} \,, \tag{5.53}$$

where P_{μ_0} is still arbitrary, we necessarily get the condition

$$\begin{aligned}&\forall\, n \in \mathbb{N}\ \forall\, \mu_0, \mu', \mu'' \quad \text{with} \quad \mu' \neq \mu'' : \\ &\mathrm{Tr}\left[P_{\mu''}(U^n P_{\mu_0} U^{\dagger\,n}) P_{\mu'} (U^n P_{\mu_0} U^{\dagger\,n}) P_{\mu''}\right] = 0 \,.\end{aligned} \tag{5.54}$$

With the definition $A := P_{\mu'}(U^n P_{\mu_0} U^{\dagger\,n}) P_{\mu''}$ Eq. (5.54) becomes $\mathrm{Tr}\left[A^\dagger A\right] = 0$. Since $A^\dagger A$ is a positive operator, this is possible if and only if $A = 0$. Hence condition (5.54) is equivalent to

$$\begin{aligned}&\forall\, n \in \mathbb{N}\ \forall\, \mu_0, \mu', \mu'' \quad \text{with} \quad \mu' \neq \mu'' : \\ &P_{\mu'}(U^n P_{\mu_0} U^{\dagger\,n}) P_{\mu''} = 0 \,.\end{aligned} \tag{5.55}$$

This condition implies

$$\sum_{\mu'} P_{\mu'}(U^n P_{\mu_0} U^{\dagger\,n}) P_{\mu''} = P_{\mu''}(U^n P_{\mu_0} U^{\dagger\,n}) P_{\mu''} \tag{5.56}$$

for any μ_0 and μ'', and arbitrary $n \in \mathbb{N}$. But since $\sum_{\mu'} P_{\mu'} = \mathbb{1}_{\mathcal{H}}$, the left hand side of the last equation must be equal to $(UP_{\mu_0}U^\dagger)P_{\mu''}$. Hence we obtain

$$P_{\mu''}(U^n P_{\mu_0} U^{\dagger\,n}) P_{\mu''} = (U^n P_{\mu_0} U^{\dagger\,n}) P_{\mu''} \tag{5.57}$$

on the one hand and by taking the adjoint of Eq. (5.57)

$$P_{\mu''}(U^n P_{\mu_0} U^{\dagger\, n}) P_{\mu''} = P_{\mu''}(U^n P_{\mu_0} U^{\dagger\, n}) \tag{5.58}$$

on the other hand, for any $n \in \mathbb{N}$ and arbitrary μ_0 and μ''. Therefore

$$(U^n P_{\mu_0} U^{\dagger\, n}) P_{\mu''} = P_{\mu''}(U^n P_{\mu_0} U^{\dagger\, n}) \tag{5.59}$$

for any $n \in \mathbb{N}$ and arbitrary μ_0, μ'', and so $\left[U^n P_{\mu_0} U^{\dagger\, n}\, ,\, P_{\mu''}\right] = 0$ for any $n \in \mathbb{N}$ and all $P_{\mu_0}, P_{\mu''} \in \{P_\mu\}$. $\square$

5.4.3 Discussion

The implication (a)⇒(c) of the theorem constitutes the main result of this section: the decoherence of histories of arbitrary length for all initial states from the set $\mathcal{S}_{\{P_\mu\}}$ implies decoherence of such histories for arbitrary initial states $\rho \in \mathcal{S}$. It should be mentioned that the set $\mathcal{S}_{\{P_\mu\}}$ can be viewed as the smallest natural set of states that is associated with our framework. It is discrete and may consist of just two elements (in the case of "yes-no" propositions). The set $\mathcal{S}$, on the other hand, contains the continuum of all possible states that are allowed in our framework.

We have thus found that if decoherence is established for arbitrary history lengths and all initial states from the discrete set $\mathcal{S}_{\{P_\mu\}}$, which is the smallest natural set induced by the given partition $\{P_\mu\}$, then any set of histories constructed from $\{P_\mu\}$ is decoherent for all possible initial states.

As far as the physical significance of this result is concerned, it is certainly of relevance to the decoherence programme [46], especially with respect to the issue concerning the emergence of classicality from the initial state of the Universe in quantum cosmology [18, 47].

As an additional result, we obtain, as a straightforward corollary of Theorem 4, a necessary single-iteration decoherence criterion for sets of arbitrarily coarse-grained histories, already stated as Theorem 3 in Section 5.3.3. The proposition of Theorem 3 follows trivially from the implication (a)⇒(b) of Theorem 4, as $\mathcal{S}_{\{P_\mu\}} \subset \mathcal{S}^{\mathrm{cl}}_{\{P_\mu\}}$.

5.5 Generalisation to approximate decoherence

5.5.1 Introduction

Condition (3.6) is the condition for *exact* decoherence. In most of physical models, however, decoherence of histories can be established only *approximately* (cf. e.g. [19]). It is therefore desirable to generalise the above results to the case of approximate decoherence. Let us first remember what is meant by approximate decoherence. An absolutely consistent assignment of probabilities to a given set of histories requires that whenever we bundle up the given histories to coarser-grained histories then the probability for each such coarser-grained history must be equal to the sum of the probabilities for its constituent finer-grained histories, and this has to be true for all possible coarse-grainings. If these probability sum rules are fulfilled only approximately, for all possible coarse-grainings of a given set of finer-grained histories, then we get an approximately consistent assignment of probabilities and call the given set of histories approximately decoherent. Quantitatively, one requires that the probability sum rules are satisfied to some order ϵ, meaning that the interference terms are suppressed by a very small factor $\epsilon \ll 1$ compared to the sums over the probabilities, for all possible coarse-grainings. Approximate decoherence to order $\epsilon \ll 1$ thus means that the probabilities are defined only up to that order. A condition that proved to be useful for approximate decoherence is (cf. Ref. [19])[20]

$$|\,\mathcal{D}_{U,\rho}\,[h_{\boldsymbol{\alpha}}, h_{\boldsymbol{\beta}}]\,| < \epsilon \Big(\mathcal{D}_{U,\rho}\,[h_{\boldsymbol{\alpha}}, h_{\boldsymbol{\alpha}}]\mathcal{D}_{U,\rho}\,[h_{\boldsymbol{\beta}}, h_{\boldsymbol{\beta}}]\Big)^{\frac{1}{2}} \quad \text{for } \; h_{\boldsymbol{\alpha}} \neq h_{\boldsymbol{\beta}}\,. \tag{5.60}$$

