A LOWER BOUND FOR THE PARTITION FUNCTION
FROM CHEBYSHEV’S INEQUALITY APPLIED TO A
COIN FLIPPING MODEL FOR THE RANDOM PARTITION

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Abstract. We use a coin flipping model for the random partition and
Chebyshev’s inequality to prove the lower bound \(\lim \log p(n) \sqrt{n} \geq C\) for the
number of partitions \(p(n)\) of \(n\), where \(C\) is an explicit constant.

A partition of size \(n \in \mathbb{N}_0\) is a decreasing sequence of natural numbers
whose sum is \(n\). Let \(p(n)\) be the number of partitions of \(n\). For example,
\(p(5) = 7\) counts the partitions \((5)\), \((4, 1)\), \((3, 2)\), \((3, 1, 1)\), \((2, 2, 1)\), \((2, 1, 1, 1)\)
and \((1, 1, 1, 1, 1)\). In this note we use a model for the random partition to
prove that for all \(\varepsilon > 0\)

\[
(*) \quad \frac{\log p(n)}{\sqrt{n}} > \frac{\sqrt{8 \log 2}}{1 + \varepsilon} \quad \text{for all } n \text{ sufficiently large.}
\]

We end with an explicit bound that replaces \(\sqrt{8 \log 2}\) with the slightly
smaller constant \(\frac{8}{3} \log 2\). The proof of (*) is self-contained and intended
to be readable by anyone knowing the basics of probability theory.

The asymptotically correct result is \(\lim_{n \to \infty} \frac{\log p(n)}{\sqrt{n}} = 2\sqrt{\pi^2/6}\). The upper bound \(\log p(n) \leq 2\sqrt{\pi^2/6} \sqrt{n}\) is relatively easy to prove—see for instance
Theorem 15.7 in [5]—but getting a tight lower bound is much more chal-
lenging. A fairly lengthy proof using only real analysis was given by Erdős
in [2]. Our proof is motivated by the model for the random partition in [1, §4.3], and by the abacus notation for partitions (see [3, page 79]). The latter
was used in [4] to prove the uniform lower bound \(p(n) \geq e^{\sqrt{n}/14}\), and in [6]
to prove the upper bound \(\log p(n) \leq C(\varepsilon)n^{1/2+\varepsilon}\) for all \(\varepsilon > 0\). The novel
feature here is to combine these motivations to give a simple proof of (*).

The proof begins with a coin flipping model for the random partition. Using
linearity of expectation it is easy to show that a partition generated by \(m\) flips has expected size about \(m^2/8\). Critically, the standard deviation is of
order \(m^{3/2}\). By Chebyshev’s inequality, most of the \(2^m\) partitions generated
by \(m\) coin flips have size within a few standard deviations of \(m^2/8\). This
leads quickly to the claimed bound.

Coin flipping model. We represent a partition \(\lambda\) of length \(\ell\) as the set of
boxes \(\{(i, j) : 1 \leq i \leq \ell, 1 \leq j \leq \lambda_i\}\), forming its Young diagram. We draw

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Young diagrams in ‘French notation’, so that the box \((i, j)\) is geometrically a unit square with diagonal from \((i - 1, j - 1)\) to \((i, j)\). For example, the partition \((6, 4, 2, 2)\) of length 4 is shown in Figure 1 above.

Let \(\Omega = \{H, T\}^m\) be the probability space for \(m\) flips of an unbiased coin in which each \(\omega \in \Omega\) has equal probability \(\frac{1}{2^m}\). Given \(\omega \in \Omega\) with exactly \(\ell\) tails, we define the boundary of a corresponding partition \(P(\omega)\) of length \(\ell\) as follows. Start at \((0, \ell)\) and step right to \((1, \ell)\). Then for each head, step one unit right, and for each tail, step one unit down. For instance, if \(m = 10\) and \(\omega = HTTHHTHHTH\) then \(P(\omega) = (6, 4, 2, 2)\); the final head corresponds to a step from \((6, 0)\) to \((7, 0)\) that is not part of a geometric box.

