ON BIDIGARE’S PROOF OF SOLOMON’S THEOREM

1. INTRODUCTION

This note gives a version of Bidigare’s proof [1] of an important theorem of Solomon [3, Theorem 1] that emphasises certain combinatorial and algebraic features of the proof. There are no essentially new ideas.

To state Solomon’s theorem we need the following definitions. A composition of \( n \in \mathbb{N}_0 \) is a sequence \((p_1, \ldots, p_k)\) of natural numbers such that \( p_1 + \cdots + p_k = n \). To indicate that \( p \) is a composition of \( n \) we write \( p \models n \).

Let \( S_n \) denote the symmetric group of degree \( n \) and let \( \mathbb{Z}S_n \) be the integral group ring of \( S_n \). Given \( p \models n \), let \( \Xi_p \in \mathbb{Z}S_n \) be the sum of all minimal length coset representatives for the right cosets \( S_p \backslash S_n \). Equivalently, if \( p = (p_1, \ldots, p_k) \), then \( \Xi_p \) is the sum of all \( g \in S_n \) such that \( \text{Des}(g) \subseteq \{p_1, p_1 + p_2, \ldots, p_1 + p_2 + \cdots + p_{k-1}\} \), where \( \text{Des}(g) = \{x \in \{1, \ldots, n-1\} : xg < (x+1)g\} \) is the descent set of \( g \).

Given compositions \( p, q \) and \( r \) of \( n \in \mathbb{N}_0 \) we define \( m_{pq}^r \) to be the number of \( k \times \ell \) matrices \( A \) with entries in \( \mathbb{N}_0 \) such that

(i) the \( i \)th row sum is \( p_i \) for each \( i \),
(ii) the \( j \)th column sum is \( q_j \) for each \( j \),
(iii) the entries, read in the order \( A_{11}, \ldots, A_{1\ell}, \ldots, A_{k1}, \ldots, A_{k\ell} \) with any zero entries ignored, form the composition \( r \).

**Theorem 1** (Solomon). If \( p, q \) and \( r \) are compositions of \( n \in \mathbb{N}_0 \) then

\[
\Xi^p \Xi^q = \sum_{r \models n} m_{pq}^r \Xi^r.
\]

2. THE PROOF

Define a set composition of \( n \) to be a tuple \((P_1, \ldots, P_k)\) such that \( P_1 \cup \cdots \cup P_k = \{1, \ldots, n\} \) and the sets \( P_1, \ldots, P_n \) are disjoint and non-empty. If \( |P_i| = p_i \) for each \( i \) then we say that \((P_1, \ldots, P_k)\) has type \((p_1, \ldots, p_k)\).

Let \( \Pi_n \) be the set of all set compositions of \( n \). There is an action of \( S_n \) on \( \Pi_n \) defined by

\[(P_1, \ldots, P_k)g = (P_1g, \ldots, P_kg) \text{ for } g \in S_n.\]

We define an associative product \( \wedge : \Pi_n \times \Pi_n \to \Pi_n \) by

\[(P_1, \ldots, P_k) \wedge (Q_1, \ldots, Q_\ell) = (P_1 \cap Q_1, \ldots, P_1 \cap Q_\ell, \ldots, P_k \cap Q_1, \ldots, P_k \cap Q_\ell)^*\]

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where the $\ast$ indicates that any empty sets in the tuple should be deleted. (Thus in forming $P \wedge Q$ we loop through the sets in $Q$ faster than the sets in $P$, and reading $P \wedge Q$ in order, if $i < j$ then we see all the elements of $P_i$ before any of the elements of $P_j$.) We record some further basic properties below.

(1) $\wedge$ is idempotent, i.e. $P \wedge P = P$ for all $P \in \Pi_n$.

(2) $\{1, \ldots, n\}$ is the identity for $\wedge$.

(3) If $P$ has type $p$ and $Q$ has type $q$ then the type of $P \wedge Q$ is a common refinement of $p$ and $q$.

(4) If $P$ has type $(1^n)$ then $P \wedge Q = P$, for any $Q \in \Pi_n$.

(5) The product $\wedge$ is $S_n$-invariant. That is, if $g \in S_n$ and $P$, $Q \in \Pi_n$ then $(P \wedge Q)g = Pg \wedge Qg$.

Thanks to (1) and (2), $\Pi_n$ is an idempotent semigroup. Note that in (3) ‘refinement’ allows for some rearrangement of parts in the case of $Q$: for example $(\{1,2\}, \{3\}) \wedge (\{3\}, \{1,2\}) = (\{1,2\}, \{3\})$ has type $(2,1)$, and is the wedge product of set compositions of types $(2,1)$ and $(1,2)$.

The $\mathbb{Z}$-algebra $\mathbb{Z} \Pi_n$ is an associative unital algebra whose product is $S_n$-invariant. Here are some of its basic properties.

(A) $\mathbb{Z} \Pi_n$ is a right $\mathbb{Z} S_n$-module by linear extension of the action of $S_n$ on $\Pi_n$.

(B) Let $\Pi_{(1^n)}$ be the collection of set compositions of type $(1^n)$. Given $P = (\{a_1\}, \ldots, \{a_n\}) \in \Pi_{(1^n)}$, let $\overline{P} \in S_n$ be the permutation sending $i$ to $a_i$ for each $i$. The map $P \mapsto \overline{P}$ is then a linear isomorphism $\mathbb{Z} \Pi_{(1^n)} \to \mathbb{Z} S_n$ of $\mathbb{Z} S_n$-modules.