In [19] it was shown that with this condition *most* (in a statistical sense) probability sum rules are satisfied to order ϵ provided the number of all possible histories $h_{\boldsymbol{\alpha}}$ is large. Condition (5.60) can be motivated by the *"Dowker-Halliwell inequality"*,

$$|\,\mathcal{D}_{U,\rho}\,[h_{\boldsymbol{\alpha}}, h_{\boldsymbol{\beta}}]\,| \leq \Big(\mathcal{D}_{U,\rho}\,[h_{\boldsymbol{\alpha}}, h_{\boldsymbol{\alpha}}]\mathcal{D}_{U,\rho}\,[h_{\boldsymbol{\beta}}, h_{\boldsymbol{\beta}}]\Big)^{\frac{1}{2}}\,, \tag{5.61}$$

[20] In [19] a weaker condition was proposed, with $|\,\mathrm{Re}\,\mathcal{D}_{U,\rho}\,[h_{\boldsymbol{\alpha}}, h_{\boldsymbol{\beta}}]\,|$ instead of $|\,\mathcal{D}_{U,\rho}\,[h_{\boldsymbol{\alpha}}, h_{\boldsymbol{\beta}}]\,|$ on the left-hand side.

also derived in the work of Dowker and Halliwell in [19]. [21] The degree of approximate decoherence can thus be regarded as the extent to which the left-hand side of the inequality (5.61) is less than the right-hand side.

Here we assume a stronger condition, which guarantees that *all* probability sum rules are satisfied to the order ϵ, for all possible coarse-grainings, namely,

$$|\,\mathcal{D}_{U,\rho}\,[h_{\boldsymbol{\alpha}}, h_{\boldsymbol{\beta}}]\,| < \epsilon\,\frac{\Big(\mathcal{D}_{U,\rho}\,[h_{\boldsymbol{\alpha}}, h_{\boldsymbol{\alpha}}]\mathcal{D}_{U,\rho}\,[h_{\boldsymbol{\beta}}, h_{\boldsymbol{\beta}}]\Big)^{\frac{1}{2}}}{|\mathcal{K}[\{P_\mu\}\,;\,k\,]\,|} \quad \text{for } \; h_{\boldsymbol{\alpha}} \neq h_{\boldsymbol{\beta}}\,, \tag{5.62}$$

where $|\mathcal{K}[\{P_\mu\}\,;\,k\,]\,|$ denotes the number of elements in the set $\mathcal{K}[\{P_\mu\}\,;\,k\,]$, which is the number of all possible histories $h_{\boldsymbol{\alpha}}$. It is bounded from above by d^k with d being the dimension of the Hilbert space, $d = \dim\mathcal{H}$.

The only difficult part in the proof of Theorem 4 was to show the implication "(a)⇒(b)". As a first step towards proving analogous results for approximate decoherence, we confine ourselves to generalizing just this part of Theorem 4. Instead of exact decoherence we now assume approximate decoherence of histories for arbitrary history lengths k and for all initial states $\rho \in \mathcal{S}_{\{P_\mu\}}$, i.e., we replace in the statement (a) of Theorem 4 the exact decoherence condition (3.6) by our approximate decoherence condition (5.62). The task now is to show that the statement (b) of Theorem 4 is still implied in some approximate sense. The derivation of this implication is done in a similar way as for exact decoherence in the proof of Theorem 4 in Sec. 5.4. Let us again state the result (cf. also Ref. [3]) as a theorem.

[21] The intuitive meaning of the Dowker-Halliwell inequality (5.61) is clear: it indicates that there can be *no interference with a history which has probability zero.*

5.5.2 Results

Theorem 5:

Let a projective partition $\{P_\mu\}$ of a finite dimensional Hilbert space $\mathcal{H}$, a unitary map U on $\mathcal{H}$, and a small $\epsilon > 0$ be given. Then

$$\forall\, \rho \in \mathcal{S}_{\{P_\mu\}}\ \forall\, k \in \mathbb{N}\ \ \forall\, h_{\boldsymbol{\alpha}}, h_{\boldsymbol{\beta}} \in \mathcal{K}[\{P_\mu\}\,;\,k\,] \ \text{ with }\ h_{\boldsymbol{\alpha}} \neq h_{\boldsymbol{\beta}}\ :$$
$$|\mathcal{D}_{U,\rho}\,[h_{\boldsymbol{\alpha}}, h_{\boldsymbol{\beta}}]| < \epsilon\, \frac{\Big(\mathcal{D}_{U,\rho}\,[h_{\boldsymbol{\alpha}}, h_{\boldsymbol{\alpha}}]\mathcal{D}_{U,\rho}\,[h_{\boldsymbol{\beta}}, h_{\boldsymbol{\beta}}]\Big)^{\frac{1}{2}}}{|\mathcal{K}[\{P_\mu\}\,;\,k\,]\,|} \tag{5.63}$$

only if

$$\forall\, P_{\mu'}, P_{\mu''} \in \{P_\mu\}\ \forall\, n \in \mathbb{N}\ :\ \|\ [(U^n P_{\mu'} U^{\dagger\, n}), P_{\mu''}]\ \|_2 \leq 2d^{\frac{3}{2}}\sqrt{\epsilon}\,, \tag{5.64}$$

where $d = \dim \mathcal{H}$ and $\|\cdot\|_2$ denotes the Hilbert-Schmidt operator norm.

Proof:

The proof is constructed as follows. Using the trivial relation

$$\Big(\mathcal{D}_{U,\rho}\,[h_{\boldsymbol{\alpha}}, h_{\boldsymbol{\alpha}}]\mathcal{D}_{U,\rho}\,[h_{\boldsymbol{\beta}}, h_{\boldsymbol{\beta}}]\Big)^{\frac{1}{2}} \leq \frac{1}{2}\Big(\mathcal{D}_{U,\rho}\,[h_{\boldsymbol{\alpha}}, h_{\boldsymbol{\alpha}}] + \mathcal{D}_{U,\rho}\,[h_{\boldsymbol{\beta}}, h_{\boldsymbol{\beta}}]\Big) \tag{5.65}$$

together with the techniques of Sec. 5.2 and Sec. 5.4, we first show that the approximate decoherence assumption (5.63) of the theorem necessarily implies that

$$\begin{aligned} &\forall\, n \in \mathbb{N}\ \forall\, \mu_0, \mu', \mu'' \quad \text{with} \quad \mu' \neq \mu''\ : \\ &\Big|\mathrm{Tr}\,\big[P_{\mu''}(U^n P_{\mu_0} U^{\dagger\, n}) P_{\mu'} (U^n P_{\mu_0} U^{\dagger\, n}) P_{\mu''}\big]\,\Big| \leq d \cdot \epsilon\,. \end{aligned} \tag{5.66}$$

In the second part of the proof we then conclude that this proposition necessarily implies the commutativity condition (5.64) of the theorem.