Let \(N\) be the size of \(P(\omega)\) and let \(X_t\) be the number of heads up to and including flip \(t\). Let \(Y = m - X_m\) be the total number of tails; this is the length of \(P(\omega)\). A move down at step \(t\) adds \(X_t - 1 + 1\) boxes to the Young diagram. Therefore setting

\[
C_t = \begin{cases} 
X_{t-1} & \text{if } \omega_t = T \\
0 & \text{if } \omega_t = H
\end{cases}
\]

we have \(N = Y + \sum_{t=1}^{m} C_t\).

**Expectation and variance.** Since \(X_t\) is distributed binomially as \(\text{Bin}(t, \frac{1}{2})\), we have \(\mathbf{E}[X_t] = t/2\) and \(\text{Var}X_t = t/4\). Hence \(\mathbf{E}[Y] = m/2\) and \(\text{Var}Y = m/4\). Observe that \(C_t\) is non-zero only if flip \(t\) is tails. Conditioning on this event shows that \(\mathbf{E}[C_t] = \frac{1}{2}\mathbf{E}[X_{t-1}] = \frac{t-1}{4}\). Hence, by linearity of expectation, \(\mathbf{E}[N] = \mathbf{E}[Y] + \sum_{t=1}^{m} \mathbf{E}[C_t] = \frac{m}{2} + \frac{1}{4} \sum_{t=1}^{m} (t-1) = m/2 + m(m-1)/8 = m(m+1)/8\).

**Lemma 1.** If \(t \leq u\) then the random variables \(C_t\) and \(X_u\) are uncorrelated.

**Proof.** Again we condition on the event that flip \(t\) is tails. In this event, \(C_t = X_{t-1}\) and \(X_u = X_{t-1} + W\), where \(W\) is the number of heads between flips \(t + 1\) and \(u\), inclusive. Since \(W\) is independent of \(X_{t-1}\),

\[
\mathbf{E}[C_t X_u] = \frac{1}{2} \mathbf{E}[X_{t-1}(X_{t-1} + W)]
\]
This is perhaps a little surprising, since the inequality whenever \( t < u \) simplifies calculations, we use the upper bound \( \text{Var} \) critically. By Lemma 1, \( E \) independent. A final conditioning argument shows that \( \text{Var} \) holds whenever \( t < u \) as required. 

As a corollary, we find that 

\[
E[C_tC_u] = \frac{1}{2} E[C_tX_{u-1}] = \frac{1}{2} E[C_t]E[X_{u-1}] = E[C_t][C_u]
\]

whenever \( t < u \). Hence \( C_t \) and \( C_u \) are uncorrelated for distinct \( t \) and \( u \). This is perhaps a little surprising, since the inequality \( C_t \leq C_u \), which holds whenever \( t < u \) and \( C_u \neq 0 \), shows that they are not in general independent. A final conditioning argument shows that \( \text{Var} C_t = E[C_t^2] - E[C_t]^2 = \frac{1}{2} E[X_{t-1}^2] - \frac{1}{4} E[X_{t-1}]^2 \), and so

\[
\text{Var} C_t = \frac{1}{2} \text{Var} X_{t-1} + \frac{1}{4} E[X_{t-1}]^2 = \frac{1}{2} \left( \frac{t-1}{4} \right) + \frac{1}{4} \left( \frac{t-1}{2} \right)^2 = \left( \frac{t-1}{16} \right)(2 + t - 1) = \frac{t^2 - 1}{16}.
\]

By Lemma 1, \( E[C_tY] = E[C_t(m - X_m)] = E[C_t]E[m - X_m] = E[C_t]E[Y] \) for all \( t \). Hence \( C_t \) and \( Y \) are also uncorrelated. If \( Z \) and \( Z' \) are uncorrelated random variables then, by a one-line calculation, \( \text{Var} (Z + Z') = \text{Var} Z + \text{Var} Z' \). We therefore have \( \text{Var} N = \text{Var} Y + \sum_{t=1}^{m} \text{Var} C_t \) and so

\[
\text{Var} N = \frac{m}{4} + \sum_{t=1}^{m} \frac{t^2 - 1}{16} = \frac{3m}{16} + \frac{m(m+1)(2m+1)}{96} = \frac{m^3}{48} + \frac{m^2}{32} + \frac{19m}{96}.
\]

