1196 A MODIFIED CHEBYSHEV POLYNOMIAL OF DEGREE n TANGENT TO THE UNIT CIRCLE AT $n-1$ POINTS.

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In College Mathematics Journal: `https://maa.tandfonline.com/doi/pdf/10.1080/07468342.2022.2026088`

Such polynomials exist. Shifting n, we shall prove that for each $n \in \mathbf{N}$ with $n \geq 3$ there exists a polynomial P_n of degree n with coefficients in the integers such that the graph of $P_n(x)$ is tangent to the unit circle at exactly $n-1$ points in the open interval $(-1,1)$. For $n=2$ we may simply take $P_2(x)=1$, which is tangent to the unit circle at 0 and has degree 0.

To define the P_n for $n \geq 3$, we need the Chebyshev polynomials of the second kind. Recall that, in the usual notation, U_m is the unique polynomial with real coefficients of degree m such that $(\sin\theta)U_m(\cos\theta) = \sin(m+1)\theta$. For instance $U_0(x)=1$, $U_1(x)=2x$, and since $\sin 3\theta = -\sin^3\theta + 3\sin\theta\cos^2\theta = \sin\theta(-\sin^2\theta+3\cos^2\theta) = \sin\theta(-1+4\cos^2\theta)$ we have $U_2(x) = 4x^2-1$. In fact each U_n has integer coefficients. For each $n \in \mathbf{N}$ with $n \geq 4$, define

$$P_n(x) = x^2 U_{n-2}(x) - 2xU_{n-3} + U_{n-4}.$$

As shown in [1, Theorem 5], the defining property of U_m and the relation $2\cos\theta\sin r\theta = \sin(r+1)\theta + \sin(r-1)\theta$ imply that if $n \geq 4$ then

$$\begin{aligned}
&(\sin\theta)P_n(\cos\theta)\\
&= (\cos^2\theta\sin\theta)U_{n-2}(\cos\theta) - 2(\cos\theta\sin\theta)U_{n-3}(\cos\theta) + (\sin\theta)U_{n-4}(\cos\theta)\\
&= \cos^2\theta\sin(n-1)\theta - 2\cos\theta\sin(n-2)\theta + \sin(n-3)\theta\\
&= (1-\sin^2\theta)\sin(n-1)\theta - \sin(n-1)\theta - \sin(n-3)\theta + \sin(n-3)\theta\\
&= -\sin^2\theta\sin(n-1)\theta.
\end{aligned}$$

Hence $P_n(\cos\theta) = -\sin\theta\sin(n-1)\theta$ for each such n. Setting $P_3(x) = 2x^3-2x$ we have $P_3(\cos\theta) = 2\cos^3\theta - 2\cos\theta = 2(\cos^2\theta-1)\cos\theta = -2\sin^2\theta\cos\theta = -\sin\theta\sin 2\theta$. Therefore

$$P_n(\cos\theta) = -\sin\theta\sin(n-1)\theta \quad \text{if } n \geq 3. \tag{$\star$}$$

Since each U_m has integer coefficients, so does each P_n.

Observe that, by $(\star)$,

$$(\cos\theta)^2 + P_n(\cos\theta)^2 = \cos^2\theta + \sin^2\theta\sin^2(n-1)\theta \leq \cos^2\theta + \sin^2\theta = 1.$$

Hence the graph of $P_n(x)$ for $-1 \leq x \leq 1$ lies inside the closed unit disc. Moreover, we have $(\cos\theta)^2 + P_n(\cos\theta)^2 = 1$ if and only if $\sin^2(n-1)\theta = 1$,

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so if and only if $\theta = \frac{(2k-1)\pi}{n-1}$ for some $k \in \mathbf{N}$. Thus if $x = \cos \frac{(2k-1)\pi}{n-1}$ and $x \in (-1, 1)$, the graph of $P_n(x)$ is tangent to the unit circle.

To get distinct values of $\cos\theta$ we may assume that $\theta \in [0, \pi]$. If $n = 2m$ is even then there are $2m-1$ distinct tangent points, obtained by taking $k = 1, \ldots, m-1, m, m+1, \ldots 2m-1$ to get x-coordinates

$$\cos \frac{\pi}{2m-1}, \ldots, \cos \frac{(2m-3)\pi}{2m-1}, \cos \frac{(2m-1)\pi}{2m-1} = -1,$$
$$- \cos \frac{2\pi}{2m-1}, \ldots, - \cos \frac{(2m-2)\pi}{2m-1}.$$

If $n = 2m+1$ is odd then there are $2m$ distinct tangent points, obtained by taking $k = 1, \ldots, m-1, m$ to get x coordinates

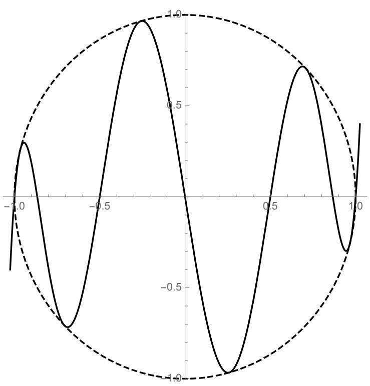
$$\cos \frac{\pi}{2m}, \ldots, \cos \frac{(2m-3)\pi}{2m}, \cos \frac{(2m-1)\pi}{2m}$$

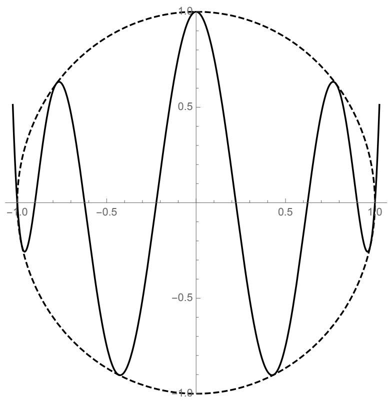
and then $k = m+1, \ldots, 2m$ to get x coordinates

$$- \cos \frac{\pi}{2m}, \ldots, - \cos \frac{(2m-3)\pi}{2m}, - \cos \frac{(2m-1)\pi}{2m}.$$

This completes the proof.

Remark. We remark that since $P_n(1) = P_n(\cos 0) = 0$ and $P_n(-1) = P_n(\cos \pi) = 0$ by $(\star)$, the graph of $P_n(x)$ meets the graph of the unit circle at $x = \pm 1$; of course since the unit circle has a vertical asymptote at these points, the graph is not tangent. Thus P_n is tangent to the unit circle at $n-1$ points and has two further intersection points. Since tangent points have multiplicity (at least) 2, this meets the bound in Bezout's Theorem, that the intersection multiplicity between the algebraic curves $y = P_n(x)$ and $x^2 + y^2 = 1$ of degrees n and 2 respectively is $2n$, and shows that each tangent point has degree exactly 2. This behaviour can be seen in the two graphs below showing $P_7(x)$ (left) and $P_8(x)$ (right).





Remark. Numerical calculations suggest that if n is even then there is an even polynomial $Q_n(x^2)$ whose graph is tangent to the unit circle at $n-1$ points, and such that two of the intersection points have triple multiplicity, again meeting the bound for Bezout's Theorem. For example,

$$Q_8(x^2) = 1 - \frac{(7x)^2}{2} + \frac{(7x)^4}{24} - \frac{(7x)^6}{864} + \frac{(7x)^8}{96768}.$$

References

[1] Milan Janjić, *On a class of polynomials with integer coefficients*, J. Integer Seq. **11** (2008), no. 5, Article 08.5.2, 9.