The first part of the proof will be accomplished by contradiction, i.e. we will assume that the proposition (5.66) is not satisfied, and then show that this assumption contradicts the approximate decoherence condition (5.63) of the theorem. The way we proceed is very similar to the proof of Theorem 4 in Sec. 5.4.

Assume condition (5.66) is not satisfied. This means there exist an integer $\tilde{n} \in \mathbb{N}$, and partition elements P_{μ_0}, $P_{\mu'}$, $P_{\mu''} \in \{P_\mu\}$, with $\mu' \neq \mu''$, such that

$$\Big|\mathrm{Tr}\,\big[P_{\mu''}(U^{\tilde{n}} P_{\mu_0} U^{\dagger\, \tilde{n}}) P_{\mu'} (U^{\tilde{n}} P_{\mu_0}\, U^{\dagger\, \tilde{n}}) P_{\mu''}\big]\,\Big| = c > d\,\epsilon\,. \tag{5.67}$$

This, as we will see, is in contradiction to the approximate decoherence condition (5.63) of the theorem. Using the trivial relation (5.65) the approximate decoherence condition (5.63) can be rewritten in the form

$$\forall\, \rho \in \mathcal{S}_{\{P_\mu\}}\ \forall\, k \in \mathbb{N}\ \ \forall\, h_{\boldsymbol{\alpha}}, h_{\boldsymbol{\beta}} \in \mathcal{K}[\{P_\mu\}\,;\,k\,] \ \text{ with } \ h_{\boldsymbol{\alpha}} \neq h_{\boldsymbol{\beta}}\ :$$

$$|\mathcal{D}_{U,\rho}\,[h_{\boldsymbol{\alpha}}, h_{\boldsymbol{\beta}}]| < \epsilon\, \frac{\Big(\mathcal{D}_{U,\rho}\,[h_{\boldsymbol{\alpha}}, h_{\boldsymbol{\alpha}}] + \mathcal{D}_{U,\rho}\,[h_{\boldsymbol{\beta}}, h_{\boldsymbol{\beta}}]\Big)}{2|\mathcal{K}[\{P_\mu\}\,;\,k\,]\,|}\ . \tag{5.68}$$

Since the length k of the histories is arbitrary, we may choose $k = q\tilde{n}$ with *arbitrary* $q \in \mathbb{N}$. Note that $\tilde{n}$ is that integer for which we assumed to have a violation of the proposition (5.66), see Eq. (5.67). Please also note that q can still be chosen arbitrarily. The two histories $h_{\boldsymbol{\alpha}}$ and $h_{\boldsymbol{\beta}}$ in (5.68) are *arbitrary* but *different* ! They are labeled by the multi-indices $\boldsymbol{\alpha} = \alpha_1 \ldots \alpha_{q\tilde{n}}$ and $\boldsymbol{\beta} = \beta_1 \ldots \beta_{q\tilde{n}}$ with $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$. In order to derive the aimed contradiction let us *choose* $h_{\boldsymbol{\alpha}}$ and $h_{\boldsymbol{\beta}}$ such that $\alpha_{\tilde{n}} \neq \beta_{\tilde{n}}$. Then $h_{\boldsymbol{\alpha}} \neq h_{\boldsymbol{\beta}}$ is guaranteed, irrespective of the values of the other indices within $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. In particular, the sub-multi-indices

$$\begin{aligned} \tilde{\boldsymbol{\alpha}} &:= \alpha_1 \ldots \alpha_{\tilde{n}-1}\alpha_{\tilde{n}+1} \ldots \alpha_{q\tilde{n}-1}\ , \\ \tilde{\boldsymbol{\beta}} &:= \beta_1 \ldots \beta_{\tilde{n}-1}\beta_{\tilde{n}+1} \ldots \beta_{q\tilde{n}-1}\ . \end{aligned} \tag{5.69}$$

can now be chosen arbitrarily and *independently from one another*. Note that $\alpha_{\tilde{n}}$ and $\beta_{\tilde{n}}$ are not yet fixed! They, too, can still be chosen arbitrarily, but with the restriction $\alpha_{\tilde{n}} \neq \beta_{\tilde{n}}$.

By summing over $\tilde{\boldsymbol{\alpha}}$ and $\tilde{\boldsymbol{\beta}}$ (which corresponds to coarse-graining) *before taking the modulus*, and using $\sum_\mu P_\mu = \mathbb{1}_{\mathcal{H}}$, we obtain on the one hand:

$$\Bigg|\sum_{\tilde{\boldsymbol{\alpha}},\tilde{\boldsymbol{\beta}}} \mathcal{D}_{U,\rho}\,[h_{\boldsymbol{\alpha}}, h_{\boldsymbol{\beta}}]\Bigg| = \Big|\mathrm{Tr}\,\Big[P_{\alpha_{q\tilde{n}}}(U^{q-1})^{\tilde{n}} P_{\alpha_{\tilde{n}}} U^{\tilde{n}}\,\rho\, U^{\dagger\,\tilde{n}} P_{\beta_{\tilde{n}}} (U^{\dagger\,q-1})^{\tilde{n}} P_{\beta_{q\tilde{n}}}\Big]\,\Big|\ . \tag{5.70}$$

On the other hand it follows from (5.68) that:

$$
\begin{aligned}
\Big| \sum_{\tilde{\boldsymbol{\alpha}},\tilde{\boldsymbol{\beta}}} \mathcal{D}_{U,\rho}\,[h_{\boldsymbol{\alpha}}, h_{\boldsymbol{\beta}}] \Big| \;&\leq\; \sum_{\tilde{\boldsymbol{\alpha}},\tilde{\boldsymbol{\beta}}} |\mathcal{D}_{U,\rho}\,[h_{\boldsymbol{\alpha}}, h_{\boldsymbol{\beta}}]| \\
&<\; \sum_{\tilde{\boldsymbol{\alpha}},\tilde{\boldsymbol{\beta}}} \epsilon \, \frac{\Big(\mathcal{D}_{U,\rho}\,[h_{\boldsymbol{\alpha}}, h_{\boldsymbol{\alpha}}] + \mathcal{D}_{U,\rho}\,[h_{\boldsymbol{\beta}}, h_{\boldsymbol{\beta}}]\Big)}{2|\mathcal{K}[\{P_\mu\}\,;\,k\,]\,|} \\
&\leq\; \epsilon \sum_{\boldsymbol{\alpha},\boldsymbol{\beta}} \frac{\Big(\mathcal{D}_{U,\rho}\,[h_{\boldsymbol{\alpha}}, h_{\boldsymbol{\alpha}}] + \mathcal{D}_{U,\rho}\,[h_{\boldsymbol{\beta}}, h_{\boldsymbol{\beta}}]\Big)}{2|\mathcal{K}[\{P_\mu\}\,;\,k\,]\,|} \\
&=\; \frac{\epsilon}{2|\mathcal{K}[\{P_\mu\}\,;\,k\,]\,|} \Bigg(|\mathcal{K}[\{P_\mu\}\,;\,k\,]\,| \sum_{\boldsymbol{\alpha}} \mathcal{D}_{U,\rho}\,[h_{\boldsymbol{\alpha}}, h_{\boldsymbol{\alpha}}] + \\
&\qquad\qquad + \; |\mathcal{K}[\{P_\mu\}\,;\,k\,]\,| \sum_{\boldsymbol{\beta}} \mathcal{D}_{U,\rho}\,[h_{\boldsymbol{\beta}}, h_{\boldsymbol{\beta}}] \Bigg) \\
&=\; \epsilon\,,
\end{aligned}
\tag{5.71}
$$