Critically \( \text{Var} N \) is cubic in \( m \), not quartic as one might naively expect. To simplify calculations, we use the upper bound \( m^3/48 + m^2/32 + 19m/96 \leq 2m^3/96 + 4m^3/96 = m^3/16 \) for \( m \geq 3 \) to get \( \text{Var} N \leq m^3/16 \).

**Lower bound.** The concentration of measure estimate in Chebyshev’s inequality

\[
P\left[ |Z - E[Z]| \geq d\sqrt{\text{Var} Z} \right] \leq \frac{1}{d^2}
\]

implies that

\[
P\left[ |N - \frac{m(m+3)}{8}| \geq d\frac{m^{3/2}}{4} \right] \leq \frac{1}{d^2}
\]

for \( m \geq 3 \) and any \( d > 0 \).
The probability space $\Omega$ has $2^m$ elements. The proportion giving partitions with $|N - m(m + 3)/8| < dm^{3/2}/4$ is more than $1 - 1/d^2$. Since distinct coin flip sequences give distinct partitions, it follows that

$$\sum_n p(n) > 2^m(1 - \frac{1}{d^2})$$

where the sum is over all $n \in \mathbb{N}_0$ such that $|n - m(m + 3)/8| < dm^{3/2}/4$. Since $p(n)$ is monotonic, we deduce that, for $m \geq 3$,

$$p\left(\frac{m(m + 3)}{8} + dm^{3/2}/4\right) > \frac{2^m}{dm^{3/2}}\left(1 - \frac{1}{d^2}\right),$$

where we extend the domain of $p$ to $\mathbb{R}$ by setting $p(x) = p(\lfloor x \rfloor)$. The function $d \mapsto \frac{1}{d}(1 - \frac{1}{d^2})$ is maximized when $d = \sqrt{3}$, where it has value $\frac{2}{3\sqrt{3}}$. Therefore we take $d = \sqrt{3}$. Let $\eta > 0$ be given. Provided $m$ is sufficiently large we have $3m/8 + \sqrt{3}m^{3/2}/4 < \eta m^2/8$. Hence

$$p\left(\frac{m^2}{8}(1 + \eta)\right) > \frac{2^m}{3\sqrt{3}m^{3/2}}$$

for all $m$ sufficiently large. Setting $n = m^2(1 + \eta)/8$ and taking logs we obtain

$$\log p(n) \geq \sqrt{\frac{8n}{1 + \eta}} \log 2 - \frac{3}{2} \log \frac{8n}{1 + \eta} + \log \frac{4}{3\sqrt{3}}$$

for all $n$ sufficiently large. Since $(\log n)/\sqrt{n} \to 0$ as $n \to \infty$ it follows that for all $\varepsilon > 0$,

$$\frac{\log p(n)}{\sqrt{n}} > \frac{\sqrt{8}\log 2}{1 + \varepsilon}$$

for all $n$ sufficiently large, as claimed in $(\star)$. The constant on the right-hand side is approximately 1.961, somewhat lower than the asymptotically correct $2\sqrt{\pi/6} \approx 2.565$. For a concrete lower bound, take $\eta = \frac{1}{8}$ and $m = 8\sqrt{n}/3$ in $(\dag)$ to get $p(n) \geq 2^{8\sqrt{n}/3}/2^{5/2}n^{3/4}$ for all $n$ sufficiently large. (One can easily check that $n \geq 10^6$ suffices.) Using a computer to check small cases one can show that in fact

$$p(n) \geq 2^{8\sqrt{n}/3}/2^{5/2}n^{3/4}$$

for all $n \geq 2$.

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REFERENCES