(C) By (3) above, $\mathbb{Z} \Pi_{(1^n)}$ is an ideal of $\mathbb{Z} \Pi_n$. Moreover, by (4), each $Q \in \Pi_n$ acts trivially on $\mathbb{Z} \Pi_{(1^n)}$ on the right.

(D) By (5), the fixed point space $(\mathbb{Z} \Pi_n)^{S_n}$ is a subalgebra of $\mathbb{Z} \Pi_n$. Given $q \models n$, let $X^q$ be the sum of all set compositions of type $q$. Then $\{X^q : q \models n\}$ is a basis of $(\mathbb{Z} \Pi_n)^{S_n}$. If $q$ has $\ell$ parts then $X^q$ is the orbit sum under the action of $S_n$ for the set composition

$$T^q = (\{1 \ldots q_1\}, \ldots, \{q_1 + \cdots + q_{\ell-1} + 1, \ldots, n\}).$$

Let $I = (\{1\}, \ldots, \{n\}) \in \Pi_n$. By (3) above $P \wedge I \in \Pi_{(1^n)}$ for each $P \in \Pi_n$.

The main step in Bidigare’s proof is the following theorem.

**Theorem 2.** The map $f \mapsto \overline{f}$ from $(\mathbb{Z} \Pi_n)^{S_n}$ to $\mathbb{Z} S_n$ defined by linear extension of $P \mapsto P \wedge I$ is a $\mathbb{Z}$-algebra homomorphism such that $X^p \wedge I = \Xi_p$.

The final claim concerning $\Xi_p$ is clear. The first part is a corollary of the following stronger proposition.

**Proposition 3.** If $f \in (\mathbb{Z} \Pi_n)^{S_n}$ and $x \in \mathbb{Z} \Pi_n$ then $\overline{f} \wedge I \overline{x} \wedge I = \overline{f} \wedge x \wedge I$. 

Proof. Let \( p \) be a composition with \( k \) parts. It suffices to prove the proposition when \( f = X^p \), the sum of all set compositions of type \( p \), and \( x = Q \), an arbitrary set composition.

Suppose that \( Q \) has type \( q \) where \( q \) has \( \ell \) parts. Let \( g = Q \wedge I \in S_n \); equivalently, \( g \) is the permutation of minimal length such that \( T^q g = Q \). We have

\[
X^p \wedge I Q \wedge I = \sum_P P \wedge I g
\]

where the sum is over all \( P \in \Pi_n \) of type \( p \). Fix such a \( P \). Set \( d_i = p_1 + \cdots + p_{i-1} \) for \( 1 \leq i < k \). Claim: \((P \wedge I)g = (P \wedge T_q)g \wedge I\). Proof of claim: The singleton sets in positions \( d_i + 1, \ldots, d_i + p_i \) on both sides are obtained by taking the entries of \( P \) in increasing order, and applying \( g \) to each. Hence

\[
X^p \wedge I Q \wedge I = \sum_P P \wedge I g
= \sum_P (P \wedge T_q)g \wedge I
= (X^p \wedge T_q)g \wedge I
= (X^p \wedge Q) \wedge I
\]

as required. \( \square \)

It follows from Theorem 2 that the span of the \( \Xi^p \) for \( p \vdash n \) is a subalgebra of \( \mathbb{Z}S_n \) isomorphic to \((\mathbb{Z}\Pi_n)^{S_n}\). To complete the proof of Theorem 1 we compute the structure constants for this algebra. The following definition will be helpful: say that \( T \in \Pi_n \) is increasing if whenever \( 1 \leq i < i' \leq \ell \) and \( x \in T_i, x' \in T_{i'} \), we have \( x < x' \). (Equivalently, \( T \) is increasing if and only if \( T = T^p \) for some \( p \vdash n \).)

**Proposition 4.** Let \( p, q \) and \( r \) be compositions of \( n \). Then the coefficient of \( X^r \) in \( X^p \wedge X^q \) is \( m_{pq}^r \).

Proof. It is equivalent to show that the coefficient of \( T^r \) in \( X^p \wedge X^q \) is \( m_{pq}^r \). If \( T^r = P \wedge Q \) where \( P \) and \( Q \) are set compositions then, since \( T^r \) is increasing, \( P \) must also be increasing. Therefore it suffices to show that if

\[
Q = \{ Q \in \Pi_n : T^p \wedge Q = T^r, \ Q \ has \ type \ q \}
\]

then \( |Q| = m_{pq}^r \). Suppose that \( p \) has \( k \) parts, \( q \) has \( \ell \) parts and that \( r \) has \( m \) parts. Given \( Q \in Q \) define \( M(Q) \) to be the \( k \times \ell \) matrix such that

\[
M(Q)_{ij} = |T^p_i \cap Q_j| \quad \text{for } 1 \leq i \leq k, 1 \leq j \leq \ell.
\]

The \( i \)th row sum of \( M(Q) \) is \( |T^p_i| = p_i \) and the \( j \)th column sum of \( M(Q) \) is \( |Q_j| = q_j \). Moreover, reading the non-zero entries of \( M(Q) \) in the order specified in (iii) gives the composition \( r \). Claim: Conversely, given a matrix \( M \) satisfying these conditions, there is a unique \( Q \) such that \( M(Q) = M \) and \( T^p \wedge Q = T^r \). Proof of claim: fix a row \( i \) and suppose inductively that
we have allocated the elements of $T_i^p$ up to and including $a$ to the sets $Q_1, \ldots, Q_{j-1}$. (For the base case $j = 1$, take $a = p_1 + \cdots + p_{i-1}$.) Then we must put $a + 1, \ldots, a + M_{ij}$ into the set $Q_j$ to have $T^p \land Q$ increasing. || □

References