where we used $\sum_{\boldsymbol{\alpha}} 1 = |\mathcal{K}[\{P_\mu\}\,;\,k\,]\,|$ and utilized the fact that $\sum_{\boldsymbol{\alpha}} \mathcal{D}_{U,\rho}\,[h_{\boldsymbol{\alpha}}, h_{\boldsymbol{\alpha}}] = 1$ (cf., e.g., Ref. [19]). Equations (5.70) and (5.71) together thus yield

$$
\Big| \mathrm{Tr}\left[P_{\alpha_{q\tilde{n}}} (U^{q-1})^{\tilde{n}} P_{\alpha_{\tilde{n}}} U^{\tilde{n}} \, \rho \, U^{\dagger\,\tilde{n}} P_{\beta_{\tilde{n}}} (U^{\dagger\,q-1})^{\tilde{n}} P_{\beta_{q\tilde{n}}} \right] \Big| < \epsilon \,. \tag{5.72}
$$

Being a necessary consequence of the approximate decoherence condition (5.68), this inequality must hold for all $q \in \mathbb{N}$, any $\rho \in \mathcal{S}_{\{P_\mu\}}$, and arbitrary $\alpha_{\tilde{n}}$, $\beta_{\tilde{n}}$, $\alpha_{q\tilde{n}}$, $\beta_{q\tilde{n}}$ with $\alpha_{\tilde{n}} \neq \beta_{\tilde{n}}$. In order to derive a contradiction, we choose $\alpha_{\tilde{n}} = \mu'$, $\beta_{\tilde{n}} = \mu''$, and $\alpha_{q\tilde{n}} = \beta_{q\tilde{n}} = \mu_0$. Since $\mu' \neq \mu''$ (see the above assumption (5.67)), this choice is possible. Furthermore we let our histories start with the initial state $\rho = P_{\mu_0}/\mathrm{Tr}\,[P_{\mu_0}]$. With this choice inequality (5.72) becomes:

$$
\Big| \mathrm{Tr}\left[P_{\mu_0} (U^{q-1})^{\tilde{n}} P_{\mu'} U^{\tilde{n}} P_{\mu_0} U^{\dagger\,\tilde{n}} P_{\mu''} (U^{\dagger\,q-1})^{\tilde{n}} P_{\mu_0} \right] \Big| < (\mathrm{Tr}\,[P_{\mu_0}])\,\epsilon \leq d\epsilon \,, \tag{5.73}
$$

where we used $\mathrm{Tr}\,[P_{\mu_0}] \leq d = \dim \mathcal{H}$. Note that q is still arbitrary, i.e., inequality (5.73) *must hold for any* $q \in \mathbb{N}$!

Now we are at the point where to make use of our *Lemma on uniform recurrence* from Section 5.2. According to the Lemma, for any given arbitrarily small $\varepsilon' > 0$ we can always find an integer $q \in \mathbb{N}$ such that $U^q = \mathbb{1}_{\mathcal{H}} + \hat{\mathcal{O}}(\varepsilon')$, where $\hat{\mathcal{O}}(\varepsilon')$ is some operator whose norm is of order ε': $\| \hat{\mathcal{O}}(\varepsilon') \| < \varepsilon'$. Please note that we mustn't confuse the two different (small) numbers ϵ and ε'. They have different meaning and are completely independent from one another! Using the submultiplicativity property of operator norms, we have

$$\| U^{-1}\hat{\mathcal{O}}(\varepsilon') \| \leq \| U^{-1} \| \times \| \hat{\mathcal{O}}(\varepsilon') \| = \| \hat{\mathcal{O}}(\varepsilon') \| \tag{5.74}$$

and hence $U^{q-1} = U^{-1} + \hat{\mathcal{O}}'(\varepsilon')$, where $\| \hat{\mathcal{O}}'(\varepsilon') \| < \varepsilon'$.

Choosing q such that $\| \; U^q - \mathbb{1}_{\mathcal{H}} \; \| < \varepsilon'$ for any given *arbitrarily small* $\varepsilon' > 0$, we thus get a situation where the expressions $(U^{q-1})^{\tilde{n}}$ and $(U^{\dagger\, q-1})^{\tilde{n}}$ in Eq. (5.73) can be replaced by $(U^\dagger + \hat{\mathcal{O}}'(\varepsilon'))^{\tilde{n}}$ and $(U + \hat{\mathcal{O}}'^{\dagger}(\varepsilon'))^{\tilde{n}}$, respectively. We now proceed basically in the same way as in the proof of Theorem 4. Using the convenient definition

$$A_{r_1,r_2,\dots,r_{\tilde{n}}} := \prod_{i=1}^{\tilde{n}} \left(U^{\dagger\, r_i} (\hat{\mathcal{O}}'(\varepsilon'))^{1-r_i} \right) \;, \tag{5.75}$$

where the operators inside the product are written out from left to right in the order of increasing index i, we have:

$$(U^\dagger + \hat{\mathcal{O}}'(\varepsilon'))^{\tilde{n}} = \sum_{r_1,\dots,r_{\tilde{n}} \in \{0,1\}} A_{r_1,\dots,r_{\tilde{n}}} \;. \tag{5.76}$$

This yields for the left hand side of inequality (5.73):

$$\begin{aligned}
&\left| \mathrm{Tr}\left[P_{\mu_0} (U^{q-1})^{\tilde{n}} P_{\mu'} U^{\tilde{n}} P_{\mu_0} U^{\dagger\,\tilde{n}} P_{\mu''} (U^{\dagger\, q-1})^{\tilde{n}} P_{\mu_0} \right] \right| = \\
&= \left| \mathrm{Tr}\left[P_{\mu_0} \left(\sum_{r_1,\dots,r_{\tilde{n}} \in \{0,1\}} A_{r_1,\dots,r_{\tilde{n}}} \right) P_{\mu'} U^{\tilde{n}} \, P_{\mu_0} \, U^{\dagger\,\tilde{n}} P_{\mu''} \left(\sum_{s_1,\dots,s_{\tilde{n}} \in \{0,1\}} A^\dagger_{s_1,\dots,s_{\tilde{n}}} \right) P_{\mu_0} \right] \right| \\
&= \left| \sum_{r_1,\dots,r_{\tilde{n}} \in \{0,1\}} \; \sum_{s_1,\dots,s_{\tilde{n}} \in \{0,1\}} \mathrm{Tr}\left[P_{\mu_0} A_{r_1,\dots,r_{\tilde{n}}} P_{\mu'} U^{\tilde{n}} \, P_{\mu_0} \, U^{\dagger\,\tilde{n}} P_{\mu''} A^\dagger_{s_1,\dots,s_{\tilde{n}}} P_{\mu_0} \right] \right| \\
&\geq \left| \mathrm{Tr}\left[P_{\mu_0} U^{\dagger\,\tilde{n}} P_{\mu'} U^{\tilde{n}} \, P_{\mu_0} \, U^{\dagger\,\tilde{n}} P_{\mu''} U^{\tilde{n}} P_{\mu_0} \right] \right| \\
&\quad - \left| \sum_{\substack{r_1,\dots,r_{\tilde{n}} \in \{0,1\} \\ r_1+\dots+r_{\tilde{n}} < \tilde{n}}} \; \sum_{\substack{s_1,\dots,s_{\tilde{n}} \in \{0,1\} \\ s_1+\dots+s_{\tilde{n}} < \tilde{n}}} \mathrm{Tr}\left[P_{\mu_0} A_{r_1,\dots,r_{\tilde{n}}} P_{\mu'} U^{\tilde{n}} \, P_{\mu_0} \, U^{\dagger\,\tilde{n}} P_{\mu''} A^\dagger_{s_1,\dots,s_{\tilde{n}}} P_{\mu_0} \right] \right| \;.
\end{aligned} \tag{5.77}$$

In the last step we used the well known triangle inequality $|x+y| \geq |x| - |y|$ (for $x, y \in \mathbb{C}$) and the fact that for $(r_1, \dots, r_{\tilde{n}}) = (1, 1, \dots, 1)$ we have $A_{1,1,\dots,1} = (U^\dagger)^{\tilde{n}}$

and $A^{\dagger}_{1,1,\dots,1} = U^{\tilde{n}}$, respectively. According to (5.73) the left hand side of this equation must be smaller than $d \cdot \epsilon$. Hence we have:

$$
\begin{aligned}
&\left|\mathrm{Tr}\left[P_{\mu_0} U^{\dagger\,\tilde{n}} P_{\mu'} U^{\tilde{n}}\, P_{\mu_0}\, U^{\dagger\,\tilde{n}} P_{\mu''} U^{\tilde{n}} P_{\mu_0}\right]\right| < \\
&< \; d\epsilon + \left| \sum_{\substack{r_1,\dots,r_{\tilde{n}}\in\{0,1\}\\ r_1+\dots+r_{\tilde{n}}<\tilde{n}}} \; \sum_{\substack{s_1,\dots,s_{\tilde{n}}\in\{0,1\}\\ s_1+\dots+s_{\tilde{n}}<\tilde{n}}} \mathrm{Tr}\left[P_{\mu_0} A_{r_1,\dots,r_{\tilde{n}}} P_{\mu'} U^{\tilde{n}}\, P_{\mu_0}\, U^{\dagger\,\tilde{n}} P_{\mu''} A^{\dagger}_{s_1,\dots,s_{\tilde{n}}} P_{\mu_0}\right]\right| \\
&\leq \; d\epsilon + \sum_{\substack{r_1,\dots,r_{\tilde{n}}\in\{0,1\}\\ r_1+\dots+r_{\tilde{n}}<\tilde{n}}} \; \sum_{\substack{s_1,\dots,s_{\tilde{n}}\in\{0,1\}\\ s_1+\dots+s_{\tilde{n}}<\tilde{n}}} \left|\mathrm{Tr}\left[P_{\mu_0} A_{r_1,\dots,r_{\tilde{n}}} P_{\mu'} U^{\tilde{n}}\, P_{\mu_0}\, U^{\dagger\,\tilde{n}} P_{\mu''} A^{\dagger}_{s_1,\dots,s_{\tilde{n}}} P_{\mu_0}\right]\right| ,
\end{aligned}
\tag{5.78}
$$

where we now used the triangle inequality $|x+y| \leq |x| + |y|$ (for $x, y \in \mathbb{C}$). Using the cyclic permutation-invariance property of the trace we rewrite the last equation as

$$
\begin{aligned}
&\left|\mathrm{Tr}\,\left[P_{\mu''}(U^{\tilde{n}} P_{\mu_0} U^{\dagger\,\tilde{n}}) P_{\mu'} (U^{\tilde{n}} P_{\mu_0} U^{\dagger\,\tilde{n}}) P_{\mu''}\right]\right| \\
&< \; d\epsilon + \sum_{\substack{r_1,\dots,r_{\tilde{n}}\in\{0,1\}\\ r_1+\dots+r_{\tilde{n}}<\tilde{n}}} \; \sum_{\substack{s_1,\dots,s_{\tilde{n}}\in\{0,1\}\\ s_1+\dots+s_{\tilde{n}}<\tilde{n}}} \left|\mathrm{Tr}\left[A^{\dagger}_{s_1,\dots,s_{\tilde{n}}} P_{\mu_0} A_{r_1,\dots,r_{\tilde{n}}} P_{\mu'} U^{\tilde{n}} P_{\mu_0} U^{\dagger\,\tilde{n}} P_{\mu''}\right]\right| .
\end{aligned}
\tag{5.79}
$$

We now employ the inequality $|\,\mathrm{Tr}[BT]\,| \;\leq \|\, B \,\|\;\; \mathrm{Tr}\sqrt{T^{\dagger}T}$ for bounded operators $B : \mathcal{H} \to \mathcal{H}$ and operators $T : \mathcal{H} \to \mathcal{H}$ with finite trace norm $\|\, T\,\|_1 := \mathrm{Tr}\sqrt{T^{\dagger}T}$, see Ref. [57]. With the definitions

$$
B^{s_1,\dots,s_{\tilde{n}}}_{r_1,\dots,r_{\tilde{n}}} \;:=\; A^{\dagger}_{s_1,\dots,s_{\tilde{n}}} P_{\mu_0} A_{r_1,\dots,r_{\tilde{n}}} \;, \tag{5.80}
$$

$$
T \;:=\; P_{\mu'} U^{\tilde{n}} P_{\mu_0} U^{\dagger\,\tilde{n}} P_{\mu''} \;\;, \tag{5.81}
$$

we thus can deduce from Eq (5.79):

$$
\begin{aligned}
&\left|\mathrm{Tr}\,\left[P_{\mu''}(U^{\tilde{n}} P_{\mu_0} U^{\dagger\,\tilde{n}}) P_{\mu'} (U^{\tilde{n}} P_{\mu_0} U^{\dagger\,\tilde{n}}) P_{\mu''}\right]\right| < \\
&< d\epsilon + \sum_{\substack{r_1,\dots,r_{\tilde{n}}\in\{0,1\}\\ r_1+\dots+r_{\tilde{n}}<\tilde{n}}} \; \sum_{\substack{s_1,\dots,s_{\tilde{n}}\in\{0,1\}\\ s_1+\dots+s_{\tilde{n}}<\tilde{n}}} \|\, B^{s_1,\dots,s_{\tilde{n}}}_{r_1,\dots,r_{\tilde{n}}} \,\|\; \mathrm{Tr}\sqrt{T^{\dagger}T}\;.
\end{aligned}
\tag{5.82}
$$

In the very same fashion as in the proof of Theorem 4 (cf. Eq. (5.51)) we can show that $\|\, B^{s_1,\dots,s_{\tilde{n}}}_{r_1,\dots,r_{\tilde{n}}} \,\| < \varepsilon'$, provided that $s_1 + \dots + s_{\tilde{n}} < \tilde{n}$ and $r_1 + \dots + r_{\tilde{n}} < \tilde{n}$. Since the case where all $s_1, \ldots, s_{\tilde{n}}$ and all $r_1, \ldots, r_{\tilde{n}}$ are equal to 1 is excluded from the sum on the right-hand side of Eq. (5.82), we may thus conclude from Eq. (5.82), using the definition $M := \mathrm{Tr}\sqrt{T^\dagger T}$:

$$\left|\mathrm{Tr}\left[P_{\mu''}(U^{\tilde{n}}P_{\mu_0}U^{\dagger\,\tilde{n}})P_{\mu'}(U^{\tilde{n}}P_{\mu_0}U^{\dagger\,\tilde{n}})P_{\mu''}\right]\right| < d\,\epsilon + 2^{2\tilde{n}}M\varepsilon' \ . \tag{5.83}$$

Since the (positive) number c in the assumption (5.67) as well as the (positive) numbers $\tilde{n}$ and M are *fixed constants*, we can always arrange $d\,\epsilon + 2^{2\tilde{n}}M\varepsilon' < c$, simply by choosing a sufficiently small $\varepsilon' > 0$. This, together with Eq. (5.83), obviously contradicts our assumption (5.67) and thus proves our proposition (5.66).

We now turn to the second part of the proof. The task is to show that the proposition (5.66) necessarily implies the commutativity condition (5.64) of the theorem. Condition (5.66) is equivalent to

$$\forall\, n \in \mathbb{N} \ \ \forall\, \mu_0, \mu', \mu'' \quad \text{with} \quad \mu' \neq \mu'' : \ \| \ P_{\mu'}(U^n P_{\mu_0} U^{\dagger\,n})P_{\mu''} \ \|_2 \leq \sqrt{d\,\epsilon}\,, \tag{5.84}$$

where $\|\,A\,\|_2 := \sqrt{\mathrm{Tr}[A^\dagger A]}$ denotes the Hilbert-Schmidt operator norm for any operator A on $\mathcal{H}$. It then follows, *for all* $n \in \mathbb{N}$ and *for all* μ_0, μ'', that

$$\begin{aligned}
&\| \ [(U^n P_{\mu_0} U^{\dagger\,n}), P_{\mu''}] \ \|_2 = \\
&\quad = \ \| \,(U^n P_{\mu_0} U^{\dagger\,n})P_{\mu''} - P_{\mu''}(U^n P_{\mu_0} U^{\dagger\,n})\,\|_2 \\
&\quad = \ \| \,(\sum_{\mu'} P_{\mu'}\,)(U^n P_{\mu_0} U^{\dagger\,n})P_{\mu''} - P_{\mu''}(U^n P_{\mu_0} U^{\dagger\,n})(\sum_{\mu'} P_{\mu'}\,)\,\|_2 \\
&\quad \leq \ \sum_{\substack{\mu' \\ \mu' \neq \mu''}} \| \ P_{\mu'}(U^n P_{\mu_0} U^{\dagger\,n})P_{\mu''} - P_{\mu''}(U^n P_{\mu_0} U^{\dagger\,n})P_{\mu'} \ \|_2 \\
&\quad \leq \ \sum_{\substack{\mu' \\ \mu' \neq \mu''}} \{ \, \| \ P_{\mu'}(U^n P_{\mu_0} U^{\dagger\,n})P_{\mu''} \ \|_2 + \| \ P_{\mu''}(U^n P_{\mu_0} U^{\dagger\,n})P_{\mu'} \ \|_2 \} \\
&\quad \leq \ 2(\,\sharp\,\mu'\,)\sqrt{d\epsilon} \leq 2d\sqrt{d\epsilon} = 2d^{\frac{3}{2}}\sqrt{\epsilon}\,,
\end{aligned} \tag{5.85}$$

i.e. $\| \ [(U^n P_{\mu_0} U^{\dagger\,n}), P_{\mu''}] \ \|_2 \leq 2d^{\frac{3}{2}}\sqrt{\epsilon}$ *for all* $n \in \mathbb{N}$ and *for all* μ_0, μ''. This is the commutativity condition (5.64) of the theorem. □

Corollary:

Let a projective partition $\{P_\mu\}$ of a finite dimensional Hilbert space $\mathcal{H}$, a unitary map U on $\mathcal{H}$, and a small $\epsilon > 0$ be given. Then

$$\forall\, \rho \in \mathcal{S}^{\mathrm{cl}}_{\{P_\mu\}}\ \forall\, k \in \mathbb{N}\ \ \forall\, h_{\boldsymbol{\alpha}}, h_{\boldsymbol{\beta}} \in \mathcal{K}[\{P_\mu\}\,;\,k\,] \ \text{ with } \ h_{\boldsymbol{\alpha}} \neq h_{\boldsymbol{\beta}} \ :$$

$$|\mathcal{D}_{U,\rho}\,[h_{\boldsymbol{\alpha}}, h_{\boldsymbol{\beta}}]| < \epsilon\, \frac{\Big(\mathcal{D}_{U,\rho}\,[h_{\boldsymbol{\alpha}}, h_{\boldsymbol{\alpha}}]\mathcal{D}_{U,\rho}\,[h_{\boldsymbol{\beta}}, h_{\boldsymbol{\beta}}]\Big)^{\frac{1}{2}}}{|\mathcal{K}[\{P_\mu\}\,;\,k\,]\,|} \tag{5.86}$$

only if the following necessary condition is fulfilled:

$$\forall\, P_{\mu'}, P_{\mu''} \in \{P_\mu\} \ : \ \|\ [U P_{\mu'} U^\dagger, P_{\mu''}]\ \|_2 \leq 2 d^{\frac{3}{2}} \sqrt{\epsilon}\,, \tag{5.87}$$

where $d = \dim \mathcal{H}$ and $\|\cdot\|_2$ denotes the Hilbert-Schmidt operator norm.

Proof:

The corollary follows trivially from Theorem 5, because of $\mathcal{S}_{\{P_\mu\}} \subset \mathcal{S}^{\mathrm{cl}}_{\{P_\mu\}}$. $\square$

5.5.3 Discussion

With Theorem 5 we have made a first step towards generalization of our previous results, obtained in Sec. 5.3 and Sec. 5.4, to the case of approximate decoherence. Theorem 5 provides a generalization of the implication (a)⇒(b) of Theorem 4 — which was the only difficult part in the proof of Theorem 4 — to approximate decoherence of histories. Whether a meaningful generalization of *all* the implications that Theorem 4 involves can be obtained, is to be investigated in future research.

The immediate relevance of the result of this section is clear: with the corollary of Theorem 5 we have obtained a further generalization with regard to our search of single-iteration necessary decoherence conditions. Namely, the above corollary of Theorem 5 provides a *single-iteration necessary condition* for approximate decoherence of sets of arbitrarily coarse-grained histories of arbitrary length starting from any classical initial state. This condition *generalizes* the single-iteration necessary criterion that was stated in Theorem 3 to the more general case of approximate decoherence — "approximate decoherence" in the sense as defined by our condition (5.62).

Chapter 6

Conclusions and outlook

In this thesis we have employed the decoherent histories formalism of quantum mechanics to provide a framework for studying quantum maps which in many respects resembles the method of classical symbolic dynamics. The latter has proven a very powerful tool in studying information-theoretic features of complex classical dynamical systems. Our investigations here were motivated by this powerful method and aimed at developing our decoherent histories framework to a similarly useful method for studying information-theoretic signatures of complex quantum dynamical systems. The research presented in this thesis is therefore to be viewed as a contribution to the challenging programme of developing a general theory of "quantum symbolic dynamics".

In one part of this thesis we have successfully applied our quantum histories framework to the study of classical predictability of the coarse-grained evolution of the quantum baker's map, which is a prototypical quantum map invented for the theoretical investigation of quantum chaos. Here we have analyzed the issue of how the predictability of the evolution is affected by the character of coarse graining chosen from a family of different coarse-grained descriptions. Our family of coarse-grainings is parameterized by the *number of scales* at which information is discarded in the symbolic representation of the quantum baker's map and by the *extent of coarse-graining* at every such scale. All members of the considered family have been shown to lead, in the classical limit $\hbar \to 0$, to approximately decoherent sets of histories, with probability distributions being peaked over histories displaying regularities in time in accordance to the shift property of the classical baker's map. With regard to the issue of predictability of the evolution we have found that the short-time entropy production is strongly affected by the number of scales at which information is lost rather than the extent of coarse-graining on any particular scale, whereas the duration of the short-time regime is determined by the extent of coarse-graining on that scales. Multi-scale coarse-grainings exhibit significantly more unpredictability than 1-scale coarse-grainings with the same degree of prior knowledge.

The research leading to these results has opened a number of interesting questions. The family of coarse-grained descriptions which has been examined here, consists of a particular type of coarse-grainings. All members are defined by projective partitions (of the Hilbert space) whose elements are partial sums over one-dimensional projectors corresponding to basis states on which the symbolic representation of the quantum baker's map is defined (cf. Eqs. (2.37), (2.38), (2.39)). In other words, we have chosen coarse-grainings which are very natural with respect to the symbolic representation of the map. It would, however, also be interesting to investigate a more general class of coarse-grainings with regard to the same issues — coarse-grained descriptions defined by projective partitions whose elements are not given by partial sums over one-dimensional projectors corresponding to the "natural" basis states on which the symbolic dynamics of the map is defined. One possibility would be to start with arbitrary *superpositions* of such "natural" basis states, and define coarse-grainings by projective partitions whose partition elements are given by partial sums over one-dimensional projectors corresponding to these superposition states. Various questions would then be worth studying. Can we establish, in the classical limit $\hbar \to 0$, decoherence for these more general types of coarse-grained descriptions? How would *decoherence* depend on the character of more general coarse-grainings? Do coarse-grained descriptions exist which, in the limit $\hbar \to 0$, lead to decoherent sets of histories on the one hand, but on the other hand to probability distributions that are peaked over histories displaying regularities in time which are not related to the classical shift property? In other words, are there different classical limits for the same quantum map possible, displaying substantially different classical regularities in time, if using substantially different coarse-grained descriptions for the histories? Or does the non-linear nature of the dynamics involve a stronger constraint on the character of coarse-graining than in the case of a linear dynamics? An analysis of all the above questions should be feasible if tackling them by means of numerical methods.

The main part of the thesis involves research on *decoherence properties* of closed quantum dynamical systems given in terms of unitary quantum maps. We have investigated decoherence properties of arbitrarily long histories constructed from a fixed projective partition of a finite dimensional Hilbert space. Again, the use of a *fixed projective partition* for all times and the interest in *arbitrarily long* histories, were *motivated* by the analogy with the method of classical symbolic dynamics, which is based on a fixed partitioning of the phase space and involves the study of infinitely long symbolic sequences. To be able to introduce information-theoretic quantities for the characterization of dynamical features — like in the theory of classical symbolic dynamics — we must be able to assign probability distributions to sets of quantum histories. Decoherence of histories is a necessary requirement for this to be possible. Only decoherent sets of histories have predictive content. Checking decoherence of very long histories using the standard way by means of the decoherence functional is normally a very difficult task, especially when the system dynamics is complex and therefore difficult to simulate, as is normally the case for chaotic quantum maps. In this thesis we have provided simpler *necessary decoherence conditions* for sets of histories within our framework — necessary criteria for decoherence that employ only a *single iteration* of a given quantum map. Furthermore, we have found a surprising result with regard to the fundamental issue of how decoherence of histories is affected by the choice of the initial state. Within the considered framework we have shown that, if decoherence is established for arbitrary history lengths and all initial states from the discrete set $\mathcal{S}_{\{P_\mu\}}$, which is the smallest natural set of states that one would normally associate with a given partition $\{P_\mu\}$, then any set of histories constructed from $\{P_\mu\}$ is decoherent for all possible initial quantum states. This result concerns the interesting question of the survival of initial classicality and decoherence, and as such it is of relevance to the decoherence program (see [46] and references therein), especially with regard to the issue of an emerging classicality from the initial quantum state of the Universe in quantum cosmology [18, 47], additionally to being a useful mathematical result within the decoherent histories framework.

A generalization of the above results to *approximate decoherence* of histories has also been addressed in the thesis. In particular, a generalization of the single-iteration necessary decoherence condition to the case of approximately decoherent sets of histories has been obtained. As for an analogous result regarding the dependence of decoherence on the initial state, more work has to be done.

Another open question still remains to be solved, namely, whether there exists a simple *single-iteration* decoherence criterion which is *both necessary and sufficient* at the same time. The trivially sufficient condition that has been discussed in the thesis has been shown to be too strong to be also a necessary condition for decoherence. The necessary condition which has been derived, on the other hand, is too weak to be also a sufficient condition. The goal of finding a single-iteration decoherence condition which is both necessary and sufficient for decoherence of arbitrarily coarse-grained histories has not yet been achieved.

A *further generalization* of our framework would be worth studying within the context of approximate decoherence. Instead of considering sets of histories that are constructed from projective partitions we could be interested in considering the so-called *"effect histories"* [32, 33], which are constructed from *"effect partitions"* [1] of the Hilbert space. Our motivation for this generalization is provided by the *conjecture* [59] that *approximate decoherence* of "sharp histories" can be viewed as *exact decoherence* of "unsharp histories". By "sharp histories" we mean histories constructed from a projective partition, whereas by "unsharp histories" we mean histories that are constructed from an effect partition. The use of the terms "sharp" and "unsharp" is motivated by the notion of "sharp observables" and "unsharp observables" used, e.g.,

[1] An *"effect partition"* of a Hilbert space $\mathcal{H}$ is defined to be a set of *positive operators* $\{E_\mu\}$ on $\mathcal{H}$ such that $\sum_\mu E_\mu = \mathbb{1}_\mathcal{H}$, i.e., a set $\{E_\mu : \mathcal{H} \to \mathcal{H} \,|\, \forall\mu :\ 0 \leq E_\mu \leq \mathbb{1}_\mathcal{H} \text{ and } \sum_\mu E_\mu = \mathbb{1}_\mathcal{H}\}$.

by Paul Busch within the context of operational quantum physics and the generalized measurement theory (see [60, 61, 62] and references therein). In [62] a *projection valued measure* (PVM) is associated with a *"sharp observable"*, whereas a *positive operator valued measure* (POVM) can be associated with an *"unsharp observable"* (if it is not a PVM). Since a projective partition $\{P_\mu\}$ of the Hilbert space is induced by some PVM and an effect partition $\{E_\mu\}$ by some POVM, we may associate some "sharp observable" with $\{P_\mu\}$ and some "unsharp observable" with $\{E_\mu\}$, which motivates calling the corresponding histories "sharp histories" and "unsharp histories". Let us briefly state the above conjecture mathematically more precisely. Again, we use our framework of *fixed* partitions, i.e., sets of histories are to be constructed from a fixed given projective partition $\{P_\mu\}$ in the case of sharp histories and a fixed given effect partition $\{E_\mu\}$ in the case of unsharp histories. Let us moreover denote by $\mathcal{K}[\{P_\mu\}\,;\,k\,] \equiv \{P_\mu\}^k$ and $\mathcal{K}[\{E_\mu\}\,;\,k\,] \equiv \{E_\mu\}^k$, respectively, the corresponding exhaustive sets of mutually exclusive histories of length k. One more definition is needed. For a given unitary map $U : \mathcal{H} \to \mathcal{H}$, the class operators for effect histories $h_{\boldsymbol{\alpha}} \in \mathcal{K}[\{E_\mu\}\,;\,k\,]$ are given by (cf. [32, 33]):

$$\begin{aligned} C[h_{\boldsymbol{\alpha}}] &:= \left(U^{\dagger\,k}\sqrt{E_{\alpha_k}}U^k\right)\left(U^{\dagger\,k-1}\sqrt{E_{\alpha_{k-1}}}U^{k-1}\right)\dots\left(U^{\dagger}\sqrt{E_{\alpha_1}}U\right) \\ &= U^{\dagger\,k}\sqrt{E_{\alpha_k}}U\sqrt{E_{\alpha_{k-1}}}U\dots\sqrt{E_{\alpha_2}}U\sqrt{E_{\alpha_1}}U\ , \end{aligned} \tag{6.1}$$

where the operators $\sqrt{E_{\alpha_j}}$ are the Kraus operators corresponding to the effects E_{α_j}. The decoherence functional for effect histories is defined in the same way as for sharp histories, namely by Eq. (3.4). Furthermore, by "exact decoherence" it is meant that the off-diagonal elements of the decoherence functional vanish, whereas by "approximate decoherence" we shall here mean that the off-diagonal elements of the decoherence functional are all smaller than some $\epsilon > 0$.

To state the above conjecture mathematically, let $\mathcal{K}[\{P_\mu\}\,;\,k\,]$ be some approximately decoherent set of sharp histories, generated by some given projective partition $\{P_\mu\}$ of the Hilbert space. The off-diagonal elements of the decoherence functional are assumed all to be smaller than some $\epsilon > 0$, i.e., $\forall h_{\boldsymbol{\alpha}}, h_{\boldsymbol{\beta}} \in \mathcal{K}[\{P_\mu\}\,;\,k\,]$ with $h_{\boldsymbol{\alpha}} \neq h_{\boldsymbol{\beta}}$ we have $|\,\mathcal{D}_{U,\rho}\,[h_{\boldsymbol{\alpha}}, h_{\boldsymbol{\beta}}]\,| < \epsilon$. ρ is some initial state from which the histories start. Then, according to our conjecture, there should exist an effect partition $\{E_\mu\}$

of the Hilbert space, such that $\forall E_{\mu'} \in \{E_\mu\}\ \exists P_{\mu''} \in \{P_\mu\}\ : \mathrm{dist}(E_{\mu'}, P_{\mu''}) < \mathcal{O}(\epsilon)$ and $\mathcal{D}_{U,\rho}\,[\tilde{h}_{\boldsymbol{\alpha}}, \tilde{h}_{\boldsymbol{\beta}}] \propto \delta_{\boldsymbol{\alpha\beta}} \equiv \prod_{j=1}^{k} \delta_{\alpha_j \beta_j}$ for all $\tilde{h}_{\boldsymbol{\alpha}}, \tilde{h}_{\boldsymbol{\beta}} \in \mathcal{K}[\{E_\mu\}\,;\,k\,]$, where $\mathcal{K}[\{E_\mu\}\,;\,k\,]$ is the "new" set of *unsharp histories* generated by the effect partition $\{E_\mu\}$. Whether this *conjecture* proves to be true is an *open question* to be investigated in future research.

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